Commentationes Mathematicae Universitatis Carolinae

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Riga p-point

Commentationes Mathematicae Universitatis Carolinae, Vol. 18 (1977), No. 4, 675--683

Persistent URL: `http://dml.cz/dmlcz/105811`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

18,4(1977)

RIGA P-POINT +)

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Abstract: For a graph G we define a G-arrow ultrafilter U by the partition relation $U \longrightarrow (U,G)^2$. We prove that every ultrafilter U is F-arrow ultrafilter for every component-finite forest F and we exhibit an example of a p-point U which is G-arrow iff G is a component finite forest.

This means that p-pointness does not induce any non-trivial partition property.

Key words: Ultrafilter, partition property.

AMS: 04A20, 02K05, 05A99 Ref. Ž.: 8.83

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§ 1. Introduction. Statement of results. An ordinal number is considered as the set of all smaller ordinals.

A graph G is a couple (V,E) where V is a set (the set of vertices) and $E \subseteq [V]^2 = \{e \subseteq V; |e| = 2\}$ (the set of edges). If G will be considered as the set of edges E only then we mean the graph $(\bigcup E, E)$.

A homomorphism $f: (V,E) \longrightarrow (V',E')$ is a mapping $f: V \longrightarrow V'$ which satisfies $\{x,y\} \in E \Longrightarrow \{f(x),f(y)\} \in E'$. An 1-1 homomorphism is called a monomorphism. If $V \subseteq V'$ and the inclusion is a monomorphism then (V,E) is called a subgraph of (V′,E′). If both f and f^{-1} are monomorphisms then (V,E)

+) This note was written during 1977-summer stay at Latvian State University, Riga.

and (V',E') are said to be isomorphic, this is denoted by $(V,E) \simeq (V',E')$.

The graph $K_\alpha = (\alpha, [\alpha]^2)$ is called the complete graph of size α. (K_α will be needed for $\alpha \leq \omega$ only).

The chromatic number $\chi(G)$ of a graph G is the minimal cardinal number α for which there exists a homomorphism $G \longrightarrow K_\alpha$. Equivalently, $\chi(G)$ is the minimal number of colours which are needed for a colouration of vertices of G in such a way that the vertices of no edge get the same colour.

A cycle of length n, $3 \leq n < \omega$, is every graph isomorphic to the graph $C_n = (n, \{\{i, i+1\}; i \in n-1\} \cup \cup \{\{1, n-1\}\})$. Clearly $C_3 = K_3$.

A forest is a graph which does not contain any cycle.

In this list of graph-theoretical notions the following is the only non-standard one: A component-finite forest is a forest each of its components is finite. Explicitly, (V,E) is a component-finite forest if there are finite forests (V_i, E_i), $i \in I$, such that V_i, $i \in I$, are pairwise disjoint sets and $V = \bigcup_{i \in I} V_i$, $E = \bigcup_{i \in I} E_i$.

Let $G = (V,E)$ be a graph, $x \in V$. Put $d_G(x) = |\{y; \{x,y\} \in \in E\}|$ (the degree of x) and $\delta(G) = \min\{d_G(x); x \in V\}$.

It is easy to see that for every finite forest F there exists a number δ_F such that every graph G with $\delta(G) \geq \delta_F$ contains a subgraph isomorphic to F. If $|F| = n$ then it suffices to put $\delta_F = n$ and to prove the statement by induction on n (every forest contains a vertex of degree 1).

All ultrafilters considered in this paper are proper non-principal ultrafilters on ω.

Definition ([4]): An ultrafilter U is called a G-*arrow ultrafilter* if for every partition $c: [\omega]^2 \longrightarrow 2$ either there exists $X \in U$ such that $c([X]^2) = \{0\}$ or there exists $G' \subseteq [\omega]^2$, $G' \simeq G$, such that $c(G') = \{1\}$. This fact is denoted by $U \longrightarrow (U,G)^2$.

The notion of G-ultrafilter refines the scale of "partition" notions which are related to ultrafilters on ω:

k-arrow ultrafilter is K_k-arrow ultrafilter,

arrow ultrafilter is an ultrafilter which is K_k-arrow for every $k < \omega$,

Ramsey ultrafilter is K_ω-ultrafilter (see [1], Theorem 2.1).

In [1] and [4] there is proved the mutual independence of notions k-arrow ultrafilter, (k + 1)-arrow ultrafilter, p-point, q-point. In particular, it is proved in [1] that there exists a p-point which is not a 3-arrow ultrafilter.

The purpose of this note is to prove:

Theorem 1: Every ultrafilter is F-arrow ultrafilter for every component-finite forest F.

