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CENTRALLY SPLITTING RADICALS

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Abstract: Recently, many authors studied centrally splitting torsion theories and their applications. Here, we present a characterization of centrally splitting radicals which covers almost all the results appeared in the literature.

Key words: Preradical, torsion theory, centrally splitting preradical.

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In what follows, R stands for an associative ring with unit and R-mod means the category of unital left R-modules. Recall that a preradical r for R-mod is a subfunctor of the identity functor, i.e. r assigns to each $M \in$ R-mod its submodule $r(M)$ in such a way that every homomorphism $f: M \longrightarrow N$ induces a homomorphism of $r(M)$ into $r(N)$ by restriction. First of all, we shall list several basic definitions and results from [3],[4] and [5] which will be used in the sequel without any explicit reference.

A non-empty class $\mathcal{M}$ of modules is called

- hereditary if it is closed under submodules and isomorphic images,
- cohereditary if it is closed under homomorphic images,
- stable if every $M \in \mathcal{M}$ has an injective presentation

in $\mathcal{M}$,

- costable if every $M \in \mathcal{M}$ has a projective presentation in $\mathcal{M}$.

For a preradical r , $\mathcal{T}_r$ ($\mathcal{F}_r$) means the class of all $M \in R$-mod with $r(M) = M$ $(r(M) = 0)$. Obviously $\mathcal{T}_r$ is a cohereditary class closed under direct sums and $\mathcal{F}_r$ is a hereditary class closed under direct products. A preradical r is said to be

- idempotent if $r(r(M)) = r(M)$ for all $M \in R$-mod ,
- a radical if $r(M/r(M)) = 0$ for all $M \in R$-mod ,
- hereditary if $r(N) = N \cap r(M)$ for all $N, M \in R$-mod , $N \subseteq M$,
- superhereditary if it is hereditary and $\mathcal{T}_r$ is closed under direct products,
- cohereditary if $r(M/N) = (r(M) + N)/N$ for all $N, M \in R$-mod , $N \subseteq M$,
- stable if every injective module splits (a module M splits if $r(M)$ is a direct summand of M),
- costable if every projective module splits,
- splitting if every module splits,
- cosplitting if it is both hereditary and cohereditary,
- centrally splitting if it is cohereditary and $r(R)$ is a ring direct summand of R .

If r and s are preradicals, we define the preradicals $r \circ s$, $r \cap s$, $r \Delta s$ and $r + s$ by $(r \circ s)(M) = r(s(M))$, $(r \cap s)(M) = r(M) \cap s(M)$, $(r \Delta s)(M)/r(M) = s(M/r(M))$ and $(r + s)(M) = r(M) + s(M)$. If $r \cap s$ is idempotent then $r \circ s = s \circ r = r \cap s$ ([3], Prop. 3(iv)), if both r and s are hereditary then $r \circ s = s \circ r = r \cap s$ ([3], Prop. 4(iii)), and if both r and s are cohereditary then $r \Delta s = s \Delta r = r + s$ ([3], Prop. 13(iii)).

Let r be a preradical. Then

- r is hereditary iff it is idempotent and $\mathcal{T}_r$ is hereditary ([4], Prop. 2.1),
- if r is hereditary then $\mathcal{F}_r$ is closed under injective hulls ([4], Prop. 2.2(i)),
- if r is a radical and $\mathcal{F}_r$ is stable then r is hereditary ([4], Prop. 2.2),
- r is cohereditary iff it is a radical and $\mathcal{F}_r$ is cohereditary ([4], Prop. 4.1),
- if r is idempotent and $\mathcal{T}_r$ is costable then r is cohereditary ([4], Prop. 4.3),
- if r is stable then $\mathcal{T}_r$ is closed under injective hulls ([5], Prop. 2.4(i)),
- if r is idempotent and $\mathcal{T}_r$ is stable then r is stable ([5], Prop. 2.4(ii)),
- if r is costable then $\mathcal{F}_r$ is costable ([5], Prop. 3.4 (i)),
- if r is a radical and $\mathcal{F}_r$ is costable then r is costable ([5], Prop. 3.4(ii)),
- r is costable iff R splits (as a module) ([5], Prop. 3.6).

Further, a hereditary preradical r is stable iff for all left ideals $I \subseteq K \subsetneqq R$ with $K/I = r(R/I)$ there is a left ideal L with $L \neq I$ and $L \cap K = I$ (see e.g. [4], [7],[14]).

