Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 20 (1979), No. 4, 697--722

Persistent URL: http://dml.cz/dmlcz/105962

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
20, 4 (1979)

METAMATHEMATICS OF THE ALTERNATIVE SET THEORY I

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Abstract: In this paper the alternative set theory (AST) is described as a formal system. We show that there is an interpretation of Kelley-Morse set theory of finite sets in a very weak fragment of AST. This result is used to the formalization of metamathematics in AST. The article is the first paper of a series of papers describing metamathematics of AST.

Key words: Alternative set theory, axiomatic system, interpretation, formalization of metamathematics, finite formula.

Classification: Primary 02K10, 02K15
Secondary 02K05

This paper begins a series of articles dealing with metamathematics of the alternative set theory (AST; see [V]).

The first aim of our work (§ 1) is an introduction of AST as a formal system - we are going to formulate the axioms of AST and define the basic notions of this theory. Doing this we limit ourselves really to the formal side of the matter and the reader is referred to [V] for the motivation of our axioms (although the author considers good motivations decisive for the whole work in AST).

In [V] P. Vopěnka lays emphasis on intuitive explanation and pedagogically convenient presentation and this conception necessitates an introduction of axioms which are superfluous in the sense that they are provable from the other axioms. On the contrary, we try to minimalize the axiomatic system of AST in our text.

The axioms we introduce are either exactly in the same form as in [V] or they are formal precisations of Vopěnka's axiom (cf. the schema of existence of classes). There is only one essential exception since the direct formalization of Vopěnka's axiom of induction is too weak, we need "induction for formal formulas" (see e.g. § 1 ch. V [V] or [S-V 1]). If we would express the axiom in question in this form we would need notions of "formal formula", "satisfaction relation" and so on before the formulation of the axiom (cf. § 5 ch. II [V]). Therefore we choose a little different approach and formulate this assumption on the base of the notion of "Gödel-Bernays class" because this approach seems to be quicker and needs less notions.

We are going to call our formal system "alternative set theory" though the term "basic alternative set theory" would be probably better. Our axioms formalize Vopěnka's alternative set theory as it is described in [V], nevertheless it seems to be possible that in the following development of the alternative set theory it will be necessary to introduce new axioms (and we will show some candidates of such statements later in our text; moreover some principles for the choice of such axioms were mentioned in [V]).

In the second section we introduce some interesting weakening of axioms of AST. Some of them were accepted in [V] as axioms of AST, the importance of the others comes out from the further metamathematical investigations. We also try to motivate these newly mentioned axioms.

We are going to deal with the connections between the usual axiomatics of set theory and AST. Let ZF (GB, KM respectively) denote Zermelo-Fraenkel (Gödel-Bernays, Kelley-Morse respectively) axiomatic system (in KM no kind of the axiom of choice is required). Let ZF_{Fin} denote Zermelo-Fraenkel set theory of finite sets, i.e. the theory ZF in which the axiom of infinity is replaced by its negation (we assume regularity in a form which is strong enough - similarly as in AST) and let ZF^- denote the theory ZF in which the power-set axiom is omitted; similarly for other theories.

In the third section we construct an interpretation of KM_{Fin} in AST. This enables us to transfer notions of ZF_{Fin} into AST in two ways. This is used to double formalization of metamathematics in AST in the same section, too.

Let us mention some result which will be proved in the following articles of this series. It is possible to show that ZF_{Fin} is equivalent to the system of those axioms of AST which concern sets only and further that KM_{Fin} is equivalent to the theory which we get from AST replacing the prolongation axiom by the axiom guaranteeing that every subclass of a set is a set.

The results which concern interpretability of the investigated theories can be drawn in the following diagram, where $\longrightarrow$ denotes the existence of an interpretation of the first theory in the second one and $\not\longrightarrow$ represents the non-existence of such interpretation.

$$\begin{tikzcd} KM_{Fin} \arrow[r, shift left] & AST \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[d, shift right] & ZF \arrow[l, shift left, "/" marking] \\ & KM^{-} \arrow[u, shift right] & \end{tikzcd}$$

We see from the above diagram that AST is "strictly stronger" than KM_{Fin} in the sense of interpretability. But the position of these theories changes if we investigate the provability of set-formulas. It can be shown that in AST there are provable exactly those set-formulas as in ZF_{Fin} (and therefore one is able to prove in AST less set-formulas than in KM_{Fin}).

The existence of an interpretation of AST in ZF gives us security that AST is consistent under the assumption that ZF is consistent. (On the other hand there is a theoretical possibility of consistency of AST even in the case that ZF would be inconsistent.)

We shall show that we really extremely minimalized our axiomatic system of AST since every axiom of AST is independent on the others (except the axiom of choice where the independence of it remains as an open problem). Moreover, we shall see that AST is not finitely axiomatizable.

We use Tarski's notion of interpretation (cf. [T-M-R]). Dealing with some language and its formalization we use the usual symbols ($\&$, $\neg$, $\in$, $=$ and so on) in both levels since we hope that this will not lead to any confusion. To indicate the theory in which the statement in question can be proved, we write it before the statement (i.e. Theorem (T). Φ is written instead of $T \vdash \Phi$).

