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ON GENERAL CONCEPT OF BASIC SUBGROUPS

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0. Introduction. The notion of basic subgroups has been introduced by Kulikov [5] and developed by Fuchs [3]. The purpose of this paper is to build up the general concept of basic subgroups of abelian groups which can be extended to any Abelian category. In the section 1, there are presented some fundamental properties of the basic subgroups, in particular the proposition 1.7 shows the equivalence of our definition with the Kulikov's one in reduced torsion abelian groups and the corollary 1.12 describes the structure of alg. compact groups with respect to its basic subgroups. The section 2 develops the concept of basic decompositions with respect to the basic subgroups, i.e. there is discussed the question of an embedding of a given ab. group into a direct product of its basic components.

Throughout the paper a group G always stands for an abelian group. Concerning the terminology and notation, we refer to [3], 282. Otherwise, if G is a group then G_t , G_n , G_p and G^1 are the torsion part, n -component of G_t , divisible part and the first Ulm's subgroup of G respec-

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tively. We shall frequently use the following notation:

\mathbb{N} - natural numbers,

$\mathbb{N}^+ = \{n \in \mathbb{N}; n > 0\}$,

\mathbb{P} - the set of all prime numbers,

\hat{G} - the \mathbb{Z} -adic completion of G ,

(m, m') - the greatest common divisor of m and m' .

If $\{G_\alpha; \alpha \in \Lambda\}$ is a family of groups then $\coprod_{\alpha \in \Lambda} G_\alpha$ stands for the direct sum and any subgroup H , where $\coprod_{\alpha \in \Lambda} G_\alpha \subset H \subset \prod_{\alpha \in \Lambda} G_\alpha$ is called the interdirect sum of $\prod_{\alpha \in \Lambda} G_\alpha$. By a superdecomposable group we mean a group with no nonzero indecomposable direct summand (see [1]). For a $K \subset \mathbb{P}$, we define that H is a K -pure subgroup of G if $\rho^m H = H \cap \rho^m G$, for $\forall \rho \in K, \forall m \in \mathbb{N}$ and G is K -divisible if $G = \rho G$, for $\forall \rho \in K$.

1. An outline of the theory.

Definition 1.1. We shall say that B is a basic subgroup of a group G if

(i) $B = \langle \{G_\alpha; \alpha \in \Lambda\} \rangle$, where $0 \neq G_\alpha$ is an indecomposable subgroup of G , for $\forall \alpha \in \Lambda$.

(ii) $\langle \{G_\alpha; \alpha \in X\} \rangle = \coprod_{\alpha \in X} G_\alpha$ and $\coprod_{\alpha \in X} G_\alpha$ is a direct summand of G for each finite $X \subset \Lambda$.

(iii) The family $\{G_\alpha; \alpha \in \Lambda\}$ is maximal with respect to the conditions (i) and (ii).

The family $\{G_\alpha; \alpha \in \Lambda\}$ is called the basic system of G corresponding to B .

Remark 1.2. The condition 1.1 (ii) implies that B is a pure subgroup of G and $B = \coprod_{\alpha \in \Lambda} G_\alpha$. In the following we shall write $B_F = \coprod_{\alpha \in \Lambda_F} G_\alpha$ and $B_R = \coprod_{\alpha \in \Lambda_R} G_\alpha$, where $\Lambda_F = \{\alpha \in \Lambda; G_\alpha \text{ is torsion-free}\}$ and $\Lambda_R = \{\alpha \in \Lambda; G_\alpha \text{ is reduced}\}$. Similarly we shall use the symbols Λ_t, Λ_n and Λ_D for B_t, B_n and B_D respectively.

Proposition 1.3. Every group contains a basic subgroup.

Proof. Let us denote the set of all the families $\{G_\alpha; \alpha \in \Lambda\}$ satisfying 1.1 (i) and (ii) in a group G by $D(G)$. Obviously $D(G) \neq \emptyset$. Let $\{D_\gamma; \gamma \in \Gamma\} \subset D(G)$ be a chain with respect to the inclusion and $C = \bigcup_{\gamma \in \Gamma} D_\gamma$. If $G_{\alpha_1}, \dots, G_{\alpha_m} \in C$ then there is a $\gamma \in \Gamma$ such that $G_{\alpha_1}, \dots, G_{\alpha_m} \in D_\gamma$. Hence $C \in D(G)$ and the Zorn's lemma immediately implies the existence of B . q.e.d.

Proposition 1.4. Let B be a basic subgroup of G . Then

- (i) If $G = H \oplus W, B \subset W$, then H is a superdecomposable group.
- (ii) G/B contains no nonzero cyclic direct summand.
- (iii) B_t is a basic subgroup of G_t .
- (iv) G/B_t is a splitting group, where $(G/B_t)_t = G_t/B_t$ is a divisible group.
- (v) G/B is a splitting group, where $(G/B)_t = (G_t + B)/B$ is a divisible group.

