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FACTOR-SPLITTING ABELIAN GROUPS OF FINITE RANK

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Abstract: A structural description of factor-splitting torsionfree abelian groups of finite rank is presented. This criterion enables us to prove that every completely decomposable torsionfree abelian group of finite rank is factor-splitting.

Key words: Factor-splitting group, completely decomposable group.

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Throughout this paper by a group it is always meant an additively written abelian group. A torsionfree group G is called factor-splitting if any of its factor group G/H splits (see [9]). We shall use the following notation: If g is an element of infinite order of a mixed group G then $h_p^G(g)$ denotes the p-height of g in the group G (see [1]). If $\alpha \neq 0$ is an integer, $\alpha = p^k \alpha'$, $(\alpha', p) = 1$ then we write $h_p(\alpha) = k$. We put $h_p(0) = \infty$ for all primes p. The symbol π will denote the set of all primes. If $\pi' \subseteq \pi$ and T is a torsion group then $T_{\pi'}$ is a subgroup of T consisting of all the elements of T the order of which is divisible by primes from π' only. If M is a subset of a torsionfree group G then $\{M\}_{\pi'}^G$, is a π'-pure closure of M in G, i.e. the greatest subgroup of

G such that $\{M\}^G_{\pi'}/\{M\}$ is π'-primary. $R_{\pi'}$ will denote the group of rationals with denominators prime to every $p \in \pi'$.

Every maximal linearly independent set of elements of a torsionfree group G is called a basis of G. A sequence $g_0, g_1, \dots$ of elements of a (mixed) group G is said to be a p-sequence of g_0 if $pg_{i+1} = g_i$, $i = 0,1,\dots$. Stratton [11] proved that a mixed group G of finite rank splits if and only if G contains a free subgroup U of the same rank as G such that for some integer $\alpha \neq 0$ the following two conditions hold:

(1) $h_p^G(px) = 1 + h_p^G(x)$ for all $x \in \alpha U$, and all $p \in \pi$,

(2) for each $p \in \pi$ there is a morphism $\mathcal{S}_p$ defined on G such that Ker $\mathcal{S}_p$ is p-free and p-pure in G and every element of $\mathcal{S}_p(\alpha U)$ has a p-sequence in $\mathcal{S}_p(G)$.

The systematical study of factor-splitting groups was begun by Procházka [9],[10]. The results obtained here generalize those of [2] and answer some questions from [9],[10]. The technique of the example is essentially the same as in [3],[4].

1. Definition: Let $B = \{g_1, \dots, g_n\}$ be a basis of a torsionfree group G and p be a prime. We say that B satisfies (FSp) if it holds: If $p^k x = \sum_{i=1}^{n} \alpha_i g_i$ for some $x \in G$, then the equation $p^k y = \sum_{i=1}^{n} \beta_i g_i$ with $h_p(\beta_i) \geq 1$, $i = 1,2,\dots,n$ and $\beta_i = \alpha_i$ whenever $h_p(\alpha_i) \geq 1$, is solvable in G.

2. Proposition: Let $B = \{g_1, \dots, g_n\}$ be a basis of a torsionfree group G. Then $G/\{B'\}$ splits for every $B' \subseteq B$ if

and only if B satisfies (FSp) for almost all primes p.

Proof: First, suppose that the condition is not satisfied. It is easily seen that there is no loss of generality in assuming the existence of an infinite set π' of primes such that $p^{k(p)}x = \sum_{i=1}^{\ell} p\,\alpha_i g_i + \sum_{i=\ell+1}^{m} \alpha_i g_i$ is solvable in G, but $p^{k(p)-1}x = \sum_{i=1}^{\ell} \alpha_i g_i + \sum_{i=\ell+1}^{m} \beta_i g_i$ is not solvable in G for every $p \in \pi'$. If we take $B' = \{g_{l+1},\dots,g_n\}$, then $G/\{B\}$ does not split since it does not satisfy Condition (1).

Now we proceed to the sufficiency. Obviously, we can suppose that $B' = \{g_{k+1},\dots,g_n\}$. Let π' be the set of all primes p for which G has (FSp). Then $\pi \dot{-} \pi'$ is finite and if H is such a subgroup of G that $H/\{B\} = (G/\{B\})_{\pi'}$ then, by [8, Theorem 6], $H/\{B'\}$ splits if and only if $G/\{B'\}$ does. Hence we can assume that $\pi' = \pi$.

