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Abstract: The main purpose of this paper is to characterize the nodal filters in lattices (in up-directed meet-semilattices) in terms of congruences. Thus J.C. Varlet's result, stated for implicative semilattices, is generalized for lattices and up-directed meet-semilattices. Further, we give the description of some well-known quotient lattices and quotient semilattices in more precise form. Finally, we compare lattice congruences, semilattice congruences and equivalence relations generated by filters of a lattice.

Key words: Congruence relation, distributive filter, lattice, meet-semilattice, nodal filter.

Classification: 06B10, 06A12

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1. Introduction and preliminaries. In [5] J.C. Varlet has introduced the notion of nodal filter of a meet-semilattice. A filter of a meet-semilattice S is said to be nodal if it is comparable with every filter of S ordered by inclusion.

In [5] the nodal filters of an implicative semilattice are studied and an interesting characterization of nodal filters in terms of congruences is obtained (see Corollary 3 of this paper). We show that J.C. Varlet's characterization of nodal filters of an implicative semilattice can be generalized to nodal filters of an arbitrary lattice and to nodal filters of an up-directed meet-semilattice.

A filter F of a meet-semilattice $\langle A, \wedge \rangle$ is a nonvoid subset of A such that $x \wedge y \in F$ if and only if $x \in F$ and $y \in F$. A filter of a lattice $\langle A, \wedge, \vee \rangle$ is defined as a filter of the meet-semilattice $\langle A, \wedge \rangle$.

The principal filter generated by an element $a \in A$ will be denoted by $[a)$, i.e. $[a) = \{x \in A; x \geq a\}$. Further, we denote by $\mathcal{F}(A)$ ($\mathcal{F}_0(A)$) the set of all filters (principal filters) of A.

Let $\langle P, \leq \rangle$ be an arbitrary poset, and let $\emptyset \neq Q \subseteq P$. An element $a \in P$ is called a node of Q if a is comparable with every element of Q.

A poset $\langle P, \leq \rangle$ is said to be up-directed if every two-element subset of P has at least one upper bound in P. The poset dual to $\langle P, \leq \rangle$ will be denoted by $\langle P, \leq \rangle^d$. $\langle P, \leq \rangle \oplus 1$ denotes the poset obtained by adding a new element 1 such that $1 > x$ for all $x \in P$.

We use a standard lattice theory terminology and refer the reader to [2] for definitions of some further notions which we will use here without defining them.

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2. Preliminary lemmas

Lemma 1. Let a be an element of a poset $\langle P, \leq \rangle$ and let M be a nonvoid subset of P such that every element of P can be expressed as a join of some elements of M. Then a is a node of P if and only if a is a node of M.

Proof. The "only if" part being trivial, assume now that x is an arbitrary element of P. Then we have $x = \bigvee_{i \in I} m_i$ for some elements $m_i \in M$, $i \in I$.

If $a \leq m_i$ holds for some $i \in I$, then we obtain $a \leq x$ immediately. In case $a \nleq m_i$ for every $i \in I$ we get $a > m_i$ for every $i \in I$ since a is a node of M. This implies $a \geq \bigvee_{i \in I} m_i = x$, which completes the proof.

Corollary 1. Let F be an arbitrary filter of a meet-semilattice S. The F is a nodal filter of S if and only if F is a node of $\mathcal{F}_0(S)$.

Proof. It is well-known that every filter F of a meet-semilattice S is the join of all principal filters $[f)$, $f \in F$. Now the corollary follows directly from Lemma 1.

Let us recall that an element a of a lattice L is called distributive if and only if $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$ for all $x, y \in L$.

Lemma 2. Every node of a lattice is distributive.

Proof. Let $a \in L$ be a node of a lattice L, and let x, y be arbitrary elements of L.

Case 1. $a \geq x$, $a \geq y$. This implies $a \geq x \wedge y$ and thus $a \vee (x \wedge y) = a$. On the other hand, we have $(a \vee x) \wedge (a \vee y) =$ $= a \wedge a = a$.

