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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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A NOTE ON SUBQUASIVARIETIES OF SOME VARIETIES OF LATTICES

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Abstract: This paper is concerned with varieties of lattices, all subquasivarieties of which are varieties.

Key words: Lattice, variety, quasivariety, primitive lattice.

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V.I. Igošin has shown in [1] that the variety of lattices defined by the inclusion $a \wedge (b \vee (c \wedge d)) \wedge (c \vee d) \leq$ $\leq b \vee (a \wedge c) \vee (a \wedge d)$ has no subquasivariety which is not a variety. We shall give also some examples of such varieties of lattices.

Given a lattice L, we denote by $\mathbb{N}(L)$ the class of all lattices that contain no sublattice isomorphic to L. Let $\mathbb{K}$ be a class of lattices. A lattice $L \in \mathbb{K}$ is called weakly $\mathbb{K}$-projective iff L can be embedded in any lattice in $\mathbb{K}$ that has a homomorphic image isomorphic to L. A lattice is said to be primitive ($\mathbb{K}$-primitive. $\mathbb{K}$ - is a variety of lattices) if the class $\mathbb{N}(L)$ ($\mathbb{N}(L) \cap \mathbb{K}$) is a variety. It is easily verified that a non-trivial subdirectly irreducible lattice L is $\mathbb{K}$-primitive if and only if L is weakly $\mathbb{K}$-projective .

Theorem 1. Let $\mathbb{K}$ be a variety of lattices. The following conditions are equivalent.

(1) Any subquasivariety of $\mathbb{K}$ is a variety.

(2) Any non-trivial subdirectly irreducible lattice in $\mathbb{K}$ is $\mathbb{K}$-primitive.

(3) Any subdirectly irreducible lattice in $\mathbb{K}$ is weakly $\mathbb{K}$-primitive.

Proof. Assume (1) and let L be a non-trivial subdirectly irreducible lattice. The class $\mathbb{N}(L) \cap \mathbb{K}$ is a subquasivariety of $\mathbb{K}$ and so by (1), it is a variety, i.e. the condition (2) holds. Evidently, (2) is equivalent to (3). Now suppose (3) and let $\mathbb{A}$ be a subquasivariety of $\mathbb{K}$ and let $\mathbb{B}$ be the variety generated by $\mathbb{A}$. We shall show $\mathbb{A} = \mathbb{B}$. Since any lattice in $\mathbb{B}$ is isomorphic to a subdirect product of subdirectly irreducible lattices from $\mathbb{B}$ and $\mathbb{A}$ is closed under the formation of products and sublattices, it suffices to prove that all subdirectly irreducible lattices of $\mathbb{B}$ belong to $\mathbb{A}$. Let $L \in \mathbb{B}$ be subdirectly irreducible. There exists a homomorphism of a lattice $M \in \mathbb{A}$ onto L and by (3) M contains a sublattice isomorphic to L. Since $\mathbb{A}$ is closed under sublattices, we have $L \in \mathbb{A}$, and this is what we were required to prove.

A class $\mathbb{K}$ of lattices is called locally finite if any finite subset of any lattice in $\mathbb{K}$ generates a finite sublattice. If $\mathbb{A}$ is a set of lattices such that for any positive integer n there exists a positive integer $\varphi(n)$

such that any n elements of any lattice in $\mathbb{A}$ generate a sublattice of cardinality $\leq \varphi(n)$, then $\mathbb{A}$ generates a locally finite variety (see [5]). Given a class of lattices $\mathbb{K}$ we shall denote by Fin ($\mathbb{K}$) the class of all finite lattices of $\mathbb{K}$.

<u>Theorem 2</u>. Let $\mathbb{K}$ be a locally finite variety of lattices. The following conditions are equivalent.

(1) Any subquasivariety of $\mathbb{K}$ is a variety.

(2) Any non-trivial finite subdirectly irreducible lattice in $\mathbb{K}$ is $\mathbb{K}$ -primitive.

(3) Any finite subdirectly irreducible lattice in $\mathbb{K}$ is weakly $\mathbb{K}$ -projective.

(4) Any finite subdirectly irreducible lattice in $\mathbb{K}$ is weakly Fin ($\mathbb{K}$)-projective.

<u>Proof</u>. It suffices to prove that (4) implies (3). Assume (4) and let $\mathbb{A}$ be a subquasivariety of $\mathbb{K}$. Denote by $\mathbb{B}$ the subvariety of $\mathbb{K}$ generated by $\mathbb{A}$. Suppose $\mathbb{A} \subsetneqq \mathbb{B}$. Then there exists a finitely generated lattice $L \in \mathbb{B}$ such that $L \notin \mathbb{A}$. Since $\mathbb{K}$ is locally finite, L is finite. The lattice L is a homomorphic image of a lattice $M \in \mathbb{A}$. We can assume that M is finitely generated and since $M \in \mathbb{K}$, we see that M is finite. L is isomorphic to a subdirect product of finite subdirectly irreducible lattices $A_\iota \in \mathbb{B}$ ($\iota \in I$) . So we get that any A_ι ($\iota \in I$) is a homomorphic image of M and by (4) M contains sublattices isomorphic to A_ι ($\iota \in I$) .