Suppose that $h_p^{G/\{B'\}}(\sum_{i=1}^{k} \alpha_i g_i + \{B'\}) = r < \infty$ and let $p^s y = \sum_{i=1}^{k} p\,\alpha_i g_i + \sum_{i=k+1}^{m} \alpha_i g_i$, $y \in G$. By (FSp), for suitable integers $\beta_{k+1},\dots,\beta_n$ the equation $p^{s-1}z = \sum_{i=1}^{k} \alpha_i g_i + \sum_{i=k+1}^{m} \beta_i g_i$ is solvable in G, so that $s = r + 1$ and Condition (1) is satisfied.

Let p be a prime. For the sake of simplicity we shall assume that the elements $g_{k+1},\dots,g_n$ are enumerated in such a way that

(3) $g_i + \{g_{i+1},\dots,g_n\}_p^G$ is of minimal p-height in $\{g_{k+1},\dots,g_n\}_p^G / \{g_{i+1},\dots,g_n\}_p^G$ for all $i = k+1,\dots,n$,

(4) $\{g_{m+1},\dots,g_n\}_p^G / \{g_{m+1},\dots,g_n\}$ is bounded by p^s

and

(5) $g_i + \{g_{i+1},\dots,g_n\}_p^G$ is an element of

$\{g_{k+1},\dots,g_n\}_p^G / \{g_{i+1},\dots,g_n\}_p^G$ of infinite p-height for all $i = k+1,\dots,m$.

From (5) we get that for every natural integer r the equation $p^r x = g_i + y$ is solvable in G for some $y \in \{g_{i+1},\dots,g_n\}_p^G$. Then $p^t y = \sum_{j=i+1}^{m} \alpha_j g_j$ and so $p^{r+t}x = p^t g_i + \sum_{j=i+1}^{m} \alpha_j g_j$ is solvable in G. Using (FSp) repeatedly we obtain that the equation

$$(6) \qquad p^r x = g_i + \sum_{j=i+1}^{m} \alpha_j g_j$$

is solvable in G for every natural integer r.

Further, suppose that $g_k + \{B'\}$ is of minimal p-height $s_k < \infty$ in $G/\{B'\}$, $p^{s_k} x_k = g_k + \sum_{j=k+1}^{m} \alpha_j^{(k)} g_j$. Assume that we have constructed the elements $x_{i+1},\dots,x_k$ such that

(7) $g_j + \{x_{j+1},\dots,x_k,B'\}$ is of minimal p-height $s_j < \infty$ in

$$G/\{x_{j+1},\dots,x_k,B'\},$$

$$(8) \qquad p^{s_j} x_j = g_j + \sum_{r=j+1}^{m} \alpha_r^{(j)} g_r, \quad j = i+1,\dots, k.$$

Now if every element of $G/\{x_{i+1},\dots,x_k,B'\}$ is of infinite p-height, we stop. In the other case, let $g_i + \{x_{i+1},\dots,x_k,B'\}$ be of minimal p-height $s_i < \infty$ in

$G/\{x_{i+1},\dots,x_k,B'\}$ (g_i are assumed to be suitably enumerated). Then $p_i^{s_i} y_i = g_i + \sum_{r=i+1}^{k} \beta_r^{(i)} x_r + \sum_{r=k+1}^{m} \gamma_r^{(i)} g_r$, $y_i \in G$ and $p^{s_i+s_{i+1}} y_i = p^{s_{i+1}} g_i + \sum_{r=i+1}^{k} p^{s_{i+1}-s_r} \beta_r^{(i)} (g_r + \sum_{j=r+1}^{m} \alpha_j^{(r)} g_j) + p^{s_{i+1}} \sum_{r=k+1}^{m} \gamma_r^{(i)} g_r$, since obviously $s_j \geq s_{j+1}$, $j = i,\dots, k-1$. Now, using (FSp) repeatedly, we get (8) for $j = i$. In this way we construct elements $x_{\ell+1},\dots,x_k$ satisfying (7), (8) such that every element of $K = \{g_1,\dots,g \ ,x_{\ell+1},\dots,x_k,B'\}/\{x_{\ell+1},\dots,x_k,B'\}$ is of infinite p-height.

