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ON THE POINTS OF MULTIVALUEDNESS OF METRIC PROJECTIONS IN SEPARABLE BANACH SPACES

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Abstract: Given a real Banach space X and a nonempty subset $M \subset X$ we consider the set A_M of all points of multivaluedness of the metric projection on M. We prove that if X is separable and strictly convex, then A_M can be covered by countably many of Lipschitz hypersurfaces. In particular, A_M is a set of the first category and of measure zero for any Gaussian measure on X.

Key words: Multivaluedness of metric projections, separable strictly convex Banach space, Lipschitz hypersurface, small sets in Banach spaces.

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1. Introduction. We will consider a real Banach space X and a non-empty subset $M \subset X$. For $x \in X$ denote by $d_M(x)$ the distance from the point x to the set M. The metric projection $P_M(x)$ on the set M is defined as the (possibly) multivalued operator $P_M(x) = \{y \in M; \|x-y\| = d_M(x)\}$. Of course, it is possible that $P_M(x) = \emptyset$ for some x. The set of all x for which $P_M(x)$ contains at least two points will be denoted by A_M.

In some Banach spaces the set A_M is always a "small" set. Erdős [2] investigated A_M in the case of n-dimensional

Euclidean space X. He proved that in this case A_m has σ-finite (n - 1)-dimensional Hausdorff measure. Stečkin [7] investigated A_m in general spaces. He proved, for example, that in any locally uniformly convex Banach space X the set A_M is always of the first category. Note also that in [7] Stečkin proved that in any normed linear space X with strictly convex norm the complement to A_M is dense in X.

In the present article we will consider the case of a separable real Banach space X only. It is the simple fact [7] that if the norm of X is not strictly convex, then there exists a hyperplane M with $A_M = X$ and therefore A_M is not small in any sense. We prove in the present article that if the norm of a separable Banach space X is strictly convex, then A_M can be covered by countably many Lipschitz hypersurfaces (Theorem 1). Since any Lipschitz hypersurface (see Definition 1 below) is obviously a nowhere dense set, we have that A_M is a set of the first category. If X is an n-dimensional Banach space, then our result implies that A_M is of σ-finite (n - 1)-dimensional Hausdorff measure. This result generalizes the Erdős' theorem stated above. In the infinite-dimensional case it is not difficult to deduce from our result that A_M is contained in a Haar zero set in the sense of Christensen [4]. We can use for this purpose the construction from Theorem 1 of Aronszajn [1] or Theorem 7.2 of Christensen [4] . It is interesting to compare our result concerning Haar zero sets with a note of Christensen [4], p. 124. We shall also prove a more strong result which asserts that A_M is of measure zero for any Gaussian measure μ in X (we consider Gaussian measures such that $\mu(G) = 0$ for any nonempty open subset G of X).

If in addition the norm of X is smooth, then we obtain Theorem 2 from which it follows that A_M belongs to the Aronszahn's system of small sets $\mathcal{U}^o$ (defined in [1]). We are not able to prove that A_M belongs to $\mathcal{U}^o$ in general separable strictly convex Banach spaces.

2. <u>Notations, definitions and lemmas</u>. If f is real-valued function defined in a Banach space X, $a \in X$ and $o \neq v \in X$, then we put

$$D_v(f,a) = \lim_{h \to 0_+} 1/h\ (f(a+hv) - f(a)).$$

If we work in a metric space, then by $U_\delta(a)$ we mean the open δ-neighbourhood of a point a.

If x, y are points of a Banach space, then by $\overline{x,y}$ we mean the closed line segment joining these points.

The symbol R denotes the set of real numbers.

For the symbols $d_M(x)$, $P_M(x)$, A_M see Introduction.

<u>Definition 1</u> ([8]). Let X be a Banach space and $o \neq v \in X$. We shall say that $A \subset X$ is a Lipschitz hypersurface associated with v if there exists a topological complement

Z of the one-dimensional space $V = \mathrm{Lin}\{v\}$ and a Lipschitz mapping $f: Z \longrightarrow V$ such that $A = \{z + f(z), z \in Z\}$.

Definition 2. Let X be a Banach space, $x \in X$ and $M \subset X$. Then we denote by contg (M,x) the set of all $o \neq v \in X$ with the following property: There exist sequences $(x_i)_{i=1}^{\infty}$, $x_i \in M$ and $(\lambda_i)_{i=1}^{\infty}$, $\lambda_i > 0$ such that $\lambda_i \longrightarrow 0$ and $1/\lambda_i \, \| x + \lambda_i v - x_i \| \longrightarrow 0$.