Proof. Let $\{G_\alpha; \alpha \in \Lambda\}$ be a basic system of G corresponding to B .

(i) If H' is an indecomposable direct summand of H then the family $\{H'\} \cup \{G_\alpha; \alpha \in \Lambda\}$ obviously satisfies 1.1 (ii) and the maximality of $\{G_\alpha; \alpha \in \Lambda\}$ implies $H' = 0$.

(ii) If $G/B = (G_1/B) \oplus (G_2/B)$, where G_1/B is a cyclic group then $G_1 = B \oplus G'_1$ since B is a pure subgroup of G_1 . Hence $G = G'_1 \oplus G_2$, where $B \subset G_2$ and (i) yields $G'_1 = 0$.

(iii) Obviously $\{G_\alpha; \alpha \in \Lambda_t\}$ satisfies 1.1 (i) and (ii) in G_t . If $\{H\} \cup \{G_\alpha; \alpha \in \Lambda_t\}$ satisfies the conditions 1.1 (i) and (ii) in G_t then $\{H\} \cup \{G_\alpha; \alpha \in \Lambda\}$ satisfies 1.1 (i) and (ii) in G . For, if $K_t \subset \Lambda_t$ and $K_F \subset \Lambda_F$ are finite then $G = \prod_{\alpha \in K_F} G_\alpha \oplus W$ and obviously $G_t \subset W$. Since $\prod_{\alpha \in K_t} G_\alpha \oplus H$ is a direct sum of a divisible and a bounded group and it is a pure subgroup of W , we can write $G = \prod_{\alpha \in K_F} G_\alpha \oplus \prod_{\alpha \in K_t} G_\alpha \oplus H \oplus W'$. Hence it contradicts the maximality of the family $\{G_\alpha; \alpha \in \Lambda\}$.

(iv) By (iii) and (ii) G_t/B_t is divisible and consequently $G/B_t = (G_t/B_t) \oplus (G'/B_t)$, where $(G'/B_t) \cong \cong (G/G_t)$ is torsion-free.

(v) $(G_t + B)/B \cong G_t/(B \cap G_t) = G_t/B_t$ and by (iv) $G/B = ((G_t + B)/B) \oplus (G'/B)$. If $q' \in G'$ and $mq' \in B$ then $mq' = m\mathcal{B}$ for some $\mathcal{B} \in B$ since B is pure in G' . Hence $(q' - \mathcal{B}) \in G_t$ i.e. $q' \in (G_t + B) \cap G' = B$ and consequently G'/B is torsion-free. q.e.d.

Proposition 1.5. $B = B_p \oplus B_R$ is a basic subgroup of a group G iff $B_p = G_p$ and B_R is a basic subgroup

of some direct complement of G_D .

Proof. (i) Let B be a basic subgroup of a group G and $\{G_\alpha; \alpha \in \Lambda\}$ be a corresponding basic system of G . Then $B_R \cap G_D = 0$. For, if $g \in B_R \cap G_D$ then $g \in$

$\bigcap_{n \in \mathbb{N}^+} n B_R = \bigcap_{\alpha \in \Lambda_R} \left(\bigcap_{n \in \mathbb{N}^+} n G_\alpha \right)$ since B_R is pure in G . On the other hand G_α is either reduced torsion-free or torsion cyclic for $\forall \alpha \in \Lambda_R$. Hence $\bigcap_{n \in \mathbb{N}^+} n G_\alpha = 0$, for $\forall \alpha \in \Lambda_R$. Since G_D is an absolute direct summand there is a direct complement R of G_D containing B_R and by 1.4 (i),

$B_D = G_D$. Now, it is sufficient to show that $\{G_\alpha; \alpha \in \Lambda_R\}$ is a basic system of R . For, the conditions 1.1 (i), (ii) are obvious and 1.1 (iii) immediately follows from the maximality of $\{G_\alpha; \alpha \in \Lambda\}$ in G and from the decomposition $G = B_D \oplus R$.

(ii) Conversely, let $G = G_D \oplus R$, B_R be a basic subgroup of R and $\{G_\alpha; \alpha \in \Lambda_R\}$ be the corresponding basic system of R . Since G_D is a direct sum of nonzero indecomposable subgroups, say $G_D = \bigoplus_{\alpha \in \Lambda_D} G_\alpha$ then the family $\{G_\alpha; \alpha \in \Lambda_D \cup \Lambda_R\}$ obviously satisfies the conditions 1.1 (i), (ii) in G . Let $\{G_\alpha; \alpha \in \Lambda_D \cup \Lambda_R\} \cup \{H\}$ be a family of subgroups satisfying 1.1 (i), (ii) in G then H is obviously a reduced subgroup and since G_D is an absolute direct summand of G there is a direct complement R' of G_D , containing $B_R \oplus H$. If $\kappa' \in R'$ then $\kappa' = (\alpha + \kappa) \in G_D \oplus R$ and the map $\varphi: R' \xrightarrow{\kappa'} R$ is an isomorphism R' onto R which is the identity on B_R . Hence $\varphi(B_R \oplus H) = B_R \oplus \varphi(H)$ and $\{G_\alpha; \alpha \in \Lambda_R\} \cup \{\varphi(H)\}$

satisfies the conditions 1.1 (i),(ii) in \mathbb{R} , which contradicts the maximality of $\{G_\alpha; \alpha \in \Lambda_{\mathbb{R}}\}$ in \mathbb{R} . q.e.d.