If I is a two-sided ideal, we shall say that I satisfies the condition (a) ((b)) if $x \in Ix$ ($x \in xI$) for all $x \in I$. This is clearly equivalent to R/I being flat as a right (left) R-module.

As it is easy to see (cf. [4], Th. 4.11), cohereditary radicals are in a one-to-one correspondence with two-sided ideals given by $r \longmapsto r(R)$ and $I \longmapsto r$, $r(M) = IM$ for all $M \in R$-mod. Similarly, superhereditary preradicals are in a one-to-one correspondence with two-sided ideals via $r \longmapsto \cap K$, $R/K \in \mathcal{T}_r$ and $I \longmapsto r$, $r(M) = \{m \in \in M \mid Im = 0\}$ (see [4], Th. 2.12).

If I is a two-sided ideal, r is the corresponding cohereditary radical and s is the corresponding superhereditary preradical then

- s is a radical iff $I^2 = I$,
- r is idempotent iff $I^2 = I$,
- r is hereditary iff I satisfies (a),
- if I is finitely generated as a right ideal then r is superhereditary ([4], Prop. 4.8(iv)).

Now let $\mathcal{A}$ be a non-empty class of modules. We define an idempotent preradical $p_{\mathcal{A}}$ and a radical $p^{\mathcal{A}}$ by $p_{\mathcal{A}}(M) = \sum \operatorname{Im} f$, $f \in \operatorname{Hom}_R(A,M)$, $A \in \mathcal{A}$ and $p^{\mathcal{A}}(M) = \bigcap \operatorname{Ker} f$, $f \in \operatorname{Hom}_R(M,A)$, $A \in \mathcal{A}$. Denote $\mathcal{B} = \{M/N \mid M \in \in R\text{-mod}$ and N is an essential submodule of $M\}$, $\mathcal{C} = \{N \in R\text{-mod} \mid N$ is a small submodule in some module $M\}$, $\mathcal{S}$ be a representative set of simple modules and define $\mathcal{Z} = p_{\mathcal{B}}$ (the singular submodule), $\mathcal{J} = p^{\mathcal{C}}$, $\operatorname{Soc} = p_{\mathcal{S}}$ (the socle) and $\mathcal{J} = p^{\mathcal{S}}$ (the Jacobson radical).

§ 1. Main results

Proposition 1: The following are equivalent for preradicals r, s :

(i) $r \circ s = \text{zer}$ and $r \triangle s = \text{id}$,

(ii) r is a radical, s is idempotent and $\mathcal{F}_r = \mathcal{T}_s$

(iii) r is a cohereditary radical and s is the superhereditary preradical corresponding to $r(R)$.

Proof: obvious.

Proposition 2: Let r be a cohereditary radical and s be the superhereditary preradical correspoinding to $I = r(R)$. Then the following are equivalent:

(i) s is stable,

(ii) r is hereditary (i.e. cosplitting),

(iii) $I^2 = I$ and $\mathcal{T}_r \subseteq \mathcal{F}_s$,

(iv) I satisfies (a).

Proof: (i) implies (ii). Obviously, $\mathcal{F}_r = \mathcal{T}_s$. However, s is a radical by [3], Prop. 2.5, so r is idempotent and consequently hereditary.

(ii) implies (iii). If $M \in \mathcal{T}_r$ then $m \in Im$ for every $m \in M$, r being hereditary, and so $s(M) = 0$.

(iii) implies (iv). Let $x \notin Ix$ and $K \subseteq I$ be maximal with respect to $x \notin K$ and $Ix \subseteq K$. Since $I^2 = I$, $I/K \in \mathcal{T}_r \subseteq \mathcal{F}_s$ and so $Ix \nsubseteq K$, a contradiction.

(iv) implies (i). Obviously, s is idempotent and $\mathcal{T}_s = \mathcal{F}_r$ is stable.

Proposition 3: Let r be a cohereditary radical and s be the superhereditary preradical corresponding to $I = r(R)$. Then the following are equivalent:

(i) r is costable,

(ii) s is cohereditary,

(iii) $I^2 = I$ and $\mathcal{F}_s \subseteq \mathcal{T}_r$,

(iv) I is a left direct summand of R .

Proof: (i) implies (ii). Since $\mathcal{T}_s = \mathcal{F}_r$ is costable, s is cohereditary.

(ii) implies (iii). Since s is a cohereditary radical, $I^2 = I$ and for each $F \in \mathcal{F}_s$, $F/r(F) \in \mathcal{F}_r \cap \mathcal{F}_s = \mathcal{T}_s \cap \mathcal{F}_s = 0$.