All results we shall use from AST can be found in [V]. The needed results of other theories and logic we try to restrict to basic knowledge - the great deal of them is contained e.g. in [Sh] (the reader has to consult also [M] and [M-S] since the proofs of these articles are not repeated in our text).

In the series we shall prove some results which were announced (without proofs) in [So 1] and [So 2].

§ 1. The basic axioms of the alternative set theory

The language of the alternative set theory is the usual language of set theory i.e. it consists of two binary predicates $\in$ and $=$ and one sort of variables. Variables of this sort are denoted by capital Latin letters and denote classes (cf. the language of GB or KM). The predicate $=$ is understood as the predicate of equality and hence we accept the usual axioms of equality for it (as a consequence of the axiom of extensionality, these requirements can be restricted to the axiom $(\forall X,Y,Z)((X = Y \,\&\, X \in Z) \longrightarrow Y \in Z))$. The unary predicate "to be a set" we define - again as it is usual in GB - as the property "to be an element", i.e.

$$\mathrm{Set}(X) \equiv (\exists Y) X \in Y$$

We reserve lower-case Latin letters for variables running through sets (supposing that sets exist, cf. the axiom A3 below).

Formulas of the language of set theory in which only set variables are quantified, are called normal formulas; normal formulas in which only set variables and constants occur, are called set-formulas; restricted formulas are those set-formulas which have only restricted quantifiers.

The theory having the following eight axioms is called the (basic) alternative set theory and it is denoted by the symbol AST.

A 1 <u>Axiom of extensionality for classes</u>. Two classes are equal iff they have the same elements, i.e.

$(\forall X,Y)((\exists Z)(Z \in X \equiv Z \in Y) \equiv X = Y)$

A 2 <u>Schema of existence of classes</u>. For <u>every</u> formula of the language of set theory $\Phi(Z,Z_1,\dots,Z_k)$ and for every $X_1,\dots,X_k$ we require the existence of a class consisting of all sets x having the property $\Phi(x,X_1,\dots,X_k)$, i.e.

$(\forall X_1,\dots,X_k)(\exists Y)(\forall x)(x \in Y \equiv \Phi(x,X_1,\dots,X_k))$

Using the above formulated axioms we are able to define the empty class 0, the unordered pair of sets (though we do not know that it is a set; this follows from the axiom A 3) and the union of classes.

A 3 <u>Axiom of existence of sets</u>. The empty class is a set and if we add to a set one element we obtain a set again, i.e.

$Set(0) \,\&\, (\forall x,y)\ Set(x \cup \{y\})$

The theory consisting of the axioms A1 - A3 is called the theory of classes and it is denoted by the symbol TC. In this theory one can define the usual notions of set theory e.g. the ordered pair, the universal class, the representation of memmebrship $E = \{\langle x,y\rangle ; x \in y\}$, the identity $Id = \{\langle x,y\rangle ; x = y\}$ and the notions of the list from p. 29 [V] and we are going to use these notions quite freely. Classes are equivalent (in symbols $X \approx Y$) iff there is a one-one mapping (possibly a proper class) of one of them onto the second one. A class is transitive (in symbols $Tran(X)$) iff $X \supseteq \bigcup X$. The symbol $P(X)$ denotes the class of all subsets of a class X. Let us keep Vopěnka's convention that the letters F,G,f and g (possibly indexed) denote functions.

Let us repeat from [V] the definitions which are specific to AST and which cannot be sensibly defined in Cantor's set theory.

A class is called semiset iff it is a subclass of a set, i.e.

$Sms(X) \equiv (\exists y) X \subseteq y$.

A class is finite iff each its subclass is a finite set from Cantor's point of view, i.e.

$Fin(X) \equiv ((\forall Y \subseteq X) Set(Y) \& (\forall Z \subseteq P(X))(0 \neq Z \longrightarrow (\exists z \in Z)(\forall y)(y \subset$

$\subset z \longrightarrow y \notin Z)))$.

A class R is called a well-ordering of a class A (in symbols We(A,R)) iff R is a linear ordering of A and each non-empty subclass of A has the R-first element. We write We(R) instead of We(dom(R),R).

A class is countable iff it is an infinite class having a well-ordering with finite segments only, i.e.

$Count(X) \equiv (\neg\, Fin(X) \& (\exists R)(We(X,R) \& (\forall x)\ Fin(R''\{x\})))$.

A class is uncountable (in symbols Uncount(X)) iff it is neither finite nor countable.

