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CONSTRUCTION OF CARTESIAN CLOSED TOPOLOGICAL HULLS

Jiří ADÁMEK, George E. STRECKER

Abstract: H. Herrlich and L.D. Nel proved that any category which has a cartesian closed topological extension, preserving finite products, has a smallest such extension, called the CCT hull. The present paper is devoted to a direct construction of this hull.

Key words: Topological category, cartesian closed category, initially complete category, CCT hull, power-closed sink.

Classification: 18D15, 18D99, 54A99

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§ 0. Introduction. There are two essential properties of a concrete category for it to be "topologically adequate":

(a) Initial completeness, i.e., the existence of an initial structure for each structured source. An initially complete category has e.g. all limits and colimits constructed on the level of sets, and a lot of other convenient properties.

(b) Cartesian closedness, i.e., the existence of well-behaved function spaces.

The category of topological spaces fails in (b); the category of compactly generated Hausdorff spaces fails in (a) and the category of compact Hausdorff spaces fails in both. We are interested in extensions of a concrete category into a category with one, or both, of the properties (a) and (b).

Systematic methods for extending concrete categories to initially complete ones have been studied by many authors; see, e.g., [$AHS_{1,2}$; An_1; $He_{2,3}$; HS_2; Hu; R]. For example, in [AHS_1] it is shown that, whenever a category $\mathcal{K}$ has any initial completion, then it has a smallest one, the so-called MacNeille completion, which can be described as the category of "closed" sinks in $\mathcal{K}$.

Cartesian closed topological categories have also been studied extensively; see, e.g., [AK; An_2; B; HN; M; N; W]. H. Herrlich and L.D. Nel proved in [HN] that whenever a category $\mathcal{K}$ has any cartesian closed topological (CCT) extension, preserving finite products, then it has a smallest one, the so-called CCT hull of $\mathcal{K}$. The existence of a CCT hull is characterized in [AK] by the condition "strictly small-fibred", explained below.

In the present paper we introduce the notion of power-closed sinks, and, analogously to the case of Mac Neille completion, we prove that for any strictly small-fibred category its CCT hull is the category of power-closed sinks.

§ 1. The definition of a CCT hull

1.1. General assumptions. Throughout the paper, we deal with concrete categories (over sets); i.e., pairs $(\mathcal{K}, \|\ )$ consisting of a category $\mathcal{K}$ and a faithful, amnestic functor $\|\ : \mathcal{K} \longrightarrow \mathrm{Set}$. (Amnesticity means that any isomorphism f in $\mathcal{K}$, such that f is the identity map, is itself an identity morphism.) We use the same symbol for a morphism $f: A \longrightarrow B$ in $\mathcal{K}$ and its underlying map $f: |A| \longrightarrow |B|$. Finally, we assume that $\mathcal{K}$ has at most one void object, i.e., an object A with $|A| = \emptyset$.

1.2. A <u>structured map</u> from a set X is a pair (A,a), where A is an object of $\mathcal{K}$ and $a:X \to |A|$ is a map; we denote it by

$$X \xrightarrow{a} |A|.$$

A family (possibly large) of structured maps from X is called a <u>structured source</u> on X. Let $\mathcal{A}$ be a structured source on X; then an object C with $|C| = X$ is the <u>initial lift</u> of the source $\mathcal{A}$ if

(i) $a:C \to A$ is a morphism for each $X \xrightarrow{a} |A|$ in $\mathcal{A}$;

(ii) given an object C' and a map $f:|C'| \to X$ such that $a \cdot f:C' \to A$ is a morphism for each $a \in \mathcal{A}$ then also $f:C' \to C$ is a morphism.

A concrete category is <u>initially complete</u> if each structured source has an initial lift.

1.3. Dually, a structured map into a set X is a map $a: :|A| \to X$; we denote it by (A,a) or $|A| \xrightarrow{a} X$. A family of structured maps into X is a <u>structured sink</u> on X. The <u>final lift</u> of a structured sink $\mathcal{A}$ is an object C with $|C| = X$ such that, given an object C' and a map $f:X \to |C|$, then $f:C \to C'$ is a morphism iff each $f \cdot a:A \to C'$, $a \in \mathcal{A}$, is a morphism. Initial completeness is equivalent to each structured sink having a final lift.

1.4. Recall from [He_1] that a concrete category $\mathcal{K}$ is <u>topological</u> if it is

(i) initially complete;

(ii) small-fibred, i.e., for every set X the collectio of all objects A with $|A| = X$ is a (small) set;

(iii) has constant morphisms, i.e., every constant map $f: |A| \to |B|$ is a $\mathcal{K}$-morphism $f: A \to B$.

1.5. A concrete category is said to have <u>concrete finite products</u> if it is finitely productive and its forgetful functor preserves finite products; (equivalently, if the product of any finite family of objects A_i, $i \in I$, is the initial lift of the source of projections

$$(X \xrightarrow{\pi_i} |A_i|)_I, \text{ where } X = \prod_{i \in I} |A_i| \text{ in Set}).$$

Particularly, each topological category has concrete finite products.