Proposition 1.6. Let G be either cotorsion or a group, where the only indecomposable direct summands are either divisible or torsion cyclic and $B \subset G$ be a subgroup satisfying 1.1 (i),(ii) in G . Then B is basic in G iff $G = H \oplus W$, $B \subset W$ implies that H is superdecomposable.

Proof. With respect to 1.4 (i) we shall prove only the sufficient condition. Let $\{G_\alpha; \alpha \in \Lambda\}$ be a family corresponding to B . If there is a larger family $\{H\} \cup \{G_\alpha; \alpha \in \Lambda\}$ satisfying 1.1 (i),(ii) in G then by the hypothesis H is alg. compact. By 1.2 $H \oplus B$ is pure in G and consequently $(H \oplus B)/B$ is pure in G/B . Hence $G/B = ((H \oplus B)/B) \oplus (G'/B)$ and consequently $G = H \oplus G'$, where $B \subset G'$. So, H being superdecomposable yields a contradiction, q.e.d.

Proposition 1.7. Let B be a subgroup of a reduced torsion group G . Then B is basic in G iff

- (i) B is a direct sum of cyclic subgroups of prime power orders,
- (ii) B is a pure subgroup,
- (iii) G/B is divisible.

Proof. If B is a basic subgroup then by 1.1 (i), 1.2 and 1.4 (iv) the conditions (i),(ii) and (iii) are satisfied. Conversely, (i) implies 1.1 (i) and since every finite direct sum of groups of a prime power orders is bound-

ded and by (ii) is pure in G , 1.1 (ii) is satisfied. If $G = H \oplus W$, $B \subset W$ then $G/B \cong H \oplus (W/B)$ and (iii) implies that $H = 0$. Hence by 1.6, B is a basic subgroup of G . *q.e.d.*

Proposition 1.8. A subgroup $B = \prod_{p \in P} B_p$ of a torsion group G is basic in G iff B_p is a basic subgroup of G_p , for $\forall p \in P$.

Proof. If B is a basic subgroup of G then B_p obviously satisfies 1.1 (i), (ii) in G_p . Let $G_p = H \oplus W$, $B_p \subset W$; then $G = H \oplus W \oplus \prod_{q \in P, q \neq p} G_q$, $B \subset W \oplus \prod_{q \in P, q \neq p} G_q$ and 1.6 implies that $H = 0$ and B_p is a basic subgroup of G_p . Conversely, B obviously satisfies 1.1 (i) and since $G = \prod_{p \in P} G_p$ and each B_p satisfies 1.1 (ii) in G_p , B satisfies 1.1 (ii) in G , too. If $G = H \oplus W$, $B \subset W$ then $G_p = H_p \oplus W_p$, where $B_p \subset W_p$ and 1.6 implies that $H_p = 0$, for $\forall p \in P$, i.e. $H = 0$ and B is a basic subgroup of G . *q.e.d.*

Proposition 1.9. If G is either alg. compact or an adjusted group and B a basic subgroup of G , then G/B is divisible.

Proof. (i) Let G be alg. compact. By 1.5, $B_p = G_p$ and there is a subgroup $G' \subset G$ such that $G = G' \oplus G_p$ and B_R is a basic subgroup of G' . Obviously $G/B \cong G'/B_R$ and G' is the reduced alg. compact group. Hence by [3], 39.1, 163, G' is complete in the Z -adic topology and since B_R is pure in G' , $G' = H \oplus (\bigcap_{m \in \mathbb{N}^+} (B_R + mG'))$ by [3], ex.2, 166. On the other hand 1.4 (i) and [3], 40.4, 169 imply $H = 0$. Hence $G' = B_R + mG'$, for $\forall m \in \mathbb{N}^+$.

(ii) Let G be an adjusted group. By [3], 55.1,237 , G/G_t is divisible. Since $B_t = B$, 1.4 (iv) implies that G_t/B is divisible. Hence $G/B = H \oplus (G_t/B)$, where $H \cong G/G_t$. q.e.d.

Corollary 1.10. If G is a cotorsion group then there exists a basic subgroup $B \subset G$ such that G/B is divisible.