Definition (TC). a) A class X is called an η-element of a class Y (in symbols $X \eta Y$) iff there is $x \in Y''\{0\}$ such that $X = (Y''\{1\})''\{x\}$.

b) If the system of all η-elements of a class S contains all sets and if this system fulfils all axioms of the group B of Gödel-Bernays set theory (cf. [G]; axiom B8 being provable) then S is called a GB-class (more precisely a code of Gödel-Bernays classes), in symbols

$GB(S) \equiv ((\forall x) x \eta S \,\&\, E \eta S \,\&\, (\forall X \eta S)(dom(X) \eta S \,\&\, X^{-1} \eta S \,\&\, \{\langle x,y,z\rangle ; \langle y,z,x\rangle \in X\} \eta S) \,\&\, (\forall X,Y \eta S)(X-Y \eta S \,\&\, X \times Y \eta S))$.

c) A class X is called nearly universal iff it contains 0 and with each set it contains also all set-successors of this set, in symbols

$Nun(X) \equiv (0 \in X \,\&\, (\forall x \in X)(\forall y)(x \cup \{y\} \in X))$.

In TC we are able to prove the existence of GB-class (cf. § 4), but in the following axiom we require more, namely the existence of a GB-class such that the universal class is its sole η-element which is nearly universal.

A 4 Axiom of GB-class. There is a GB-class without nearly universal η-element different from V, i.e.

$(\exists S)(GB(S) \,\&\, (\forall X \eta S)(Nun(X) \rightarrow X = V))$

We shall define the satisfaction in TC and in the fourth section we shall see that A4 is equivalent to the induction for all (finite) formal set-formulas φ ,i.e. to the statement

$V \models ((\varphi(0) \,\&\, (\forall x,y)(\varphi(x) \rightarrow \varphi(x \cup \{y\}))) \rightarrow (\forall x)\varphi(x))$.

A 5 Prolongation axiom. Every countable function is a subclass of a function which is a set, i.e.

$(\forall F)(Count(F) \rightarrow (\exists f)\ F \subseteq f)$

A 6 Axiom of choice. The universal class can be well-ordered, i.e.

$(\exists R) We(V,R)$

A 7 Axiom of cardinalities. Every two uncountable cardinalities are equivalent, i.e.

$(\forall X,Y)((Uncount(X) \,\&\, Uncount(Y)) \rightarrow X \approx Y)$

A 8 Schema of regularity. If there is a set satisfying a set-formula Φ then there is even a set satisfying Φ such that none of its elements satisfies Φ , i.e.

$$(\exists x)\Phi(x) \longrightarrow (\exists x)(\Phi(x) \,\&\, (\forall y \in x)\neg\Phi(y)).$$

The axiomatic system consisting of the axioms A1 - A8 is called the (basic) alternative set theory. For easier expressing let us put moreover the convention that AST_{-i} denotes the theory AST in which the axiom Ai is omitted.

§ 2. More about the axiomatic system of AST

This section deals with axioms which arise by weakening of the axioms of AST. Formulating these axioms we simultaneously explain reasons for the investigation of such axioms and add a few remarks to the meaning of the axioms of AST.

The first axiom is quite formal - it is the axiom of extensionality restricted to sets, nevertheless it was already formulated in § 1 ch. 1 [V] as an axiom of the alternative set theory.

A 11 Axiom of extensionality for sets.

$$(\forall x,y)((\forall z)(z \in x \equiv z \in y) \equiv x = y)$$

Essentially more interesting are the following two axioms which are evidently consequences of the schema of existence of classes.

A 21 Schema of existence of normally definable classes. For every normal formula $\Phi(Z,Z_1,\dots,Z_k)$ we accept the axiom

$$(\forall X_1,\dots,X_k)(\exists Y)(\forall x)(x \in Y \equiv \Phi(x,X_1,\dots,X_k)).$$

A 22 Axiom of existence of the class of all finite sets.

$$(\exists X)(\forall y)(y \in X \equiv \mathrm{Fin}(y)).$$

Let BTC (Bernay's theory of classes) be the theory with axioms A1, A21 and A3. Then BTC is finitely axiomatizable as follows from the famous Bernay's metatheorem (cf. [B],[G] or [V-H]).

Metatheorem. The formula A21 is provable in the theory consisting of A1, the pairing axiom and axioms of the group B of Gödel-Bernays set theory.

Later we shall see that AST is not finitely axiomatizable and hence AST is strictly stronger than AST_{-2} + A21 + A22. On the other hand the last mentioned theory is strong enough for building of mathematics in the same way as it is done in ch. 1 - ch. 4 [V] (but of course we have to be more careful in the formulations of metatheorems).

The theories TC and BTC seem to be convenient bases for the investigation of theories with sets and classes. Therefore we are going to formulate our statements in these theories (and their extensions) not taking account of the possibility to prove results in weaker theories.

A 41 Schema of induction. For every set-formula $\Phi(z)$ we accept the axiom

$$(\Phi(0) \,\&\, (\forall x,y)(\Phi(x) \rightarrow \Phi(x \cup \{y\}))) \rightarrow (\forall x)\Phi(x)$$

The schema of induction was formulated in § 1 ch. I [V] and it played the basic role in the investigation of the behaviour of sets in the alternative set theory, e.g. the fact that all our sets are finite from the Cantor's point of view, is a consequence of it. Under the axiom A41 our definition of finite classes coincides with Vopěnka's one. The axiom A41 follows from A4 according to Bernay's metatheorem.

The prolongation axiom seems to be the most specific and powerful principle of the alternative set theory. It expresses simultaneously the idea of the existence of collections consisting of elements of a given property which cannot be represented by a list of their members (proper semisets) and the idea of approximation of infinity by finite sets.