The class $\mathbb{A}$ is closed under the formation of sublattices and products and thus we get that all A_ι ($\iota \in I$) are in $\mathbb{A}$ and so L is also in $\mathbb{A}$; a contradiction.

Let L be a lattice. Define a lattice L^* in this way: L is a sublattice of L^* , $L^* \setminus L$ contains exactly three elements a, u, v ; v is the smallest element of L^* , u is the greatest element of L^* and a is comparable with no element of L . Given a finite lattice L we denote by L^0 a lattice which is obtained from L by adding exactly one element comparable only with the greatest and the smallest element of L .
Let $\mathbb{K}$ be a class of lattices. A lattice $L \in \mathbb{K}$ will be called semi $\mathbb{K}$ -projective if the following condition holds: whenever φ is a homomorphism of $A \in \mathbb{K}$ onto L then there exists a homomorphism ψ of L into A such that $\varphi \circ \psi(x) = x$ for all $x \in L$, i.e. $\varphi \circ \psi = id_L$.

Lemma 1. Let $\mathbb{K}$ be a class of lattices and let $L \in \mathbb{K}$ and $L^* \in \mathbb{K}$. If L is weakly $\mathbb{K}$ -projective, then L^* is weakly $\mathbb{K}$ -projective. If L is semi $\mathbb{K}$ -projective, then L^* is also semi $\mathbb{K}$ -projective.

Proof. Let φ be a homomorphism of a lattice $A \in \mathbb{K}$ onto L^* . Let $a \in L^*$ be aomparable with no element of L and denote by b the smallest and by c the greatest element of L . There exist $a', b', c' \in A$ such that $\varphi(a') = a$, $\varphi(b') = b$, $\varphi(c') = c$. Put $v' = b' \vee a'$, $c'' = (c' \wedge v') \vee b'$, $u' = c'' \wedge a'$ and $b'' = b' \vee u'$. One can easily show that $u' < b'' < c'' < v'$, $c'' \wedge a' = u'$,

$b'' \vee a' = v'$ and $\varphi(c'') = c$, $\varphi(b'') = b$. Since the interval $I = \{x \in A ; b'' \leq x \leq c''\}$ is mapped by φ onto L, we have that it contains a sublattice L' isomorphic to L. It is easy to verify that the set $L' \cup \{a', u', v'\}$ forms a sublattice of A isomorphic to L^*. If L is semi $\mathbb{K}$-projective, then there exists a homomorphism ψ of L into I such that $\varphi \circ \psi = id_L$. Let u and v be the greatest and the smallest element of L^*. Define a mapping $\overline{\psi}$ of L^* into I by $\overline{\psi}(x) = \psi(x)$ for all $x \in L$, $\overline{\psi}(u) = u'$, $\overline{\psi}(v) = v'$ and $\psi(a) = a'$. One can easily show that $\overline{\psi}$ is a homomorphism of L^* into I such that $\varphi \circ \overline{\psi} = id_{L^*}$.

<u>Lemma 2</u>. Let $\mathbb{K}$ be a class of finite lattices and let L be a semi $\mathbb{K}$-projective lattice. If L^o is in $\mathbb{K}$, then L^o is also semi $\mathbb{K}$-projective.

<u>Proof</u>. Let φ be a homomorphism of a lattice $A \in \mathbb{K}$ onto L^o. Let u be the greatest and v the smallest element of L. Denote by u_o the smallest element of A that is mapped by φ onto u and by v_o the greatest element of A that is mapped by φ onto v. Let b be an element in A such that $\varphi(b) = a \in L^o \setminus L$. The interval $I = \{x \in A ; v_o \leq x \leq u_o\}$ is mapped by φ onto L and thus there exists a homomorphism ψ of L into I such that $\varphi \circ \psi = id_L$. Define $b' = (b \vee v_o) \wedge u_o$. Evidently $\varphi(b') = a$. It is easy to show that a mapping $\overline{\psi}$ of L^o into I defined by $\overline{\psi}(x) = \psi(x)$ for all $x \in L$ and $\overline{\psi}(a) = b'$ is a homomorphism of L^o into I such that $\varphi \circ \overline{\psi} = id_{L^o}$.

The class of all lattices will be denoted by $\mathbb{L}$ and the class of all finite lattices will be denoted by $\mathrm{Fin}(\mathbb{L})$. For any positive integer $n \geq 3$ we shall denote by M_n the lattice of dimension 2 and cardinality $n + 2$.

Corollary 1. The lattices M_n are semi $\mathrm{Fin}(\mathbb{L})$-projective.