(iii) implies (iv). Obviously s is a radical, hence $I \cdot R/(0:I)_r = {}^{R}/_{(0:I)_r}$ and $I + (0:I)_r = R$.

(iv) implies (i). Obviously.

<u>Proposition 4</u>: Let r be a cohereditary radical and $I = r(R)$. Then the following are equivalent:

(i) r is superhereditary,

(ii) I is a right direct summand of R ,

(iii) I satisfies (a) and it is finitely generated as a right ideal.

Proof: (i) implies (ii). Clearly, $I = (0:K)_r$ for some two-sided ideal K with $R/K \in \mathcal{T}_r$. Hence $I \cdot R/K = R/K$ and so $I + K = R$. However, $K \subseteq (0:I)_l$.

(ii) implies (iii) and (iii) implies (i) trivially.

<u>Proposition 5</u>: The following are equivalent for a preradical r :

(i) r is stable and cosplitting,

(ii) r is splitting and cosplitting,

(ii) r is costable and cosplitting,

(iv) there is a preradical s with $r \cap s = zer$ and

$r + s = id$,

(v) r is centrally splitting.

Proof: (i) implies (ii). Let $F \in \mathcal{F}_r$ and $T \in \mathcal{T}_r$. Then $E(T)/T \in \mathcal{T}_r$ and $\mathrm{Hom}_R(F, E(T)/T) = 0$, hence $\mathrm{Ext}_R(F,T) = 0$, as desired.

(ii) implies (iii) trivially.

(iii) implies (iv). Let s be the superhereditary preradical corresponding to $r(R)$. Clearly $r \circ s = zer$, and since r is hereditary, $r \cap s = zer$. On the other hand, $r(R) = Re$, $e^2 = e$, so $r(R)(1 - e) = 0$ and $1 \in$ $\in (s + r)(R)$.

(iv) implies (v). Obviously $r(R)$ is a ring direct summand and for every $M \in R\text{-mod}$, $M = r(R)M \oplus s(R)M =$ $= r(M) \oplus s(M)$. Now the inclusions $r(R)M \subseteq r(M)$ and $s(R)M \subseteq s(M)$ show that r is cohereditary.

(v) implies (i). $r(R)$ satisfies (a) since $r(R) = Re$ for some central idempotent e and hence r is hereditary. Finally, $M = eM \oplus (1 - e)M = r(M) \oplus (1 - e)M$ for all $M \in R\text{-mod}$.

<u>Theorem</u>: Let r be a cohereditary radical and s be the superhereditary preradical corresponding to $I = r(R)$. Then the following are equivalent:

(1) $r \cap s$ is idempotent and $r + s$ is a radical,

(2) $r \circ s = s \circ r$ and $r \Delta s = s \Delta r$,

(3) $s \circ r = zer$ and $s \Delta r = id$,

(4) r is hereditary and s is cohereditary,

(5) $r \cap s = zer$ and $r + s = id$,

(6) both r and s are splitting,

(7) both r and s are costable,

(8) both I and $(0:I)_r$ are left direct summands of R,

(9) s is cohereditary and costable,

(10) s is cohereditary and splitting,

(11) s is centrally splitting,

(12) s is cohereditary and stable,

(13) I satisfies (a) and is a left direct summand,

(14) $I^2 = I$ and $\mathcal{T}_r = \mathcal{F}_s$,

(15) I satisfies (a) and $\mathcal{F}_s \subseteq \mathcal{T}_r$,

(16) I is a left direct summand of R and $\mathcal{T}_r \subseteq \subseteq \mathcal{F}_s$,

(17) r is hereditary and costable,

(18) r is hereditary and splitting,

(19) r is centrally splitting,

(20) r is hereditary and stable,

(21) I satisfies (a) and for every left ideal K with $I + K \neq R$ there is a left ideal $L \neq K$ such that $(I + K) \cap L = K$,

(22) both r and s are stable,

(23) $r(R) \cap s(R) = 0$ and $r(R) + s(R) = R$,

(24) r is costable and $r(R) \cap s(R)$ contains no non-zero nilpotent ideal,

(25) I satisfies (a,(b) and it is principal left ideal,

(26) I satisfies (a),(b) and it is finitely generated as a left ideal,

(27) I satisfies (a),(b) and it is principal right ideal,

(28) I satisfies (a),(b) and it is finitely generated as a right ideal,

(29) I satisfies (b) and it is a right direct summand,

(30) I satisfies (b) and r is superhereditary.