For mathematics in AST it seems to be fruitful to investigate the following weakening of the prolongation axiom A51. When the alternative set theory was built there were attempts to replace the prolongation axiom by the axiom A51, but it is too weak. On the other hand we are going to show that A51 is strong enough for some metamathematical purposes - we shall show that in theories with this axiom it is possible to interpret the whole AST, but without this axiom - e.g. with the axiom A52 only - it is no longer possible. The axiom A52 was already postulated in [V] as an axiom of the alternative set theory.

A 51 **Weaker form of the prolongation axiom.** **There is** a countable class X so that every subclass of X can be obtained as an intersection of X with a set, i.e.

$(\exists X)(\mathrm{Count}(X) \,\&\, (\forall Y \subseteq X)(\exists y)(Y = X \cap y))$

A 52 Axiom of existence or proper semisets.

$(\exists X)(\mathrm{Sms}(X) \,\&\, \neg\, \mathrm{Set}(X))$.

To prove the implication A5 $\longrightarrow$ A51 in TC + A41 we use the following definitions.

It is well-known that the class N of all natural numbers (with usual properties) can be constructed even in TC + A41. The class FN of all finite natural numbers is defined by

$\mathrm{FN} = \{x \in N; \mathrm{Fin}(x)\}$.

We use α, β, γ (possibly indexed) as variables running over natural numbers; the letters n,m, possibly indexed, will be used as variables for finite natural numbers.

Let us proceed in TC + A41. The class FN is infinite and therefore it is countable. (Let us note that in AST the class N itself is uncountable.) If $Y \subseteq \mathrm{FN}$ then assuming A5 there is a set f with $f(n) = 1 \equiv n \in Y$ and hence $Y = FN \cap f^{-1}``\{1\}$. We have proved the implication A5 $\longrightarrow$ A51. To prove A51 $\longrightarrow$ A52 it is sufficient to realize that the formula $(\exists X)(Sms(X) \,\&\, Count(X)) \longrightarrow (\exists X)(Sms(X) \,\&\, \neg\, Set(X))$ is provable in the theory in question (cf. § 4 ch. I [V]).

The formulation of our axiom of choice is formally similar to a form of AC in the usual set theory, but the meaning is a little different. Let us emphasize that the class of all natural numbers is not well-ordered by $\in$. The axiom of choice plays in AST an important role even in cases when AC is not used in Cantor's set theory e.g. if we want to consider real numbers as sets, we have to use A5 (see § 2 ch. 2 [V]).

Our axiom of choice is rather formal and technical but there are much deeper philosophical reasons for the acceptance of the following weaker (?) form of the axiom of choice. We shall see that the axiom A61 is important from the metamathematical point of view, too. The implication A6 $\longrightarrow$ A61 is trivially provable in TC.

A 61 Axiom of countable choices of sets. From every relation with countable domain we can choose a function with the same domain, i.e.

$(\forall X)(Count(dom(X)) \longrightarrow (\exists F \subseteq X)(dom(F) = dom(X)))$.

The reformulation of the axiom A61 for classes with finite domain i.e. the statement $(\forall X)(Fin(dom(X)) \longrightarrow (\exists F \subseteq X)(dom(F) = dom(X)))$ is evidently provable in TC + A41. Therefore the axiom of countable choices of sets expresses our endeavour to project properties which are verifiable for finite classes even on countable classes. Hence the axiom A61 is a formalization of a principle of transcending the horizon (similarly as the prolongation axiom). We can paraphrase Vopěnka's motivation of the prolongation axiom and motivate the axiom of countable choices of sets as follows: Imagine that we find ourselves on a long straight road lined with large stones set at regular distances. The stones reach as far as we can see. Then it is natural to suppose that the stones reach the horizon.

The axiom of cardinalities guarantees that there are only two infinite cardinalities. Hence the acceptance of this axiom implies a considerable reverse for a mathematician accustomed to Cantor's set theory. This fact contributes to doubts about fitness of the axiom of cardinalities. However, omitting this axiom we weaken the alternative set theory considerably. Therefore it is natural to look for an axiom using which we can save a deal of statements in proofs of which the axiom of cardinalities is used. As such an axiom, the following one can serve.

A 71 <u>Weaker form of the axiom of cardinalities</u>. Every two infinite sets are equivalent, i.e.

$(\forall x,y)((\neg \mathrm{Fin}(x) \,\&\, \neg \mathrm{Fin}(y)) \longrightarrow x \approx y)$.

It is well-known that the schema of regularity (i.e. infinitely many axioms) is equivalent in ZF_{Fin} to the conjunction of the following two of its particular cases.

A 81 <u>Axiom of regularity for sets</u>. Every non-empty set has an element disjoint with it, i.e.

$(\forall x)(x \neq 0 \longrightarrow (\exists y \in x)(x \cap y = 0))$.

A 82 <u>Axiom of transitive closure</u>. To every set there is its transitive superset, i.e.