Proof. By [3], 55.5, 238 , $G = A \oplus C \oplus G_D$, where C is the adjusted part and A is alg. compact. If B_A and B_C are basic subgroups of A and C respectively then $B_A \oplus B_C \oplus G_D$ satisfies 1.1 (i),(ii) in G and there is a basic subgroup B of G such that $B_A \oplus B_C \oplus G_D \subset B$. By 1.9, $G/(B_A \oplus B_C \oplus G_D) \cong (A/B_A) \oplus (C/B_C)$ is divisible hence G/B is divisible, too. q.e.d.

Theorem 1.11. Let B be a basic subgroup of a group G . If $G^1 = 0$ and G/B is divisible then G can be embedded as a pure subgroup in \hat{B} . Moreover, if G is torsion-free then $\hat{B} \cong \text{Ext}(G/Z, B)$.

Proof. Consider the following map $\varphi: G \rightarrow \hat{B}$. Since G/B is divisible, there is a sequence $\{x_n\}_{n=1}^{\infty}$ for $\forall q \in G$, such that $(q - x_n) \in n!G$ and $x_n \in B$, for $\forall n \in \mathbb{N}^+$, therefore $q = \lim \{x_n\}_{n=1}^{\infty}$ in Z -adic topology of G . Now, let $\varphi(q) = \lim \{x_n\}_{n=1}^{\infty} \in \hat{B}$. Since G is Hausdorff in Z -adic topology, \hat{B} is Hausdorff by [4], 29, 27 and B is pure in G , φ is a well-defined homomorphism. If $\lim \{x_n\}_{n=1}^{\infty} = q \in G$ and $\lim \{x_n\}_{n=1}^{\infty} = 0 \in \hat{B}$ then $x_n \in n\hat{B}$ for $\forall n \in \mathbb{N}^+$

$\forall n \geq m$. By [41, 30, 28], \mathcal{B} is pure in $\hat{\mathcal{B}}$ hence $b_n \in e n \mathcal{B} \subset n \mathcal{G}$ and consequently $q = 0$. Therefore, φ is an embedding and without loss of generality we can assume that $\mathcal{G} \subset \hat{\mathcal{B}}$. Let $q = m \hat{x}$, where $q \in \mathcal{G}$ and $\hat{x} \in \hat{\mathcal{B}}$. The divisibility of \mathcal{G}/\mathcal{B} implies that $q = \mathcal{L} + m \mathcal{H}$, for some $\mathcal{L} \in \mathcal{B}$, $\mathcal{H} \in \mathcal{G}$. Therefore $\mathcal{L} = m(\hat{x} - \mathcal{H}) = m \mathcal{L}'$, for some $\mathcal{L}' \in \mathcal{B}$, since \mathcal{B} is pure in $\hat{\mathcal{B}}$. Hence $q = m(\mathcal{L}' + \mathcal{H})$. Moreover, if \mathcal{G} is torsion-free then $\hat{\mathcal{B}}$ is torsion-free, complete in the \mathbb{Z} -adic topology ([41, 29, 27]) and reduced. Hence by [31, 39.1, 163] $\hat{\mathcal{B}}$ is cotorsion and $\hat{\mathcal{B}}/\mathcal{B}$ is torsion-free, divisible by [41, 30, 28]. The following exact sequence

$$0 = \text{Hom}(\mathcal{Q}/\mathbb{Z}, \hat{\mathcal{B}}/\mathcal{B}) \rightarrow \text{Ext}(\mathcal{Q}/\mathbb{Z}, \mathcal{B}) \rightarrow \text{Ext}(\mathcal{Q}/\mathbb{Z}, \hat{\mathcal{B}}) \cong \hat{\mathcal{B}} \rightarrow \text{Ext}(\mathcal{Q}/\mathbb{Z}, \hat{\mathcal{B}}/\mathcal{B}) = 0$$

immediately implies the desired result. q.e.d.

Corollary 1.12. Let \mathcal{B} be a basic subgroup of a group \mathcal{G} , where $\mathcal{G}' = 0$. Then \mathcal{G} is alg. compact iff $\mathcal{G} \cong \hat{\mathcal{B}}$.

Proof. Let \mathcal{G} be alg. compact. According to 1.9, \mathcal{G}/\mathcal{B} is divisible and hence it is sufficient to show that the embedding $\varphi: \mathcal{G} \rightarrow \hat{\mathcal{B}}$ from 1.11 is onto. If $\hat{x} \in \hat{\mathcal{B}}$ then $\hat{x} = \lim \{b_m\}_{m=1}^{\infty}$ in $\hat{\mathcal{B}}$ for some $\{b_m; m \in \mathbb{N}^+; \} \subset \mathcal{B}$. Since $\{b_m\}_{m=1}^{\infty}$ is a Cauchy-sequence in \mathcal{B} i.e. in \mathcal{G} and by [31, 39.1, 163] \mathcal{G} is complete in the \mathbb{Z} -adic topology i.e. there is a $q \in \mathcal{G}$ such that $\varphi(q) = \hat{x}$. The converse of the corollary immediately follows from [31, 39.1, 163]. q.e.d.