Proof: (1) implies (2). Obvious.

(2) implies (3). It is easily seen that $r \circ s = zer$ and $r \Delta s = id$.

(3) implies (4) and (4) implies (5) by Proposition 1.

(5) implies (1). Obvious.

(5) implies (6) by Proposition 5.

(6) implies (7) and (7) implies (8) obviously.

(8) implies (9) by Proposition 3.

(9) implies ((10), (10) implies (11) and (11) implies (12) by Proposition 5.

(12) implies (4) by Proposition 2.

(12) is equivalent to (13) and (13) implies (14) by Propositions 2 and 3.

(14) implies (15) by Proposition 2.

(15) implies (16) and (16) implies (17) by Propositions 2 and 3.

(17) implies (18) and (18) implies (19) by Proposition 5.

(19) implies (13). Obvious.

(19) is equivalent to (20) by Proposition 5.

(20) is equivalent to (21) by Proposition 2.

Thus the conditions (1) - (21) are equivalent. Furthermore, (6) implies (22) trivially and (22) implies (20) by Proposition 2.

(5) implies (23) trivially.

(23) implies (24) by Proposition 3.

(24) implies (19). There is an idempotent e with $I = Re$. If $(1-e)a = be$ for some $a, b \in R$ then $ebe = 0$ and $be = (1-e)be$. Thus $(1-e)R \cap Re = (1-e)Re \subseteq s(R) \cap r(R)$ and so $(1-e)Re = 0$. Hence $(1-e)R \subseteq R(1-e)$ and $R(1-e)$ is two-sided.

(19) implies (25) and (25) implies (26) trivially.

(26) implies (16) by Proposition 2 and the dual of Proposition 4.

(19) implies (27) and (27) implies (28) trivially.

(28) implies (29) by Proposition 4.

(29) implies (19). Obviously, $I + (0:I)_1 = R$ and (b) yields $I \cap (0:I)_1 = 0$.

(30) is equivalent to (28) by Proposition 4.

The proof is now complete.

<u>Corollary</u>: Let r, s be preradicals with $r \circ s = s \circ r = \mathrm{zer}$ and $r \triangle s = \mathrm{id} = s \triangle r$. Then both r and s are centrally splitting.

§ 2. <u>Some preradicals are centrally splitting</u>

<u>Proposition 6</u>: The following conditions are equivalent for a ring R:

(i) Every superhereditary radical is centrally splitting,

(i´) every superhereditary cohereditary radical for mod-R is centrally splitting,

(ii) every costable cohereditary radical is centrally splitting,

(ii$'$) every costable cohereditary radical for mod-R is centrally splitting,

(iii) every two-sided ideal which is a left direct summand is a ring direct summand,

(iv) every two-sided ideal which is a right direct summand is a ring direct summand.

Proof: (i) is equivalent to (ii) by Proposition 3 and Theorem.

(ii) is equivalent to (iii) obviously.

(iii) implies (iv). If $I = fR$ is two-sided and $f^2 = f$ then $R(1 - f) = Re$ for some central idempotent e, and consequently f is central.

(iv) implies (iii) similarly. The rest is obvious.

Proposition 7: The following are equivalent:

(i) $\mathfrak{Z}$ is centrally splitting,

(ii) $\mathfrak{Z}$ is cohereditary,

(iii) $\mathfrak{Z}$ = zer ,

(iv) R is completely reducible,

(v) Soc = id ,

(vi) every preradical is centrally splitting,

(vii) Soc is centrally splitting,

(viii) Soc is cohereditary.

Proof: (i) implies (ii), (iv) implies (v), (vi) implies (vii), (vii) implies (viii) and (vi) implies (i) trivially.

(ii) implies (iii). Let K be a left ideal maximal with respect to $\mathcal{Z}(R) \cap K = 0$. Then $\mathcal{Z}(R) \oplus K = R$ and so $\mathcal{Z}(R) = 0$.

(iii) implies (iv). Every left ideal is a direct summand since no proper left ideal is essential.

(v) implies (vi). Since every module is completely reducible, every preradical is splitting, hereditary and cohereditary. Now it suffices to use Proposition 5.

(viii) implies (iv). Clearly, $I + \mathrm{Soc}(R) = R$ for every maximal left ideal I.