$(\forall x)(\exists y)(x \in y \,\&\, \mathrm{Tran}(y))$

In fact let $\Phi(z)$ be a set-formula and proceeding in ZF_{Fin} let us suppose that the formula $\Phi(v)$ holds. Thus there is y with $v \in y \,\&\, Tran(y)$ and we put $x = \{u \in y; \Phi(u)\}$. In this case we have $0 \neq x$ and hence there is $q \in x$ with $q \cap x = 0$. Since $q \in y$ we get $(\forall u \in q) \neg \Phi(u)$.

At the end of this section let us try to answer the question whether our formal theory can be held as a formalization of the more intuitively taken alternative set theory so as it is described in [V]. As far as the axioms are concerned, the situation is clear since all axioms of [V] are provable in our formal theory and conversely. This follows from the above mentioned analysis of weakening of axioms and further from the fact that the axiom of sets as particular classes (cf. § 2 ch. I [V]) is a consequence of our definition of sets. The axiom of choice is implied by the axiom of extensional coding (see §§ 5,6 ch. I [V]); on the other hand the axiom of extensional coding is a trivial consequence of the axiom of choice.

It is essentially more complicated - but also more important - to give an answer if we formalized by a convenient way the basic notions of the alternative set theory - the notions of "set" and the notion of "class". It is obvious that our notion of "set" corresponds to Vopěnka's notion "element of the universe of sets" and that our notion of "class" agrees with Vopěnka's notion "object from the extended universe". Even the choice of notations of variables was made in harmony with this interpretation. The notion of "property" in Vopěnka's axiom of existence of classes was formalized - and it seems conveniently enough - by the notion "property describable by a formula of the language of set theory".

On the other hand, in our formalization we do not keep the idea that there are sets containing proper classes as their elements (cf. § 1 ch. I [V] where e.g. the set $\{X,Y\}$ exists even in the case that X and Y are proper classes). Neither do we assume that every property of classes describes a class (cf. § 2 ch. I [V]). Nevertheless in the whole of [V] we can restrict ourselves to "codable classes" and this object can be grasped also in our theory since the system of all η-elements of a class S is coded by the coding pair $\langle S''\{0\}, S''\{1\}\rangle$ (cf. § 5 ch. I [V]). The notion of codable classes exceed only symbols $\bigcup\{X;\ \Phi(X)\}$ and $\bigcap\{X;\ \Phi(X)\}$ from § 2 ch. I [V] but these notions can be taken as only abbreviations.

In our conception there are "more" formal formulas than the metamathematical ones and therefore the axiom A4 expressing the induction for (finite) formal set-formulas is stronger than the scheme of induction for metamathematical set-formulas (A41) - cf. the different approach in § 5 ch. II [V].

§ 3. Finite formulas

The first aim of this section is to construct a convenient interpretation of KM_{Fin} in TC(and hence even in AST). To do it we need some notions.

Let us note that TC is a very weak theory and hence our construction will be rather complicated but the existence of such an interpretation enables us to formulate equivalent statements to some axioms of AST even in TC. Moreover it is easier to interpret TC than AST in a theory and this brings other advantages.

A class X is called a η-pair of classes Y and Z (in symbols $X = \{Y,Z\}^{\eta}$) iff $X = 2\times\{0\}\cup(Y\times\{0\}\cup Z\times\{1\})\times\{1\}$. We have evidently $(\forall Q)(Q\,\eta\,X \equiv (Q = Y \vee Q = Z))$; moreover by the definition of X we are able to order the pair. Analogical definitions can be done for every finite number of classes.

A η-triplet $\mathcal{A} = \{A,\tilde{E},\tilde{I}\}^{\eta}$ is called a model (of the language of set theory) iff the formulas $A \neq 0$ and $(\tilde{E}\cup\tilde{I}) \subseteq A^2$ hold.

If $\mathcal{A} = \{A,E,I\}^{\eta}$ is a constant denoting a model then the symbol $\mathcal{a}$ denotes the interpretation determined by the formulas (Tarski's possible definition; cf. [T-M-R])

$$\mathrm{Cls}^{a}(X) \equiv (X\subseteq A \;\&\; (\forall x,y)((x\in X \;\&\; \langle x,y\rangle \in \tilde{I}) \rightarrow y\in X))$$

$$X^{a}\in^{a} Y^{a} \equiv (\exists x\in Y^{a})(X^{a} = \tilde{E}''\{x\})$$

$$X^{a} =^{a} Y^{a} \equiv X^{a} = Y^{a}$$

and the symbol ϕ^{a} denotes the formula assigned by the interpretation a to a formula ϕ (of the language of set theory).

Let us note that if $\tilde{I}$ is the equality restricted to A then the formula defining $\mathrm{Cls}^{a}(X)$ simplifies itself to the formula $X\subseteq A$. If moreover $\tilde{E}$ equals to $E\upharpoonright A$ then the symbol ϕ^{a} and the symbol ϕ^{A} (cf. § 1 ch. V [V]) coincide and the definition of $X^{a}\in^{a} Y^{a}$ simplifies itself to the formula $X^{a}\in Y^{a}$

Now we are going to define the notion of hereditarily finite set; our definition agrees with the usual definition of this notion in Cantor's set theories. The class of all hereditarily finite sets is called finite universe and denoted by FV. Defining $\mathcal{FV} = \{FV, E\upharpoonright FV, Id\upharpoonright FV\}^{\eta}$ we shall see that $\mathcal{FV}$ is an interpretation of KM_{Fin} in TC.