Proposition 1.13. There are neither nonzero cotorsion, slender nor nonzero cotorsion, \aleph_1 -free groups.

Proof. If $\mathcal{G} \neq 0$ is a cotorsion, slender group then

by [4], 154, 111 G is torsion-free, reduced and there is no copy of π -adic integers in G . Hence by [3], 54.5, 235 G is alg. compact and ([3], 40.4, 169) implies a contradiction.

If $G \neq 0$ is a cotorsion, π_1 -free group then G is torsion-free, reduced and consequently G is alg. compact. Hence by [3], 40.4, 169 G contains a copy of π -adic integers J_π and since $G_\pi \subset J_\pi$, it contradicts the hypothesis. q.e.d.

2. On basic decompositions.

Definition 2.1. Let B be a subgroup of G satisfying 1.1 (i),(ii). Then the intersection of all the direct complements of $\coprod_{\alpha \in K} G_\alpha$, where K runs through the finite subsets of Λ , is called the B -residual subgroup of G , denoted by R_B^G .

Proposition 2.2. Let B be a subgroup of G satisfying 1.1 (i),(ii). Then R_B^G is the greatest fully invariant subgroup disjoint with B .

Proof. Denote by W_K the intersection of all the direct complements of $\coprod_{\alpha \in K} G_\alpha$, for a finite $K \in \Lambda$. By [3], 9.6, 48, W_K is fully invariant subgroup and R_B^G as an intersection of fully invariant subgroups is fully invariant, too, and obviously $R_B^G \cap B = 0$. If A is a fully invariant subgroup such that $A \cap B = 0$ and $G = (\coprod_{\alpha \in K} G_\alpha) \oplus W$ a direct decomposition for a finite $K \in \Lambda$, then by [3], 9.3, 47 $A \subset W$. Hence $A \subset R_B^G$. q.e.d.

The proof of the following proposition is straightforward and hence omitted.

Proposition 2.3. Let G be a group. Then the following are equivalent:

- (i) 0 is a basic subgroup,
- (ii) G is superdecomposable,
- (iii) 0 is the unique basic subgroup,
- (iv) if B is a basic subgroup, then $R_B^G = G$,
- (v) there is a basic subgroup B such that $R_B^G = G$.

Lemma 2.4. Let D and M be subgroups of a group G such that M is D -high in G . If $y \in G \setminus D \oplus M$ and $ky \in M$, for some $k \in \mathbb{N}^+$ then there exists a $m \in \mathbb{N}^+$ such that $1 < m < k$, $m | k$ and $my \in D \oplus M \setminus M$. In particular, k is not a prime.

Proof. Since $D \cap \langle M, y \rangle \neq 0$, consider a $d = m + ly$, where $d \in D$, $d \neq 0$, $m \in M$ and $l \in \mathbb{N}^+$. Let $m = (l, k)$, $k = k'm$ and $l = l'm'$. Since $ul' + vk' = 1$, for some $u, v \in \mathbb{Z}$ we get $ud = um + my - vk'y$ and hence $my \in M \oplus D$ and consequently $m > 1$. If $my \in M$ then obviously $ud = 0$, i.e. $\sigma(d) | u$. Furthermore, $k'd = k'm + l'ky$, so $k'd = 0$ and $\sigma(d) | k'$. Hence $\sigma(d) = 1$ implies a contradiction and $my \notin M$, $m < k$ follows immediately. q.e.d.

Next we shall use the following notation. If $B = \coprod_{\alpha \in \Lambda} G_\alpha$, where each G_α is a nonzero reduced indecomposable torsion group then $B_{p,m}$ stands for $\coprod_{\alpha \in \Lambda_{p,m}} G_\alpha$, where $\Lambda_{p,m} = \{ \alpha \in \Lambda ; G_\alpha \cong \mathbb{Z}(\pi^m) \}$. Consider $K_B = \{ p \in \mathbb{P} ; B_p \neq 0 \}$ in the

natural order. In case that $K_B = \{\pi_1, \dots, \pi_2\}$, take $\pi_b = 1$ for each $b > 2$. Define the following sequence from \mathbb{N}^+ by $\lambda_0 = 1$ and $\lambda_{m+1} = \lambda_m \cdot \pi_1 \cdots \pi_{m+1}$ and the corresponding sequence of subgroups by $B_0 = 0$, $B_{m+1} = \prod_{i=1}^{m+1} B_{\pi_i, m+2-i}$, for each $m \in \mathbb{N}$. Then obviously $B = \prod_{m \in \mathbb{N}} B_m$ and $\lambda_m B \subset \prod_{i > m} B_i$.