Case 2. $a \leq x$, $a \leq y$. Then we get $a \leq x \wedge y$ and thus $a \vee (x \wedge y) = x \wedge y$. Further, we have $(a \vee x) \wedge (a \vee y) = x \wedge y$.

Case 3. $x \leq a \leq y$. Then $a \vee (x \wedge y) = a \vee x = a$ and $(a \vee x) \wedge (a \vee y) = a \wedge y = a$.

Case 4. $x \geq a \geq y$. See Case 3.

A filter F of a lattice L is said to be distributive if F is distributive, as an element of $\mathscr{F}(L)$.

Corollary 2. Every nodal filter of a lattice is distributive.

3. Congruences generated by filters. We denote by $\Theta_{\mathfrak{A}}[F]$ the congruence relation of an algebra $\mathfrak{A}$ generated by a subset F of $\mathfrak{A}$, i.e. $\Theta_{\mathfrak{A}}[F]=\bigcap\{\theta \in \mathfrak{C}(\mathfrak{A});F\times F\subseteq\theta\}$. Further, we write $\Theta_A[F]$ instead of $\Theta_{\langle A,\emptyset\rangle}[F]$, the equivalence relation of A generated by F.

The following theorem gives a characterization of $\Theta_{\langle S,\wedge\rangle}[F]$ whenever F is a filter of an up-directed meet-semilattice $\langle S,\wedge\rangle$.

Theorem 1. Let F be an arbitrary filter of an up-directed meet-semilattice $\langle S,\wedge\rangle$. Then
$S/\Theta_{\langle S,\wedge\rangle}[F]\cong\langle\{F\vee[s);s\in S\},\subseteq\rangle^d$; this isomorphism is given by $[s]\Theta_{\langle S,\wedge\rangle}[F]\longmapsto F\vee[s)$ for $s\in S$.

Proof. First, the mapping $h:s\longmapsto F\vee[s), s\in S$ is a meet-homomorphism of S onto $\langle\{F\vee[s);s\in S\},\subseteq\rangle^d$ since
$h(a\wedge b) = F\vee[a\wedge b) = F\vee([a)\vee[b)) = (F\vee[a))\vee(F\vee[b)) =$
$= h(a)\vee h(b)$ for every $a,b\in S$.

Further, by the Homomorphism Theorem (see [3; p. 57]), it holds $S/\mathrm{Ker}\,h\cong\langle\{F\vee[s);s\in S\},\subseteq\rangle^d$ and this isomorphism is given by $[s]\mathrm{Ker}\,h\longmapsto F\vee[s)$ for $s\in S$.

Finally, we claim that $\mathrm{Ker}\,h=\Theta_{\langle S,\wedge\rangle}[F]$. Clearly $\Theta_{\langle S,\wedge\rangle}[F]\subseteq\mathrm{Ker}\,h$ since $F\times F\subseteq\mathrm{Ker}\,h$. Conversely, let a, b be such elements of S that $(a,b)\in\mathrm{Ker}\,h$, i.e. $F\vee[a)=F\vee[b)$. This implies that $f\wedge a = f\wedge b$ for some element $f\in F$. Denote by u an upper bound of elements a, b, f; obviously $u \in F$. Then we get $(f,u) \in F \times F \subseteq \Theta_{\langle S,\wedge\rangle}[F]$ and thus $a=u\wedge a \equiv f\wedge a = f\wedge b \equiv u\wedge b=b \quad (\Theta_{\langle S,\wedge\rangle}[F])$. Hence we have also $\Theta_{\langle S,\wedge\rangle}[F] \supseteq \operatorname{Ker} h$. Summary, $\Theta_{\langle S,\wedge\rangle}[F] = \operatorname{Ker} h$ holds and the proof is complete.

The following theorem is a slight modification of the well-known result concerning distributive ideals of a lattice (see [1; Lemma 2.5] and [2; Ch. III, § 3, Theorem 4]).