Definition (TC). A set is called hereditarily finite iff it is an element of the domain of a finite well-ordering keeping $\in$. The class of all hereditarily finite sets is denoted FV i.e.

$FV = \{x;(\exists \trianglelefteq)(We(\trianglelefteq) \& Fin(\trianglelefteq) \& E \restriction dom(\trianglelefteq) \subseteq \trianglelefteq \& x \in dom(\trianglelefteq)\}$.

Lemma (TC). The finite universe is transitive and it is the minimal class containing 0 and saturated w.r.t. those set-successors which are its subclasses i.e.

$Tran(FV) \& 0 \in FV \& (\forall x,y \in FV)(x \cup \{y\} \in FV) \& (\forall X)((0 \in X \&$
$\& (\forall x,y \in X)(x \cup \{y\} \subseteq X \rightarrow x \cup \{y\} \in X)) \rightarrow FV \subseteq X)$.

Proof. The statements Tran(FV) and $0 \in FV$ follow immediately from the definition of the class FV. The proof of the remaining statements is also very easy but rather long:

If X is finite then for every set u, the class $X \cup \{u\}$ is finite, too. In fact if $Y \subseteq X \cup \{u\}$ then $Y - \{u\} \subseteq X$ and hence $Y - \{u\}$ is a set and therefore Y is also a set according to the axiom A3. If $0 \neq Z \subseteq P(X \cup \{u\})$ then there is a minimal (w.r.t. inclusion) element q of the class $\{v - \{u\}; v \in Z\}$ and then either q or $q \cup \{u\}$ is a minimal (w.r.t. inclusion) element of the class Z.

If $\trianglelefteq$ is a finite well-ordering and if $z \in dom(\trianglelefteq)$ then either z is the $\trianglelefteq$-first element or there is a set q with $\{v; v \trianglelefteq z\} = \{v; v \trianglelefteq q\} \cup \{z\}$. To prove it it is sufficient to consider the class $\{\{\langle v,z \rangle; w \trianglelefteq v < z\}; w < z\}$ and its minimal element w.r.t. inclusion.

If $\trianglelefteq$ is a finite well-ordering keeping $\in$ and if $u \subseteq$ $\subseteq dom(\trianglelefteq)$ then the well-ordering $\trianglelefteq \cup \{\langle v,u \rangle; v = u \vee (u \notin$ $\notin dom(\trianglelefteq) \& v \in dom(\trianglelefteq))$ keeps $\in$ and moreover by induction using the previous parts of the proof we get that the well-ordering in question is finite.

Let $x \in dom(\leq_1)$ and $y \in dom(\leq_2)$ and let $\leq_1$ and $\leq_2$ be finite well-orderings keeping $\in$. Using the above proved statements (and the fact that $(\forall z \in dom(\leq_2))(z \subseteq \{q; q <_2 z\})$) we can prove by induction w.r.t. $\leq_2$ that there is a finite well-ordering keeping $\in$ so that $dom(\leq_1) \cup dom(\leq_2)$ is a subclass of its domain. Hence even $x \cup \{y\}$ is a subset of its domain and therefore the above mentioned construction gives us a finite well-ordering keeping $\in$ such that $x \cup \{y\}$ is an element of its domain. This proves the third statement of our theorem.

Let us assume that the formula $0 \in X \,\&\, (\forall z, y \in X)(x \cup \{y\} \subseteq X \rightarrow x \cup \{y\} \in X)$ holds and let $x \in FV - X$. In this case there would be a finite well-ordering $\leq$ keeping $\in$ with $x \in dom(\leq)$. The class $\{y \in dom(\leq); (\exists z)(z \notin X \,\&\, z \subseteq \{q; q < y\})\}$ would have the $\leq$-first element and this leads to a contradiction.

Metatheorem. The interpretation $\mathcal{FV}$ is an interpretation of KM_{Fin} in TC.

Demonstration. The formula $A1^{FV}$ follows immediately from the last lemma and from the axiom A1; moreover the last lemma implies even $A3^{FV}$. The statement $A2^{FV}$ is a trivial consequence of the axiom A2 since we admit all subclasses of FV as $\mathcal{FV}$ -classes.

If $\Phi(z)$ is a set-formula such that the statement $((\forall x,y)(\Phi(x) \rightarrow \Phi(x \cup \{y\})))^{FV}$ holds and if $F \subseteq FV^2$ then the classes $\{x \in FV; \Phi^{FV}(x)\}$ and $\{x \in FV; F''x \in FV\}$ equal to the class FV by the last lemma. Hence we have proved the statement $A41^{FV}$ and the $\mathcal{FV}$ -axiom of replacement. In § 1 ch. I [V] all axioms of ZF_{Fin} were proved from the axioms A11, A3 and A41, A8 and therefore it remains to say only a few words to the statement $A8^{FV}$.