Theorem 2.5. Let G be a group, $B \subset G$ a reduced torsion subgroup satisfying 1.1 (i), (ii) in G and $F \subset G$ be a K -pure torsion-free subgroup, where $K_B \subset K \subset P$. Then

1) For $\forall m \in \mathbb{N}$, there is a subgroup $W_m \subset G$ such that

$$(i) \quad G = B_1 \oplus \dots \oplus B_m \oplus W_m,$$

$$(ii) \quad W_m = B_{m+1} \oplus W_{m+1},$$

(iii) W_{m+1} is a B_{m+1} -high subgroup of W_m containing $(\prod_{i > m+1} B_i + F + \lambda_{m+1} G)$.

2) There is a homomorphism $\varphi: G \rightarrow \prod_{\substack{\pi \in K_B \\ m \in \mathbb{N}^+}} B_{\pi, m}$ whose image is an interdirect sum of $\prod B_{\pi, m}$ and

$$(i) \quad G' \subset \bigcap_{\substack{\pi \in K_B \\ m \in \mathbb{N}}} \pi^m G = \bigcap_{m \in \mathbb{N}} \lambda_m G \subset R_B^G \subset \ker \varphi = \bigcap_{m \in \mathbb{N}} W_m,$$

$$(ii) \quad \bigcap_{m \in \mathbb{N}} (F + \lambda_m G) \subset \bigcap_{\substack{\pi \in K_B \\ m \in \mathbb{N}}} (F + \pi^m G) \subset \ker \varphi.$$

3) If $G/B \oplus F$ is K -divisible then

$$\ker \varphi = \bigcap_{\substack{\pi \in K \\ m \in \mathbb{N}}} (F + \pi^m G) = \bigcap_{m \in \mathbb{N}} (F + \lambda_m G) \quad \text{and}$$

$\varphi(G)$ is K -pure in $\prod B_{\pi, m}$.

In particular, if G/B is K -divisible then

$$R_B^G = \text{ker } \varphi = \bigcap_{\substack{\mu \in \mathbb{K} \\ m \in \mathbb{N}}} \pi^m G.$$

Proof. 1) $B_m \cap (\prod_{i \geq m} B_i + F + \mathfrak{K}_m G) = 0$, for $\forall m \in \mathbb{N}$. For, let $l = l' + f + \mathfrak{K}_m g$, where $l \in B_m$, $l' \in \prod_{i \geq m} B_i$, $f \in F$, $g \in G$ and $\nu(l - l') = m$, then $m = \pi \pi_i^{\alpha_i}$, where each $\pi_i \in \mathbb{K}_B$. Since F is \mathbb{K}_B -pure there is a $f' \in F$ such that $-mf = m\mathfrak{K}_m g = m\mathfrak{K}_m f'$. Hence $f = -\mathfrak{K}_m f'$, $l - l' = \mathfrak{K}_m (g - f') = \mathfrak{K}_m l'' \in \prod_{i \geq m} B_i$ since B is pure in G and consequently $l = 0$.

For $m = 0$, define $W_0 = G$ and W_1 as a B_1 -high subgroup of W_0 containing $(\prod_{i \geq 1} B_i + F + \mathfrak{K}_1 G)$. Obviously $W_0 \supset W_1 \oplus B_1$ and since $\pi_1 g \in W_1$, for every $g \in W_0$, 2.4 implies $W_0 = B_1 \oplus W_1$. Suppose that the theorem holds for an $n - 1 \geq 0$. Hence $W_n \supset B_{n+1} \oplus (\prod_{i \geq n+1} B_i + F + \mathfrak{K}_{n+1} G)$. Denote by W_{n+1} the B_{n+1} -high subgroup of W_n containing $(\prod_{i \geq n+1} B_i + F + \mathfrak{K}_{n+1} G)$. Let $g \in W_n \setminus B_{n+1} \oplus W_{n+1}$, then obviously $\mathfrak{K}_{n+1} g \in W_{n+1}$. By 2.4, let κ_1 be the greatest natural number such that $1 < \kappa_1 < \mathfrak{K}_{n+1}$, $\kappa_1 | \mathfrak{K}_{n+1}$ and $\kappa_1 g \in B_{n+1} \oplus W_{n+1} \setminus W_{n+1}$.

Then $\kappa_1 g = l + w$, where $l \in B_{n+1}$, $w \in W_{n+1}$ and there is an $i \in \{1, \dots, n+1\}$ such that $\pi_i \kappa_1 g \in W_{n+1}$. Hence $\pi_i \kappa_1 g - \pi_i w = \pi_i l \in W_{n+1}$ and consequently $\pi_i l = 0$, i.e. $l \in B_{\pi_i, n+2-i}$ and $l = \pi_i^{n+1-i} l'$, for some $l' \in B_{\pi_i, n+2-i}$. Let $\kappa_1 = \pi_i^a \kappa_1'$, where $(\kappa_1', \pi_i) = 1$, then $a \leq n+1-i$. Obviously $l' = \kappa_1' l''$, where $l'' \in B_{\pi_i, n+2-i}$. Hence $l = \pi_i^{n+1-i} \kappa_1' l'' = \kappa_1 l_1$,

$g - \ell_1 \notin B_{m+1} \oplus W_{m+1}$ and $\kappa_1(g - \ell_1) = w \in W_{m+1}$. By the repeated use of 2.4 we get finite sequences ℓ_1, \dots, ℓ_r and $\kappa_{m+1} > \kappa_1 > \dots > \kappa_r > 1$, where $\kappa_j \in \mathbb{P}$, such that $(g - \ell_1 - \dots - \ell_r) \notin B_{m+1} \oplus W_{m+1}$ and $\kappa_r(g - \ell_1 - \dots - \ell_r) \in W_{m+1}$, which contradicts 2.4.