Theorem 2 [P(x)]: There exists an ultrafilter U with the following properties:

i. U is a p-point,

ii. U is a G-arrow ultrafilter iff G is a component-finite forest.

Theorem 1 is proved in ZFC. Theorem 2 is proved here under CH and it follows from the machinary developed in [1] (independently, a similar procedure was found by P. Simon, see [8],[9]) that Theorem 2 is valid under the following consequence of Martin's axiom:

[P(c)] If F is a set of infinite subsets ω, $|F| < 2^{\aleph_0}$, such that finite intersections of elements of F are infinite, then there exists an infinite set $A \subseteq \omega$ such that $A \setminus B$ is a finite set for all $B \in F$.

(In the terminology of [1] one has to realize only that the function $X \longrightarrow \chi(X)$ is a " $\mathcal{D}$-norm" which "handles the p-point condition"; this is, essentially, proved by statements 0 - 4 stated below in the proof of Theorem 2.)

I thank to B. Balcar, F. Galvin and P. Simon for stimulating discussions and correspondence.

Added in proof: Independently, a theorem similar to Theorem 2 was proved by F. Galvin (who proved the existence of a p-point U which satisfies $U \longrightarrow (U, C_3 \vee C_4 \vee \ldots \vee C_n)^2$ for every n).

§ 2. <u>Proof of Theorem 1</u>. Let F be a fixed component-finite forest. Denote by F_i, $i \in \omega$, all the components of F, let F_i have n_i vertices (we may assume that F has infinite many components).

Let U be an ultrafilter. We prove $U \longrightarrow (U, F)^2$.

Let $c: [\omega]^2 \longrightarrow 2$ be a partition and assume that $c([X]^2) \neq \{0\}$ for every $X \in U$.

Consider the graph $G = (\omega, c^{-1}(1)) = (\omega, E)$.

It is $\chi(G) = \omega$ as if there exists a homomorphism $f: : G \longrightarrow K_k$ for a $k < \omega$ then there exists $i \in k$ such that

$f^{-1}(i) \in U$ and (as f is a homomorphism) $[f^{-1}(i)]^2 \subseteq c^{-1}(0)$ which is a contradiction.

Now there exists a family of finite subgraphs $G_i = (V_i', E_i')$ of G with the following properties:

1. V_i' are pairwise disjoint sets, $i \in \omega$;

2. $\chi(V_i', E_i') = n_i + 1$, $i \in \omega$;

3. $\chi(V^{\vee}, E^{\vee}) \leq n_i$ for every proper subgraph of (V_i', E_i'), $i \in \omega$.

(The existence of graphs G_i', $i \in \omega$, may be seen as follows: According to a compactness argument, see [2], there exists a subgraph of G with chromatic number $n_1 + 1$ and if we take any minimal subgraph G_1' of G with this property then G_1' satisfies 2 and 3. Put $G_1 = G - G_1' = (\omega \setminus V_1', \{e \in E;\ e \cap V_1' = \emptyset\})$. It is again $\chi(G_1) = \omega$ and therefore we may proceed for G_1 analogously as for g.)

It is well known that every graph which satisfies conditions 2 and 3 above satisfies also

4. $\delta(G_i') \geq n_i \geq \delta_{F_i}$, $\quad i \in \omega$.

From 1 and 4 follows that the graph G contains a subgraph F' isomorphic to F. Consequently $F' \subseteq c^{-1}(1)$.

§ 3. Proof of Theorem 2. In the proof we use a construction of a general type described in [1]. However, as we are not interested in any generalizations we give a self-contained description of the desirable ultrafilter. In this particular case the construction is also simpler.

The following is a non-trivial combinatorial fact which will be used (see [3],[5],[6]):

Proposition: For every $3 \leq n \in \omega$, $3 \leq k \in \omega$, there exists a graph $G_{n,k}$ with the following properties:

1. $\chi(G_{n,k}) = n$;
2. $G_{n,k}$ does not contain cycles of length $3,\dots,k$.

Put $G_{n,k} = (V_{n,k}, E_{n,k})$.

Let $\{D_3, D_4, \dots, D_n, \dots\} = \mathcal{D}$ be a partition of ω such that $|D_i| = |V_{i,i}|$. Without loss of generality let us assume $D_i = V_{i,i}$. Put $E = \bigcup_{i=3}^{\omega} E_{i,i}$. For the sake of brevity we put $\chi(X) = \chi(X, [X]^2 \cap E)$ for every $X \subseteq \omega$.

The desirable ultrafilter will be constructed by means of the following sets:

A set $X \subseteq \omega$ is said to be large if $\chi(X) = \omega$.