Let $\Phi(z)$ be a set-formula and let us have $\Phi^{FV}(x) \& x \in \in \mathrm{dom}(\trianglelefteq)$ where $\trianglelefteq$ is a finite well-ordering keeping $\in$. Let y be the $\trianglelefteq$-first element z with $\Phi^{FV}(z)$. We have obviously $(\forall z \in y) \neg \Phi^{FV}(z)$ which completes the demonstration.

By the last metatheorem the formula $(\forall x \in FV)(\exists y,z \in \in FV)(y = P^{FV}(x) \& z = \cup^{FV} x)$ is provable in TC. Moreover for every $x \in FV$ we have $P^{FV}(x) = P(x)$ and $\cup^{FV} x = \cup x$. Thus if $\Phi(x)$ is a set-formula equivalent to a formula $\Psi(x, \mathcal{F}(x))$ where $\Psi(z_1, z_2)$ is a restricted formula and $\mathcal{F}$ is an operatopm constructed from P and $\cup$ then for every $x \in FV$ the formula $\Phi^{FV}(x) \equiv \Phi(x)$ holds.

Theorem (TC + A41). $N^{\mathcal{FV}} = FN$.

Proof. In TC + A41 we can define natural numbers as sets z satisfying the formula $\mathrm{Tran}(z) \& (\forall x,y \in z)(x \in y \vee x = y \vee y \in x) \& \& (\forall u \subseteq z)(u \neq 0 \longrightarrow (\exists q \in u)(q \cap u = 0))$ and hence the consideration mentioned above assures us that $N^{\mathcal{FV}} = FV \cap N$.

By the last lemma we have $x \in FV \longrightarrow \mathrm{Fin}(x)$ and from this we immediately get the inclusion $N^{\mathcal{FV}} \subseteq FN$. Further $\mathrm{Tran}(FV \cap N)$ and therefore if $\alpha \notin FV$ then $FV \cap N \subseteq \alpha$. Since $\beta \in FV \cap N \longrightarrow \beta + 1 \in FV \cap N$, the class $FV \cap N$ cannot be a set and from this the formula $\alpha \notin FN$ follows.

There are many important notions defined in ZF_{Fin}. Since AST is stronger than ZF_{Fin} we are able to define these notions in AST in the same way. However, there is moreover another method how we can construct notions of ZF_{Fin} in the alternative set theory. The interpretation $\mathcal{FV}$ induces namely an interpretation of ZF_{Fin} in AST and hence we are able to construct notions of ZF_{Fin} according to this interpretation (i.e. if $\Phi(z)$ define a notion in ZF_{Fin} then we define the corresponding notion in AST by the formula $\Phi^{\mathcal{F}\mathcal{V}}(z)$).

The notions defined by the first way are called by the same terms as in ZF_{Fin} (i.e. without any attribute); to notions obtained by the second way we add the attribute "finite" and in the notation we add to them the index "F". In accordance with this convention were defined e.g. the notions of natural and finite natural numbers, the notions of rational and finite rational numbers and the notions of the universal class and the finite universe (finite universal class). We are going to use this convention quite freely. Let us note that there is one exception from this convention since the term "finite set" is used for all sets the cardinality of which is a finite natural number and not only for hereditarily finite sets.

At the first view it is rather surprising that the notions defined by the second way play a more important role in AST than those defined by the first way but this becomes clear if we realize that we interpret the intuitive notion "finite" in the alternative set theory by the predicate "to be finite" and therefore the interpretation of the collection of all hereditarily finite sets from Cantor's point of view onto finite universe is in some sense more natural than its interpretation onto the whole universe.

For some results it will be essential that the construction of notions by the second way does not require all axioms of AST and hence that we can define these notions even in TC.

Now we are going to use these two processes of defining notions for double formalizing of metamathematics in AST. Let us note that we are forced to define some notions (e.g. notions depending on proper classes) directly in TC since they cannot be defined in ZF_{Fin}.

As usual we can define in ZF_{Fin} the notions of (formal) formula, proof, provability (in symbols $\vdash$), consistency (in symbols Con) and so on. We admit infinitely many constants, however, we restrict ourselves in the following to formulas with predicates $\in$ and $=$ only; but this restriction is, of course, unessential.

Thus we have all above mentioned notions even in AST. By the use of the second way we define in TC the notions of finite formula, finite proof, finite probability (in symbols $\vdash_F$), finite consistency (in symbols Con_F) and so on. By the consideration stated above we obtain in AST some connections between notions without attribute and notions with attribute "finite". For example we see that a formula (proof respectively) is a finite formula (finite proof respectively) iff it is a hereditarily finite set; further every free variable occuring in a finite formula is a finite free variable occuring in it and so on.

Let us emphasize that the length of any finite proof is a finite natural number.