Note. The geometrical sense of the preceding definition is clear. It is essentially a natural generalization of the well-known notion of the contingent of a set M in a point x defined in Euclidean spaces. We shall use the notion contg (M,x) only in the connection with the following simple lemma which is an easy generalization of the well-known proposition concerning contingents in Euclidean spaces (cf.[5], Lemma 3.1, p. 264). In this point we follow in the present article the method of Erdős [2].

Lemma 1. Let M be a subset of a Banach space X and $o \neq v \in X$ a vector. Then the set A of all points $x \in X$ for which $v \notin$ contg (M,x) can be covered by countably many Lipschitz hypersurfaces associated with v.

Proof. Put $V = \mathrm{Lin}\{v\}$. Let Z be a topological complement of V. Denote by π_V (resp. π_Z) the projection of X on V (resp. Z) parallel to Z (resp. V). If $x \in A$ it is easy to see that there exists a positive integer n such that

(1) $\| \pi_Z(y-x) \| > 1/n \; \| \pi_V(y-x) \|$ whenever $y \in M$ and $\pi_V(y-x) = t\,v$ for some $0 < t < 1/n$.

Let A_n be the set of all points $x \in A$ for which (1) holds. Choose further for each n a sequence of sets $\{A_{nm}\}_{m=1}^{\infty}$ such that $A_n = \bigcup_{m=1}^{\infty} A_{nm}$ and $\|\pi_V(y-x)\| < 1/n \|v\|$ whenever $x \in A_{nm}$ and $y \in A_{nm}$. Obviously $A = \bigcup_{n,m=1}^{\infty} A_{nm}$.

Let now n, m be fixed and x, y be distinct elements of A_{nm}. Without loss of generality we can suppose that $\pi_V(y-x) = t \cdot v$ for $t \geq 0$ and therefore by (1)

$$\|\pi_V(y) - \pi_V(x)\| < n \|\pi_Z(y) - \pi_Z(x)\| .$$

Thus the set $\{(\pi_Z(x), \pi_V(x));\ x \in A_{nm}\}$ is the graph of a Lipschitz mapping $f: Z \longrightarrow V$ defined on a subset of Z. Since any Lipschitz function defined on a subset of a metric space has a Lipschitz extension on the whole space, we obtain that A_{nm} is a subset of a Lipschitz hypersurface associated with v. The proof is complete.

Lemma 2. Let v be a continuous convex function defined on a Banach space X, $x \in X$ and $o \neq v \in X$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$-D_{-v}(f,x) - \varepsilon < D_v(f,y) < D_v(f,x) + \varepsilon$$

for any $y \in U_\delta(x)$ (δ-neighbourhood of x).

Proof: Suppose that there exists $\varepsilon > 0$ and a sequence $y_n \longrightarrow x$ such that $D_v(f,x) + \varepsilon \leq D_v(f,y_n)$. Then for any $t > 0$ we have $1/t\ (f(y_n+tv) - f(y_n)) \geq D_v(f,x) + \varepsilon$. By the continuity of f we have $1/t\ (f(x+tv) - f(x) \geq D_v(f,x) + \varepsilon$ for any $t > 0$ and this is a contradiction. Thus "the right inequality" is proved. "The left inequality" follows from the right one in which we replace v by -v.

<u>Lemma 3</u>. Let f be a continuous function defined on a Banach space X, $0 \neq v \in X$, $x \in X$, $a > 0$ and $K \in R$. Let $D_v(f,z) < K$ for any $z \in \overline{x,x+av}$ (see Notations). Then $f(x+av) - f(x) < aK$.

<u>Proof</u>: If we define $g(t) = f(x+vt)$ for $t \in R$ we see that Lemma is an immediate consequence of well known theorems from the real analysis (e.g. of Theorem 7.2 or Theorem 7.3 from [5], p. 204).

3. Theorems

<u>Theorem 1</u>. Let X be a separable Banach space with a strictly convex norm p and let $M \subset X$. Let A_M be the set of points of multivaluedness of the metric projection P_M. Then A_M can be covered by countably many Lipschitz hypersurfaces.

In particular, A_M is a set of the first category and of measure zero for any Gaussian measure μ in X (with $\operatorname{supp} \mu = X$). Consequently A_M is a subset of a Haar zero set in the sense of Christensen.