The following facts about large sets hold:

0. ω is large;
1. X large, $Y \supseteq X \Longrightarrow Y$ large;
2. X, Y not large $\Longrightarrow X \cup Y$ not large;
3. X finite $\Longrightarrow X$ not large;
4. If $\{A_n;\ n \in \omega\}$ is a partition of ω then there exists a large set X such that either $X \subseteq A_n$ for some n or $X \cap A_n$ is finite for every $n \in \omega$.

(Only 4 has to be mentioned: If $\chi(A_n) \in \omega$ for all $n \in \omega$ then for every $n \in \omega$ choose $\overline{n}$ such that $\chi(D_{\overline{n}} \setminus \bigcup_{i=1}^{m} A_i) \geq n$ and put $X = \bigcup_{m=3}^{\omega} (D_{\overline{n}} \setminus \bigcup_{i=1}^{m} A_i)$. X is large and $X \cap A_n$ is finite for every $n \in \omega$.)

Now let $\{P_\alpha;\ \alpha < 2^{\aleph_0}\}$, $P_\alpha = \{A_n^\alpha;\ n \in \omega\}$, be an enumeration of all partitions of ω .

Put $X_0 = \omega$. Let X_ι , $\iota < \alpha$ be already chosen large sets with the property that any finite intersection of X_ι is again large and that for every $\iota < \alpha$ the following holds:

(*) either $\exists n(X \subseteq A_n^{\iota})$ or $\forall n\ (|x \cap A_n^{\iota}| < \omega)$.

In this situation we find a large set X'_{α} such that $X'_{\alpha} \setminus X_{\iota}$ is a finite set for all $\iota < \alpha$. The existence of X'_{α} is easy to see: for each n we choose i(n) as the minimal i for which $\chi(X_1 \cap \ldots \cap X_n \cap D_i) \geq n$ and we put $X'_{\alpha} = \bigcup_{m=3}^{\omega}(D_{i(n)} \cap \bigcap_{j=3}^{m} X_j)$.

Using 4 there exists a large set $X_{\alpha} \subseteq X'_{\alpha}$ such that (*) holds for the partition P_{α}.

Summing up it follows that $\{X_{\alpha};\ \alpha < 2^{\aleph_0}\}$ is a family of large sets with all its finite intersections again large such that (*) holds for every $\alpha < 2^{\aleph_0}$. Let U be an ultrafilter generated by this family.

U is a p-point by (*) and $U \longrightarrow (U,F)^2$ for every component-finite forests follows from Theorem 1.

Let G be a graph which fails to be a component-finite forest. We distinguish two cases:

a) G contains an infinite component. Consider the partition $c: [\omega]^2 \longrightarrow 2$ defined by $c(e) = 1$ iff $e \in E$. Then $c([X]^2) \neq \{0\}$ for every $X \in U$ (every $X \in U$ is large and consequently $|[X]^2 \cap E| = \aleph_0$). Moreover G fails to be a subgraph of E.

b) G contains a cycle of length n. Consider the partition $c: [\omega]^2 \longrightarrow 2$ defined by $c(e) = 1$ iff $e \in E_{i,i}$, $i > n$. It is again $c([X]^2) \neq 0$ for every $X \in U$. Moreover a cycle of length n fails to be a subgraph of $\bigcup_{i > n} E_{i,i}$.

§ 4. <u>Concluding remarks</u>. The notion of G-arrow ultrafilter effectively refines the hierarchy provided by k-arrow ultrafilters. It can be shown the non-validity of the following implications:

1. $U \longrightarrow (U,G)^2$ for every graph without K_k implies $U \longrightarrow (U,K_k)^2$, $k \geq 3$;

2. $U \longrightarrow (U,K_k)^2$ implies $U \longrightarrow (U,G)^2$ for every graph without K_k, $k \geq 3$.

While the above proof used only large sets defined by means of chromatic number , these theorems use combinatorial partition theorems of the type described in [7]. 1 is implicitly proved in [1] and stated in [4], 2 will appear in a joint paper with V. Rödl. Theorems and methods given in [7] do not imply 2.

Finally, let us remark that for partitions of triples similar theorems do not hold. One can prove that the following three statements about an ultrafilter U are equivalent:

a. U is a Ramsey ultrafilter.

b. $U \longrightarrow (U,K_3^4)^3$ here $K_3^4 = (\{1,2,3,4\}, [\{1,2,3,4\}]^3)$.

c. $U \longrightarrow (U,T)^3$ for every triple system T which does not contain K_3^4.

(The equivalence a and b is proved in [1]. A more detailed discussion of the statement c is going to appear elsewhere.)

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(Oblatum 16.8. 1977)