2) Consider the following map

$$\varphi: G \longrightarrow \prod B_{\pi, m}$$

$$g \longmapsto (\ell_{\pi_1, 1}; \ell_{\pi_1, 2}, \ell_{\pi_2, 1}; \ell_{\pi_1, 3}, \ell_{\pi_2, 2}, \ell_{\pi_3, 1}; \dots),$$

where $\ell_{\pi_i, j}$ are the direct components of g in the decompositions $G = B_{\pi_1, 1} \oplus \dots \oplus (\prod_{i=1}^m B_{\pi_i, m+1-i}) \oplus W_m$, for each $m \in \mathbb{N}$. With respect to the equality $W_m = B_{m+1} \oplus W_{m+1}$, φ is a well-defined map and a homomorphism as well.

Since $B = \prod_{\substack{\pi \in K_B \\ m \in \mathbb{N}^+}} B_{\pi, m}$ and \mathcal{G}/B is an identity,

$\text{im } \varphi$ is an interdirect sum of $\prod B_{\pi, m}$. For the rest it is sufficient to prove 2) (ii). In particular,

$\bigcap_{\substack{\pi \in K_B \\ m \in \mathbb{N}}} (F + \pi^m G) \subset \ker \varphi$. For, if $g \in \bigcap (F + \pi^m G)$ and $m \in \mathbb{N}^+$ then $g = f_i + \pi_i^{m+1-i} h_i$, for some $f_i \in F$, $h_i \in G$ and an arbitrary $i = 1, \dots, m$ ($\kappa_m = \prod_{i=1}^m \pi_i^{m+1-i}$, $\pi_i \in K_B$).

Since $G = B_1 \oplus \dots \oplus B_m \oplus W_m$, we can write $h_i = \ell_i + w_i$, where $\ell_i \in \prod_{j=1}^m B_j$ and $w_i \in W_m$, $i = 1, \dots, m$. Now, $g = \pi_i^{m+1-i} \ell_i + (f_i + \pi_i^{m+1-i} w_i)$, $i = 1, \dots, m$ and since $\ell = \pi_i^{m+1-i} \ell_i = \pi_j^{m+1-j} \ell_j$, for $i, j = 1, \dots, m$, $\ell = 0$ and $g \in W_m$.

3) $\ker \varphi = \bigcap_{\substack{j, n \in \mathbb{N} \\ \rho \in \mathbb{K}} (F + \rho^{\dot{j}} \mathfrak{h}_m G)$. For, if $q \in \ker \varphi$ then for $\forall (\rho \in \mathbb{K}) \forall (j, n \in \mathbb{N})$, $q = \rho + \varepsilon + \rho^{\dot{j}} \mathfrak{h}_m h$ for some $\rho \in \mathbb{B}$, $\varepsilon \in F$ and $h \in G$ since $G/\mathbb{B} \oplus F$ is \mathbb{K} -divisible. Since $q \in W_m$, $\rho \in \prod_{i=m+1}^{m+t} B_i$, for some $t \in \mathbb{N}$ and similarly $q = \rho' + \varepsilon' + \rho^{\dot{j}} \mathfrak{h}_{m+t} h'$, where $\rho' \in \prod_{i \geq m+t} B_i$, $\varepsilon' \in F$ and $h' \in G$. If $\sigma(\rho - \rho') = m$ then $m = \prod \rho_i^{\alpha_i}$, where each $\rho_i \in \mathbb{K}_{\mathbb{B}}$ and since F is \mathbb{K} -pure there is a $\varepsilon'' \in F$ such that $m(\varepsilon - \varepsilon') = m \rho^{\dot{j}} \mathfrak{h}_m (h'' - h) = m \rho^{\dot{j}} \mathfrak{h}_m \varepsilon''$, where $\mathfrak{h}_m h'' = \mathfrak{h}_{m+t} h'$. Hence $\varepsilon - \varepsilon' = \rho^{\dot{j}} \mathfrak{h}_m \varepsilon''$, $\rho - \rho' = \rho^{\dot{j}} \mathfrak{h}_m (h'' - h - \varepsilon'') = \rho^{\dot{j}} \mathfrak{h}_m \rho''$, for some $\rho'' \in \mathbb{B}$, since \mathbb{B} is pure in G and consequently $\rho = \rho^{\dot{j}} \mathfrak{h}_m \rho_1$, for some $\rho_1 \in \prod_{i \geq m} B_i$, implies $q \in F + \rho^{\dot{j}} \mathfrak{h}_m G$. Hence $\ker \varphi \subset \bigcap_{\substack{n \in \mathbb{N} \\ \rho \in \mathbb{K}}} (F + \rho^n G)$, $\ker \varphi \subset \bigcap_{n \in \mathbb{N}} (F + \mathfrak{h}_n G)$.