We have admitted in finite formulas as parameters only elements of FV, however, it is possible to extend this definition admitting all parameters. The class of all finite formulas with parameters in a class C is denoted by FL_C and its elements are called (finite) formulas of the language FL_C (they can be coded by pairs $\langle \varphi, f \rangle$ where φ is a finite formula without parameters and f is a one-one mapping such that elements of dom(f) are finite variables free in φ and

$\mathrm{rng}(f) \subseteq C$). We put $FL = FL_0$. The class of all formulas with parameters in C is denoted by L_C and we put again $L = L_0$.

Evidently we are able to extend usual notions also to elements of FL_C. For example a sequence of elements of FL_C is called a finite proof in predicate calculus iff there is a one-one mapping f transferring (some) elements of C into finite constants such that our sequence is transferred to a finite proof in predicate calculus.

Subclasses of FL_C are called theories (of the language FL_C). Let us emphasize that in accordance to our preference of notions with attribute "finite", theories contain only finite formulas of the investigated language. For theories which are proper classes we extend the notion of finite provability (finite inconsistency, provability and inconsistency respectively) defining $\mathcal{T} \vdash_F \varphi (\neg \mathrm{Con}_F(\mathcal{T}), \mathcal{T} \vdash \varphi$ and $\neg \mathrm{Con}(\mathcal{T})$ respectively) iff there is a subset $\mathcal{T}_0$ of $\mathcal{T}$ with $\mathcal{T}_0 \vdash_F \varphi$ ($\neg \mathrm{Con}_F(\mathcal{T}_0)$, $\mathcal{T}_0 \neg \varphi$ and $\neg \mathrm{Con}(\mathcal{T}_0)$ respectively).

If $\mathfrak{A} = \{A, \tilde{E}, \tilde{I}\}^{\eta}$ is a model then for every finite sentence FL_A we define by induction (using the axiom A2)

$\mathfrak{A} \models a \in b$ iff $\langle a,b \rangle \in \tilde{E}$

$\mathfrak{A} \models a = b$ iff $\langle a,b \rangle \in \tilde{I}$

$\mathfrak{A} \models (\varphi \,\&\, \psi)$ iff $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \psi$

$\mathfrak{A} \models (\neg \varphi)$ iff $\neg\, \mathfrak{A} \models \varphi$

$\mathfrak{A} \models (\exists X)\, \varphi(X)$ iff $(\exists a \in A)\, \mathfrak{A} \models \varphi(a)$

Let us note that we have defined satisfaction only for finite formulas. For models which are sets can be the definition of satisfaction extended for all formulas in the obvious way. Such a definition in the general case seems to be impossible (cf. [S-V 2]).

Two models $\mathcal{A}$ and $\mathcal{B}$ are called elementarily equivalent iff they satisfy the same finite sentence (without constants) i.e. iff

$(\forall \varphi \in \mathrm{FL})((\mathcal{A} \models \varphi) \equiv (\mathcal{B} \models \varphi))$.

If $\mathcal{T}$ is a theory then we write $\mathcal{A} \models \mathcal{T}$ instead of $(\forall \varphi \in \mathcal{T})\, \mathcal{A} \models \varphi$

Quite analogically as in the classical case we can prove by induction $(\mathcal{T} \vdash_{F} \varphi \;\&\; \mathcal{A} \models \mathcal{T}) \longrightarrow \mathcal{A} \models \varphi$ and therefore we have $(\exists \mathcal{A})(\mathcal{A} \models \mathcal{T}) \longrightarrow \mathrm{Con}_{F}(\mathcal{T})$. Moreover, the Gödel's proof can be repeated in AST and thence we have also the converse implication, i.e. the statement $\mathrm{Con}_{F}(\mathcal{T}) \longrightarrow (\exists \mathcal{A})\, \mathcal{A} \models \mathcal{T}$.

In the special case that $\widetilde{I}$ is the identity on A we are going to drop it in the notation; if moreover $\widetilde{E}$ equals to $E \cap A^2$ we write $A \models \varphi$ instead of $\langle A, \widetilde{E} \rangle \models \varphi$.

For every metamathematical hereditarily finite object (e.g. natural number, formula or proof) there is its usual formalization in ZF_{Fin} and therefore using our second method of defining of notions of ZF_{Fin} we obtain its formalization in TC. (By metamathematical induction can be proved that both our ways give in AST the same for such objects.)

If φ is a formalization of a set-formula Φ in TC then one can prove by easy metamathematical induction that $\Phi \equiv V \models \varphi$.

If T is a primitively recursive (metamathematical) theory then $\overline{T}$ and $\mathcal{T}$ denote its formalizations in AST and TC given by the first and second way respectively (we suppose that a convenient description of T is chosen). Evidently in AST we can prove $\mathcal{T} = \overline{T} \cap FV$ and moreover we get $\mathcal{T} = \overline{T}$ for every (metamathematically) finite T.

Applying the classical results of logic we obtain two kinds of statements in dependence of the choice of way but sometimes one of such statements is a consequence of the second one. For example if we investigate the applications of Gödel's theorem on consistency proofs we see that relevant is only the statement

If T is a consistent theory stronger than TC then T + $+ \neg \mathrm{Con}_F(\mathcal{T})$ is consistent, too.

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(Oblatum 5.5. 1979)