Proposition 8: The following are equivalent:

(i) $\mathcal{Y} = \mathrm{id}$,

(ii) $\mathcal{Y}$ is centrally splitting,

(iii) $\mathcal{Y}$ is hereditary,

(iv) R is a V-ring,

(v) $\mathcal{J}$ is hereditary,

(vi) $\mathcal{J} = \mathrm{zer}$,

(vii) $\mathcal{J}$ is centrally splitting,

(viii) $\mathcal{J}$ is cohereditary and costable,

(ix) $\mathcal{Y}$ is splitting,

(x) $\mathcal{Y}(C) = 0$ for every cyclic module C.

Proof: (i) implies (ii), (ii) implies (iii), (vi) implies (vii) and (vii) implies (viii) trivially.

(iii) implies (iv). Let, on the contrary, $M \neq E(M)$ for some simple module M. Then M is small in $E(M)$ and hence $\mathcal{Y}(M) = 0$. Further, if $N \subseteq E(M)$ and $E(M)/N$ is small in $E(E(M)/N)$ then $N \neq 0$ and so $M \subseteq N$. Thus $M \subseteq \mathcal{Y}(E(M))$, hence $\mathcal{Y}(M) = M$, a contradiction.

(iv) implies (v). Since $\mathcal{J} = p^{\mathcal{G}}$, $\mathcal{F}_{\mathcal{J}}$ is stable.

(v) implies (vi). If $M \in R$-mod and $x \in \mathcal{J}(M)$ then $\mathcal{J}(R/(0:x)) = R/(0:x)$, so $x = 0$.

(viii) implies (ix). Since $\mathcal{J}(R) = 0$, $\mathcal{J} = \text{zer}$.

(ix) implies (x). If C is cyclic then $C = \mathcal{J}(C) \oplus X$ and $\mathcal{J}(C)$ is a $\mathcal{J}$-torsion cyclic module.

(x) implies (i). Obviously, every left ideal is an intersection of maximal left ideals, so R is a V-ring and 0 is the only cocyclic module small in its injective hull.

§ 3. Applications

From our characterization of centrally splitting radicals, almost all the results concerning central splitting from [2],[8],[9],[11],[12],[13],[15] can be deduced as simple corollaries. As an illustration, we present the description of n-torsion theories.

Recall that if $\mathcal{A}_1,\dots, \mathcal{A}_n$ are non-empty classes of modules, we shall say that $(\mathcal{A}_1,\dots, \mathcal{A}_n)$ is an n-torsion theory if $(\mathcal{A}_i, \mathcal{A}_{i+1})$ is a torsion theory, $i = 1, 2,\dots,n-1$. $\mathcal{A}_1$ is said to be a ttf-class if it is hereditary, cohereditary and closed under extensions and direct products.

Proposition 9: The following conditions are equivalent for a torsion theory $(\mathcal{T},\mathcal{F})$:

(i) $(\mathcal{T},\mathcal{F})$ is centrally splitting,

(ii) $(\mathcal{T},\mathcal{F})$ is cosplitting and stable,

(iii) $(\mathcal{T},\mathcal{F})$ is cosplitting and costable,

(iv) $(\mathcal{T},\mathcal{F})$ is cosplitting and splitting,

(v) $(\mathcal{F}, \mathcal{T})$ is a torsion theory,

(vi) there is a ring direct summand I of R such that $\mathcal{T} = \{M \in R\text{-mod} \mid IM = M\}$ and $\mathcal{F} = \{M \in R\text{-mod} \mid IM = 0\}$,

(vii) $\mathrm{Ext}_R(T,F) = \mathrm{Ext}_R(F,T)$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Proof: The equivalence of Conditions (i) - (vi) and the implication (i) implies (vii) follow immediately from Theorem. If (vii) holds then obviously $(\mathcal{T}, \mathcal{F})$ is splitting. Further, suppose that there are $T \in \mathcal{T}$, $F \in \mathcal{F}$, $A \subsetneqq C \subseteq F$, $N \subseteq T$ with $r(N) \neq N$ and with $C/A = r(F/A)$ (r is the idempotent radical corresponding to $(\mathcal{T}, \mathcal{F})$). With respect to the hypothesis, the exact sequences $0 \to N/r(n) \to T/r(N) \to T/N \to 0$ and $0 \to A \to C \to C/A \to 0$ split, a contradiction.