<u>Proof</u>: Recall that $p(x) = \|x\|$. For each $x \in A_M$ choose two distinct points $y_1(x)$, $y_2(x)$ from $P_M(x)$. Put $z_1(x) = y_1(x) - x$, $z_2(x) = y_2(x) - x$, $v(x) = y_2(x) - y_1(x) = z_2(x) - z_1(x)$. Since $p(z_1(x)) = p(z_2(x))$ and p is strictly convex there exists $h(x) > 0$ such that

$$D_{v(x)}(p,z_1(x)) + h(x) < -D_{-v(x)}(p,z_2(x)). \tag{2}$$

Choose by Lemma 2 $\eta(x) > 0$ so small that $D_{v(x)}(p,z) < D_{v(x)}(p,z_1(x)) + 1/3\, h(x)$ for any $z \in U_{4\eta(x)}(z_1(x))$ and $-D_{-v(x)}(p,z) > -D_{-v(x)}(p,z_2(x)) - 1/3\, h(x)$ for any

$z \in U_{4\eta(x)}(z_2(x))$.

The set $C = \{(z_1(x),\ z_2(x));\ x \in A_M\}$ is a subset of the separable metric space $X \times X$. Therefore there exists a sequence $(x_i)_{i=1}^{\infty} \subset A_M$ such that

$$C \subset \bigcup_{i=1}^{\infty} U_{\eta(x_i)}(z_1(x_i)) \times U_{\eta(x_i)}(z_2(x_i)).$$

Let A_i be the set of all points $x \in A_M$ for which

$$(z_1(x),\ z_2(x)) \in U_{\eta(x_i)}(z_1(x_i)) \times U_{\eta(x_i)}(z_2(x_i)).$$

Then $A_M = \bigcup_{i=1}^{\infty} A_i$ and it is sufficient to prove that each set A_i can be covered by countably many Lipschitz hypersurfaces. By Lemma 1 it is sufficient to prove that for any i and for any $x \in A_i$ we have $-v(x_i) \notin \operatorname{contg}(A_i,x)$. For this purpose fix i and suppose that there exists $x \in A_i$ such that

$$-v(x_i) \in \operatorname{contg}(A_i,x). \tag{3}$$

Put $v = v(x_i)$, $z_1 = z_1(x_i)$, $z_2 = z_2(x_i)$, $h = h(x_i)$, $\eta = \eta(x_i)$. By (3) we can choose $a > 0$ and $x^* \in A_i$ such that $\|av\| < \eta$ and

$$\|x^* - y\| < \min(\eta, 1/6\ ah), \text{ where } y = x - av. \tag{4}$$

Now we shall find a lower and an upper bound for $d_M(x^*) = \|z_2(x^*)\| = \|z_1(x^*)\|$.

Obviously $\|z_2(x^*)\| = \|y_2(x^*) - y + y - x^*\| \geq$

$\geq \|y_2(x^*) - y\| - \|y - x^*\|$ and $\|y_2(x^*) - y\| =$ $= \|(y_2(x^*) - x) + (x - y)\|$. Since $y_2(x^*) - x^* = z_2(x^*) \in$ $\in U\eta(z_2)$ and $\|x^* - x\| < 2\eta$, we have that $y_2(x^*)$ -

$- x \in U_{3\eta}(z_2)$ and therefore $D_v(p, y_2(x^*) - x) > -D_{-v}(p, z_2) - 1/3\, h$. Since $(x-y) = av$, we have

$\|(y_2(x^*) - x) + (x-y)\| \geq \|y_2(x^*) - x\| + a(-D_{-v}(p,z_2) - 1/3\, h)$. Since $y_2(x^*) \in M$, we have $\|y_2(x^*) - x\| \geq d_M(x)$ and therefore

$$(5) \qquad d_M(x^*) \geq d_M(x) - a\, D_{-v}(p,z_2) - 1/3\, ah - \|y - x^*\| >$$
$$> d_M(x) - a\, D_{-v}(p,z_2) - 1/2\, ah.$$

On the other hand,

$$\|z_1(x^*)\| = \|y_1(x^*) - x^*\| \leq \|y_1(x) - x^*\| =$$
$$= \|(y_1(x)-x) + (x-y) + (y-x^*)\| \leq \|z_1(x) + av\| + \|y - x^*\|.$$

Since $\|av\| < \eta$ and $z_1(x) \in U_\eta(z_1)$, we have

$$\overline{z_1(x),\ z_1(x) + av} \subset U_{2\eta}(z_1)$$

and therefore for any $y \in \overline{z_1(x),\ z_1(x) + av}$

$D_v(p,y) < D_v(p,z_1) + 1/3\, h$.
From Lemma 3 it follows that

$$\|z_1(x) + av\| < d_M(x) + a(D_v(p,z_1) + 1/3\, h)$$

and therefore by (4)

$$(6) \quad d_M(x^*) < d_M(x) + a\, D_v(p,z_1) + 1/3\, ah + \|y - x^*\| <$$
$$< d_M(x) + a\, D_v(p,z_1) + 1/2\, ah.$$

From (5) and (6) we obtain that $-D_{-v}(p,z_2) < D_v(p,z_1) + h$ and this is a contradiction with (2).