Proof. For any positive integer $n \geq 3$ the lattice M_n can be obtained in a finite number of steps from the three element chain by application of $\circ$.

Lemma 3. Let $\mathbb{K}$ be a locally finite variety of lattices generated by a class $\mathbb{A}$ of lattices. If $L \in \mathbb{K}$ is a finite subdirectly irreducible lattice, then L is a homomorphic image of a sublattice of a lattice $B \in \mathbb{A}$.

Proof. By [3] L is a homomorphic image of a sublattice C of an ultraproduct of lattices from $\mathbb{A}$. We can suppose that C is finitely generated and since $\mathbb{K}$ is locally finite, we have that C is finite. The class $\mathbb{N}(C)$ is closed under the formation of ultraproducts and thus there exists a lattice $B \in \mathbb{A}$ that contains a sublattice isomorphic to C.

Theorem 3. Let $\mathbb{A}$ be a class of lattices such that the following conditions hold:

(1) The variety $\mathbb{V}$ generated by $\mathbb{A}$ is locally finite.

(2) Any finite subdirectly irreducible lattice which is a homomorphic image of a sublattice of a lattice from $\mathbb{A}$ is weakly $\mathrm{Fin}(\mathbb{V})$-projective.

Then any subquasivariety of $\mathbb{V}$ is a variety.

Proof. If L is a finite subdirectly irreducible lattice of $\mathbb{V}$, then by Lemma 3 there exists a lattice $B \in \mathbb{A}$ such that L is a homomorphic image of a sublattice of B. By (2) L is Fin($\mathbb{V}$)-projective. Now, Theorem 3 follows from Theorem 2.

Corollary 2. Let $\mathbb{M}$ be a finite set of semi Fin ($\mathbb{L}$)-projective lattices and let any subdirectly irreducible lattice which is a homomorphic image of a sublattice of a lattice from $\mathbb{M}$ be semi Fin ($\mathbb{L}$)-projective. Let $\mathbb{N}$ be the set of all lattices which can be obtained from a lattice of $\mathbb{M}$ in a finite number of steps by applications of $*$ and $\circ$. Then any subquasivariety of the variety $\mathbb{V}$ generated by $\mathbb{N}$ is a variety.

Proof. One can easily show that the conditions (1) and (2) hold.

Corollary 3. Let $\mathbb{M}$ be the class of all lattices that can be obtained in a finite number of steps starting from a lattice L_i $(i = 1,2,\dots,7)$ in Fig. 1 by applications $*$ and $\circ$. Then all subquasivarieties of the variety $\mathbb{V}$ generated by $\mathbb{M}$ are varieties.

Proof. The lattices L_1 - L_6 are primitive (see [2]) and so they are sublattices of the free lattice and thus L_1 - L_6 are projective (see [6]) . The lattice $L_7 = M_3$ is semi Fin ($\mathbb{L}$)-projective by Corollary 1. Now one can easily show that the conditions (1) and (2) of Theorem 3 hold.

Corollary 4. (Igošin [1] .) All subquasivarieties of the variety $\mathbb{V}_0$ of lattices defined by the inclusion

$$a \wedge (b \vee (c \wedge d)) \wedge (c \vee d) \leq b \vee (a \wedge c) \vee (a \wedge d)$$

are varieties.

Proof. $\mathbb{V}_0$ is generated by the set of lattices $\{ M_n \; ; \; 3 \leq n < \omega \}$ (see [4]) and thus we have that $\mathbb{V}_0$ is a subvariety of the variety $\mathbb{V}$ in Corollary 3.

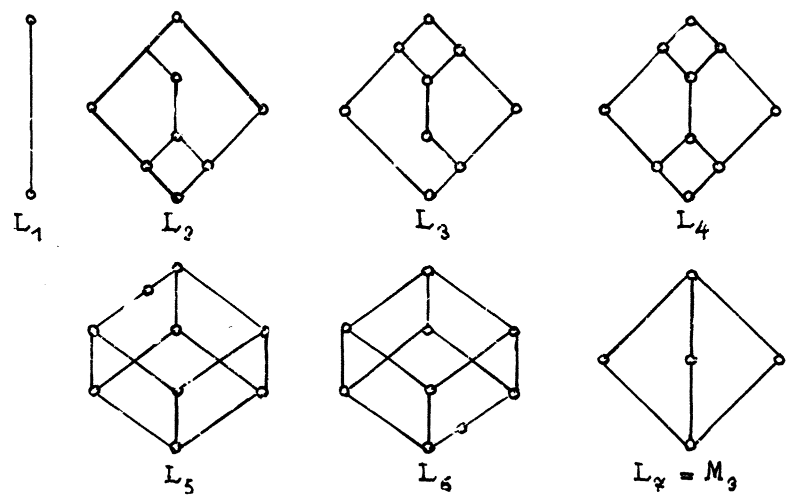

Figure 1

R e f e r e n c e s

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