<u>Theorem 2</u>. Let F be a filter of a lattice $\langle L,\wedge,\vee\rangle$. Then the following three conditions are equivalent:

(1) F is a distributive filter;

(2) $L/\Theta_{\langle L,\wedge,\vee\rangle}[F] \cong \langle \{F\vee[a);a\in L\}, \subseteq\rangle^d$, this isomorphism is effected by

$[a]\Theta_{\langle L,\wedge,\vee\rangle}[F] \longmapsto F\vee[a)$ for $a\in L$;

(3) $\Theta_{\langle L,\wedge,\vee\rangle}[F] = \Theta_{\langle L,\wedge\rangle}[F]$.

Proof (1) implies (2): The proof of this part goes along the same line as the proof of Theorem 1 and is therefore omitted.

(2) implies (3): Combining (2) and Theorem 1, we obtain (3).

(3) implies (1): By Theorem 1 and hypothesis, we get

$$\Theta_{\langle L,\wedge,\vee\rangle}[F] = \{(a,b)\in L\times L; F\vee[a)=F\vee[b)\} =$$
$$= \{(a,b)\in L\times L; f\wedge a=f\wedge b \text{ for some element } f\in F\}.$$

By the dual of [2; Ch. III, § 3, Theorem 4] , F is a distributive filter of the lattice L. The proof of Theorem 2 is completed.

Now we are going to give the above-mentioned characterization of nodal filters in terms of congruences. First, we present a result characterizing the nodal filters of up-directed meet-semilattices.

Theorem 3. Let F be an arbitrary filter of an up-directed meet-semilattice $\langle S, \wedge \rangle$. Then the following three conditions are equivalent:

(1) F is a nodal filter;

(2) $S/\Theta_{\langle S,\wedge\rangle}[F] \cong \langle S \setminus F, \leq \rangle \oplus 1$, this isomorphism is effected by

$[s]\, \Theta_{\langle S,\wedge\rangle}[F] \longmapsto s$ if $s \in S \setminus F$, and

$[s]\, \Theta_{\langle S,\wedge\rangle}[F] \longmapsto 1$ otherwise;

(3) $\Theta_{\langle S,\wedge\rangle}[F] = \Theta_S[F]$.

Proof. (1) implies (2): By Theorem 1, we have $S/\Theta_{\langle S,\wedge\rangle}[F] \cong \langle \{F \vee [s); s \in S\}, \subseteq \rangle^d$. Further, $\langle \{F \vee [s); s \in S\}, \subseteq \rangle^d = \langle \{F\} \cup \{[s); s \in S \setminus F\}, \subseteq \rangle^d \cong \langle S \setminus F, \leq \rangle \oplus 1$ since F is comparable with every filter $[s), s \in S$. Analogously, by hypothesis and Theorem 1, we obtain the explicit description of $\Theta_{\langle S,\wedge\rangle}[F]$.

(2) implies (3): Immediate.

(3) implies (1): Assume that $\Theta_{\langle S,\wedge\rangle}[F] = \Theta_S[F]$ and choose $a \in F$, $x \in S \setminus F$. Clearly, $[a \wedge x]\, \Theta_{\langle S,\wedge\rangle}[F] = [a]\, \Theta_{\langle S,\wedge\rangle}[F] \wedge [x]\, \Theta_{\langle S,\wedge\rangle}[F]$ holds (in the quotient semilattice $S/\Theta_{\langle S,\wedge\rangle}[F]$).

Further, denote by u an upper bound of elements a, x. Then $x \leq u$ implies the inequality $[x]\, \Theta_{\langle S,\wedge\rangle}[F] \leq [u]\, \Theta_{\langle S,\wedge\rangle}[F]$ (in the quotient semilattice $S/\Theta_{\langle S,\wedge\rangle}[F]$) and $a \leq u$ implies $u \in F = [a]\, \Theta_S[F] = [a]\, \Theta_{\langle S,\wedge\rangle}[F]$, i.e. $[a]\, \Theta_{\langle S,\wedge\rangle}[F] = [u]\, \Theta_{\langle S,\wedge\rangle}[F]$.