Let now μ be a Gaussian measure in X such that supp μ = X. By H. Sato [6] μ can be considered as an abstract Wiener measure. Therefore by L. Gross [3] there exists a dense subset $H \subset X$ such that μ is equivalent with any measure μ_h, $h \in H$ ($\mu_h(A) = \mu(h+A)$). It is easy to prove that for any Lipschitz hypersurface L there exists $h \in H$ such that L is a Lipschitz hypersurface associated with h and therefore $\mu(L) = 0$. Thus we have $\mu(A_M) = 0$. From this assertion we immediately obtain that A_M is a subset of a Haar zero set in the sense of Christensen (it is easy to give a direct proof of this fact, see Introduction).

Corollary. Let X be an n-dimensional Banach space with a strictly convex norm. Then A_M is always a set of σ-finite (n-1)-dimensional Hausdorff measure.

Theorem 2. Let X be a separable Banach space with a norm p which is strictly convex and smooth. Let M be a subset of X and let $(x_n)_{n=1}^{\infty}$ be a complete sequence of nonzero vectors in X. Then $A_M \subset \bigcup_{n,m=1}^{\infty} L_{nm}$ where each L_{nm} is a Lipschitz hypersurface associated with x_n.

In particular, A_M belongs to the Aronszajn's class $\mathfrak{A}^0$.

Proof: The proof of Theorem 1 works if we for $x \in A_M$ instead of $v(x) = y_2(x) - y_1(x)$ define $v(x)$ as a vector of the form x_n or $-x_n$ for which $D_{v(x)}(p, z_2(x)) < D_{v(x)}(p, z_1(x))$. The existence of a such vector $v(x)$ follows from the properties of the norm p. In fact, if no such $v(x)$ exists, then

$$D_{x_n}(p, z_2(x)) = D_{x_n}(p, z_1(x)) \text{ for any } n,$$

and from the smoothness of p and completeness of $(x_n)_{n=1}^{\infty}$ we obtain

$$D_{z_2(x)-z_1(x)}(p,z_2(x)) = D_{z_2(x)-z_1(x)}(p,z_1(x)).$$

This is a contradiction with the strict convexity of p.

The consequence concerning the class $\mathcal{U}^0$ is obvious.

Note: When the present article was written we became acquainted with the paper "S.V. Konjagin, Approksimativnye svojstva proizvolnych množestv v Banachovych prostranstvach, Dokl. Akad. Nauk SSSR 239(1978), No 2, 261-264." The results stated in that paper overlap with our results. In Theorem 1 of that paper the following proposition is contained: If X is strictly convex n-dimensional Banach space, then A_M can becovered by countably many of (n - 1)-dimensional surfaces with finite (n - 1)-dimensional Hausdorff measure. Theorem 4 asserts that in any strictly convex separable Banach space the set A_M is always a set of the first category. In the Konjagin's paper a number of further results is contained. In particular, the points of A_M are classified by degree of singularity and also descriptive properties of A_M are investigated. No results concerning measure or the possibility of the covering of A_M by surfaces in infinite-dimensional spaces are stated in the Konjagin's paper.

References

[1] N. ARONSZAJN: Differentiability of Lipschitzian mappings between Banach spaces, Studia Math. 57 (1976), 147-190.

[2] P. ERDŐS: On the Hausdorff dimension of some sets in Euclidean space, Bull. Amer. Math. Soc. 52

(1946), 107-109.

[3] L. GROSS: Potential theory on Hilbert space, J. Functional Analysis 1(1967), 123-181.

[4] J.P.R. CHRISTENSEN: Topology and Borel Structure, North-Holland Mathematics Studies 10, 1974.

[5] S. SAKS: Theory of the Integral, New York 1937.

[6] H. SATO: Gaussian measures on a Banach space and abstract Wiener spaces, Nagoya Math. J. 36(1969), 65-83.

[7] S. STEČKIN: Approximation properties of sets in normed linear spaces, Rev. Math. Pures Apl. 8(1963), 5-18 (Russian).

[8] L. ZAJÍČEK: On the points of multiplicity of monotone operators, Comment. Math. Univ. Carolinae 19 (1978), 179-189.

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