Proposition 10: The following are equivalent:

(i) $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ is a 3-torsion theory,

(ii) $\mathcal{A}_2$ is a ttf-class, $\mathcal{A}_1 = \mathcal{A}_2^+$ and $\mathcal{A}_3 = \mathcal{A}_2^*$,

(iii) there is an idempotent two-sided ideal I such that $\mathcal{A}_1 = \{M \in E\text{-mod} \mid IM = M\}$, $\mathcal{A}_2 = \{M \in R\text{-mod} \mid IM = 0\}$ and $\mathcal{A}_3 = \{M \in R\text{-mod} \mid Im \neq 0 \text{ for all } 0 \neq m \in M\}$.

Proof: Easy.

Proposition 11: The following are equivalent:

(i) $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4)$ is a 4-torsion theory,

(ii) both $\mathcal{A}_2$ and $\mathcal{A}_3$ are ttf-classes, $\mathcal{A}_1 = \mathcal{A}_2^+$, $\mathcal{A}_3 = \mathcal{A}_2^*$ and $\mathcal{A}_4 = \mathcal{A}_3^*$,

(iii) there is a two-sided ideal I which is a right direct summand such that $\mathcal{A}_1 = \{M \in R\text{-mod} \mid (0:I)_1 M = M\}$, $\mathcal{A}_2 = \{M \in R\text{-mod} \mid (0:I)_1 M = 0\} = \{M \in R\text{-mod} \mid IM = M\}$, $\mathcal{A}_3 = \{M \in R\text{-mod} \mid IM = 0\}$ and $\mathcal{A}_4 = \{M \in R\text{-mod} \mid Im \neq 0$ for all

$0 \neq m \in M\}$,

(iv) there is a two-sided ideal K which is a left direct summand such that $\mathcal{A}_1 = \{M \in R\text{-mod} \mid KM = M\}$, $\mathcal{A}_2 = \{M \in R\text{-mod} \mid KM = 0\} = \{M \in R\text{-mod} \mid (0:K)_r M = M\}$, $\mathcal{A}_3 = \{M \in R \mid (0:K)_r M = 0\}$ and $\mathcal{A}_4 = \{M \in R\text{-mod} \mid (0:K)_r m \neq 0$ for all $0 \neq m \in M\}$,

(v) $(\mathcal{A}_1, \mathcal{A}_2)$ is costable cohereditary torsion theory, $(\mathcal{A}_2, \mathcal{A}_3)$ is a superhereditary cohereditary torsion theory, $(\mathcal{A}_3, \mathcal{A}_4)$ is a stable superhereditary torsion theory and $\mathcal{A}_2 \subseteq \mathcal{A}_4$.

Proof: (ii) implies (iii). It suffices to put $I = r(R)$, where r is the superhereditary cohereditary radical corresponding to $(\mathcal{A}_2, \mathcal{A}_3)$.

(iii) implies (iv). Take $K = (0:I)_l$,

(iv) implies (v). Since K is a left direct summand, $(0:K)_r$ is a right direct summand. Hence $(\mathcal{A}_1, \mathcal{A}_2)$ is costable by Proposition 3 and $(\mathcal{A}_3, \mathcal{A}_4)$ is stable by Proposition 2.

The rest is obvious.

Proposition 12: The following are equivalent for every $n \geq 5$.

(i) $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is an n-torsion theory,

(ii) $(\mathcal{A}_1, \mathcal{A}_2)$ is a centrally splitting torsion theory, $\mathcal{A}_1 = \mathcal{A}_3 = \mathcal{A}_5 = \dots$ and $\mathcal{A}_2 = \mathcal{A}_4 = \mathcal{A}_6 = \dots$,

(iii) there is a ring direct summand I of R such that $\{M \in R\text{-mod} \mid IM = M\} = \mathcal{A}_1 = \mathcal{A}_3 = \dots$ and $\{M \in R\text{-mod} \mid IM = 0\} = \mathcal{A}_2 = \mathcal{A}_4 = \dots$.

Proof: Obviously, only the implication (i) implies

(ii) needs the proof. However, $(\mathcal{a}_3, \mathcal{a}_4)$ is stable hereditary by Proposition 11 and consequently stable cosplitting, $\mathcal{a}_4$ being cohereditary.

<u>Corollary</u>: There are only four types of n-torsion theories, namely

(i) torsion theories which cannot be extended to a 3-torsion theory,

(ii) 3-torsion theories which cannot be extended to a 4-torsion theory,

(iii) 4-torsion theories which cannot be extended to a 5-torsion theory,

(iv) centrally splitting torsion theories.

<u>Corollary</u>: There is a one-to-one correspondence between

- 3-torsion theories and ttf-classes,
- 4-torsion theories and costable ttf-classes,
- centrally splitting torsion theories and stable costable ttf-classes.

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