Consider the element

$$(9) \qquad \sum_{i=1}^{\ell} \alpha_i g_i + \{x_{\ell+1},\dots,x_k,B'\} .$$

By hypothesis there are elements $y_r \in G$ with

$$p^{r+s} y_r = \sum_{i=1}^{\ell} \alpha_i g_i + \sum_{j=\ell+1}^{k} \beta_j^{(r)} x_j + \sum_{j=k+1}^{m} \gamma_j^{(r)} g_j$$

and with respect to (6) we can assume that $\gamma_{k+1}^{(r)} = \dots = \gamma_m^{(r)} = 0$. Then the equality

$$p^{r+s}(py_{r+1} - y_r) = \sum_{j=\ell+1}^{k} (\beta_j^{(r+1)} - \beta_j^{(r)}) x_j + \sum_{j=m+1}^{m} (\gamma_j^{(r+1)} - \gamma_j^{(r)}) g_j$$

yield $p^{r+s} / \beta_j^{(r+1)} - \beta_j^{(r)}$ by the construction of x_j' s and consequently $p^r / (\gamma_j^{(r+1)} - \gamma_j^{(r)})$ by (4). It follows now that $\{p^s y_r + \{x_{\ell+1},\dots,x_k,B'\}\}_{r=1}^{\infty}$ is a p-sequence of the element

(9). Thus every element of K has a p-sequence, and hence $G/\{B'\}$ satisfies (2) by [11, Lemma 3.3, 3.4].

3. Theorem: A torsionfree group G of finite rank is factor-splitting if and only if every basis of G satisfies (FSp) for almost all primes p.

Proof: By [9, Lemma 2.6] G is factor-splitting if and only if G/U splits for every free subgroup U of G. Now it suffices to use Proposition 2.

The following example shows that the (FSp)-property for one basis and almost all primes is generally not sufficient for the factor-splitting of G.

4. Example: Put $U = \{a\} \oplus \{b\} \oplus \dot{\sum}_{p \in \pi} \{a_p\}$; $V = \{p^3 a_p - (p-1)a - b, p \in \pi\}$ and $G = U/V$. It is easy to see that $a + V$ and $b + V$ are of zero p-height in G for all primes p and consequently $\{a + V, b + V\}$ satisfies (FSp) for all primes p. For $x = a + V$, $y = a - b + V$ we have $px - y = p^3 a_p + V$, $p \in \pi$ while the assumption $px + p\lambda y = p^3(\alpha a + \beta b + \sum_q \gamma_q a_q) + \sum_q \eta_q (q^3 a_q - (q-1)a - b)$ (finite sums) leads to the equalities

$$p + p\lambda = p^3\alpha - \sum_q (q-1)\eta_q$$

$$-p\lambda = p^3\beta - \sum_q \eta_q$$

$$0 = p^3\gamma_q + q^3\eta_q .$$

Hence $p = p^3(\alpha + \beta) - \sum_q q\eta_q$ and so $p^2 / (1 + \eta_p)$. The second equality now leads to a contradiction $-1 - p\lambda =$

$= p^3\beta - \sum_{q \neq p} \eta_q - (1+\eta_p)$. Thus $\{x,y\}$ satisfies (FSp) for no p.

5. <u>Lemma:</u> Let $\pi = \bigcup_{i=1}^{m} \pi_i$ and let G be a torsion-free group of finite rank. If $G \otimes R_{\pi_i}$, $i = 1,2,\dots,m$ is factor-splitting then G is factor-splitting.

Proof: Let $B = \{g_1,\dots,g_n\}$ be an arbitrary basis of G. Since $G \otimes R_{\pi_i}$ is factor-splitting, B has (FSp) for almost all primes $p \in \pi_i$ by Theorem 3. Hence B has (FSp) for almost all $p \in \pi$ and G is factor-splitting.

6. <u>Theorem</u>: Every completely decomposable torsionfree group of finite rank is factor-splitting.

Proof: Let $G = \sum_{i=1}^{n} J_i$ be a complete decomposition of G, $h_i \in J_i$. For any permutation $\varphi \in S_n$ define π_φ to be the set of all primes p with $h_p^G(h_{\varphi(1)}) \geq h_p^G(h_{\varphi(2)}) \dots \dots \geq h_p^G(h_{\varphi(n)})$. Now $G \otimes R_{\pi_\varphi}$ is a completely decomposable group with ordered type set so that it is factor-splitting by [9, Theorem 7]. Lemma 5 now finishes the proof.

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