Summary, we get $[x]\, \Theta_{\langle S,\wedge\rangle}[F] \leq [a]\, \Theta_{\langle S,\wedge\rangle}[F]$ and thus

$[a]\,\Theta_{\langle S,\wedge\rangle}[F] \wedge [x]\,\Theta_{\langle S,\wedge\rangle}[F] = [x]\,\Theta_{\langle S,\wedge\rangle}[F]$. However, $[x]\,\Theta_{\langle S,\wedge\rangle}[F] = \{x\}$ since $x \in S \setminus F$. This means that $a \wedge x = x$ which is equivalent to $a \geq x$ for every $a \in F$. Hence we have $F \subseteq [x)$ for all $x \in S \setminus F$.

By Corollary 1, we conclude that F is a nodal filter of S.

An immediate consequence of Theorem 3 is

Corollary 3 (J.C. Varlet [51]). Let F be an arbitrary filter of an implicative semilattice $\langle S,\wedge,\Rightarrow,1\rangle$. Then the following two conditions are equivalent:

(1) F is a nodal filter;

(2) $\Theta_{\langle S,\wedge,\Rightarrow,1\rangle}[F] = \Theta_S[F]$.

Proof. It is well-known that $\Theta_{\langle S,\wedge,\Rightarrow,1\rangle}[F] = \Theta_{\langle S,\wedge\rangle}[F]$ for every filter F of an implicative semilattice $\langle S,\wedge,\Rightarrow,1\rangle$ (see, e.g., [4]). Applying Theorem 3, we obtain that (1) is equivalent to (2).

Now we direct our attention to the nodal filters of lattices.

Theorem 4. Let F be an arbitrary filter of a lattice $\langle L,\wedge,\vee\rangle$. Then the following four conditions are equivalent:

(1) F is a nodal filter;

(2) $L/\Theta_{\langle L,\wedge,\vee\rangle}[F] \cong \langle L\setminus F;\ \leq\rangle \oplus 1$, this isomorphism is effected by

$[a]\,\Theta_{\langle L,\wedge,\vee\rangle}[F] \longmapsto a$ if $a \in L\setminus F$, and

$[a]\,\Theta_{\langle L,\wedge,\vee\rangle}[F] \longmapsto 1$ otherwise;

(3) $\Theta_{\langle L,\wedge,\vee\rangle}[F] = \Theta_{\langle L,\wedge\rangle}[F] = \Theta_L[F]$;

(4) $\Theta_{\langle L,\wedge\rangle}[F] = \Theta_L[F]$.

Proof. (1) implies (2): Let F be a nodal filter. By Corollary 2, F is distributive and thus $L/\Theta_{\langle L,\wedge,\vee\rangle}[F] \cong$ $\cong \langle \{F \vee [a); a \in L\}, \subseteq \rangle^{d}$ holds. The rest of the proof is very similar to the proof of Theorem 3, so it can be omitted.

(2) implies (3): Clearly, $\Theta_{\langle L,\wedge,\vee\rangle}[F] \supseteq \Theta_{\langle L,\wedge\rangle}[F] \supseteq$ $\supseteq \Theta_{L}[F]$ holds. Applying hypothesis, we find that $\Theta_{\langle L,\wedge,\vee\rangle}[F] = \Theta_{\langle L,\wedge\rangle}[F] = \Theta_{L}[F]$.

(3) implies (4): Obvious.

(4) implies (1): Applying Theorem 3 to the up-directed meet-semilattice $\langle L, \wedge \rangle$, we get that F is a nodal filter and the proof of Theorem 4 is complete.

The following simple example shows that for an arbitrary meet-semilattice Theorem 1 and Theorem 3 are false.

Example. The diagram of the meet-semilattice $\langle S, \wedge \rangle$ is shown in Fig. 1. Let us consider that $F = \{a\}$. Clearly, we have $\Theta_{\langle S,\wedge\rangle}[F] = \Theta_{S}[F] = \omega_{S}$. However,

(i) $S/\Theta_{\langle S,\wedge\rangle}[F] \cong \langle \{F \vee [s); s \in S\}, \subseteq \rangle^{d}$ does not hold;

(ii) $F = \{a\}$ is not a nodal filter.

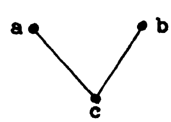

Fig. 1.

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