$\varphi(G)/\varphi(B) \cong G/\ker \varphi \oplus B$ is \mathbb{K} -divisible, i.e. $\varphi(G) = \varphi(B) + \rho^n \varphi(G)$, for $\forall (\rho \in \mathbb{K}), \forall (n \in \mathbb{N})$. Let $\rho^n x = h \in \varphi(G)$, for some $x \in \prod B_{\rho, n}$ and $\rho \in \mathbb{K}$, then $h = \rho + \rho^n h'$, for some $\rho \in \varphi(B)$ and $h' \in \varphi(G)$. Since $\varphi(B) = \prod B_{\rho, n}$ is pure in $\prod B_{\rho, n}$, there is a $\rho' \in \varphi(B)$ such that $\rho = \rho^n \rho'$ and hence $h = \rho^n (\rho' + h')$. q.e.d.

Definition 2.6. Under the conditions of the theorem 2.5 we shall call the sequence $\{W_m; m \in \mathbb{N}\}$ the \mathbb{B} -sequence of G and the corresponding homomorphism φ the

B -homomorphism of G .

Corollary 2.7. If G is either a reduced torsion or an adjusted group and B is a basic subgroup of G , then G/G^1 is isomorphic to a pure interdirect sum H of $\prod_{\substack{n \in \mathbb{N} \\ m \in \mathbb{N}^+}} B_{p,m}$. Moreover, in case of an adjusted group, H is a direct summand.

Proof. With respect to 1.7, 1.9 and 2.5 it is sufficient to show that H is a direct summand provided that G is adjusted. By [3], 54.3, 235 $H \cong G/G^1$ is alg. compact and hence a direct summand. q.e.d.

Proposition 2.8. Let B be a basic subgroup of a reduced group G . Then $(R_B^G)_t = (R_{B_t}^G)_t = (\ker \varphi)_t = G_t^1$ for every B_t -homomorphism φ of G .

Proof. Let $\{W_n; n \in \mathbb{N}\}$ be a B_t -sequence of G . Since G_t is fully invariant in G , $G_t = B_1 \oplus \dots \oplus B_n \oplus (W_n \cap G_t)$, for $\forall n \in \mathbb{N}$ and the divisibility of G_t/B_t (see 1.4 (iv)) implies $W_n \cap G_t = \coprod_{i \geq n} B_i + A_n G_t$. Hence $(\ker \varphi)_t = G_t \cap (\bigcap_{n \in \mathbb{N}} W_n) = \bigcap (W_n \cap G_t) = \ker \varphi'$, where $\varphi' = \varphi/G_t$ and 2.5 and 1.4 (iv) imply $\ker \varphi' = G_t^1$.

The converse inclusion $G_t^1 \subset (R_B^G)_t \subset (R_{B_t}^G)_t \subset (\ker \varphi)_t$ follows immediately from 2.2 and 2.5. q.e.d.

Proposition 2.9. The adjusted part of $\prod_{\substack{n \in \mathbb{P} \\ m \in \mathbb{N}}} H_{p,m}$, where $H_{p,m} = \coprod_{\alpha \in A_{p,m}} H_\alpha$, $H_\alpha \cong Z(p^n)$, for $\forall \alpha \in A_{p,m}$, is a minimal direct summand of $\prod H_{p,m}$ containing $\coprod H_{p,m}$.

Proof. By 1.8, 1.7 and [2], 29.6, 100, $\coprod H_{p,m}$ is a basic subgroup of $(\prod H_{p,m})_t$. Let G be the adjusted part

of $\Pi H_{p,n}$, then $\Pi H_{p,n}$ is a basic subgroup of G . For, $\Pi H_{p,n} \subset (\Pi H_{p,n})_t \subset G$ and $\Pi H_{p,n}$ obviously satisfies 1.1 (i),(ii) in G . If $A \oplus \Pi H_{p,n}$ also satisfies 1.1 (i),(ii) in G , where A is a nonzero indecomposable subgroup of G , then $A \oplus \Pi H_{p,n}$ also satisfies 1.1 (i),(ii) in $\Pi H_{p,n}$ since G is a direct summand by [3], 55.5, 238 and consequently in $(\Pi H_{p,n})_t$, which implies a contradiction since $\Pi H_{p,n}$ is basic in $(\Pi H_{p,n})_t$. Now, if $G = G' \oplus W$, where $\Pi H_{p,n} \subset G'$ then 1.4 (i) implies $W = 0$. q.e.d.

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