1.6. Let A and B be objects of a category $\mathcal{K}$ with finite concrete products. Their (canonical) <u>power-object</u>

$$B^A$$

is an object on the set of all morphisms from A to B:

$$|B^A| = \hom(A,B)$$

with the following universal property. Given an object D and a map $f: |D| \to \hom(A,B)$ then

$$f: D \to B^A \text{ is a } \mathcal{K}\text{-morphism}$$

iff

$$\hat{f}: D \times A \to B \text{ is a } \mathcal{K}\text{-morphism},$$

where $\hat{f}$ is the map defined by: $\hat{f}(d,a) = [f(d)](a)$. This notion has been introduced by P. Antoine [An_2].

1.7. A <u>cartesian closed topological</u> (shortly CCT) <u>category</u> is a topological category such that each pair of its objects has a power-object. Equivalently, a CCT category is a topological category $\mathcal{K}$ such that, for each object A, the functor

$$A \times - : \mathcal{K} \longrightarrow \mathcal{K}$$

has a right adjoint (namely, the functor $X \mapsto X^A$). This equivalence, and other important properties of CCT categories, are proved in [He_1].

1.8. Given a concrete category $\mathcal{K}$, we are interested in its finitely productive CCT extensions. (I.e., in CCT categories $\mathcal{L}$, containing $\mathcal{K}$ as a full, concrete subcategory +), closed under finite products.) Note that if $\mathcal{K}$ has such an extension then it has

(i) concrete finite products,

(ii) constant morphisms.

This follows from the fact that each CCT category has both. (These two conditions are not sufficient.) Even a topological category can fail to have such an extension, as proved in [AK].

On the other hand, if $\mathcal{K}$ has a finitely productive CCT extension then it has a smallest one, called the <u>CCT hull</u> of $\mathcal{K}$. It can be characterized as a finitely productive CCT extension contained in each such extension. Also, it can be characterized internally as follows.

1.9. <u>Definition</u> [HN]. Let $\mathcal{K}$ be a concrete category with finite concrete products and constant morphisms. Its <u>CCT hull</u> is a CCT category $\mathcal{L}$, in which $\mathcal{K}$ is a full, concrete subca-

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+) A concrete subcategory of a concrete category $\mathcal{L}$ is a subcategory $\mathcal{K}$ with an underlying functor arising as a restriction of that of $\mathcal{L}$.

tegory such that

(i) $\mathcal{K}$ is finally dense $^{+)}$ in $\mathcal{L}$.

(ii) The power-objects of $\mathcal{K}$-objects are initially dense $^{+)}$ in $\mathcal{L}$.

1.10. Remarks. (a) It is easy to check that, since $\mathcal{K}$ is finally dense in $\mathcal{L}$, all initial lifts in $\mathcal{K}$ are also initial lifts in $\mathcal{L}$. Particularly, since $\mathcal{L}$ has finite concrete products, it follows that $\mathcal{K}$ is closed to finite products in $\mathcal{L}$.

(b) In [HN] the condition (i) is stated in a seemingly stronger way: each $\mathcal{L}$-object L is a final lift of an episink $(|A_i| \xrightarrow{a_i} X)$ of $\mathcal{K}$-objects, i.e., a sink such that $X = \bigcup a_i(|A_i|)$. Since $\mathcal{K}$ has constant morphisms, these two conditions are equivalent: enlarging a given sink by arbitrary constant structured maps does not change the final lift.

(c) The least initial completion of $\mathcal{K}$, called the Mac Neille completion, is characterized analogously: it is an initially complete category $\mathcal{L}$ in which $\mathcal{K}$ is a full, concrete subcategory which is both finally and initially dense. See [AHS_1].

1.11. <u>Definition.</u> Let $\mathcal{K}$ be a concrete category with finite concrete products. Two structured maps $|A| \xrightarrow{a} X$ and $|A'| \xrightarrow{a'} X$ are said to be <u>productively equivalent</u>, in symbols

$$(A,a) \approx_X (A',a'),$$

if for an arbitrary map

+) A class $\mathcal{C}$ of objects of a concrete category $\mathcal{L}$ is initially dense if each $\mathcal{L}$-object is the initial lift of a source of structured maps into $\mathcal{C}$-objects. Dually: finally dense.

$$h: X \times |B| \longrightarrow |C|, \text{ where } B, C \in \mathcal{K},$$

we have:

$$h \cdot (a \times 1_{|B|}): A \times B \longrightarrow C \text{ is a } \mathcal{K}\text{-morphism}$$

iff

$$h \cdot (a' \times 1_{|B|}): A' \times B \longrightarrow C \text{ is a } \mathcal{K}\text{-morphism.}$$

1.12. Definition. A concrete category is said to be strictly small-fibred if it has finite concrete products and for each set X the productive equivalence $\approx_X$ is small (i.e., it has a small set of representatives).

1.13. Theorem [AK]. Let $\mathcal{K}$ be a concrete category with finite concrete products and with constant morphisms. Then $\mathcal{K}$ has a CCT hull iff $\mathcal{K}$ is strictly small-fibred.

Remark. "Usual" topological categories (and all of their full subcategories) are strictly small-fibred. Nevertheless, a topological category is constructed in [AK] which fails to be strictly small-fibred.

§ 2. The category of power-closed sinks

2.1. Throughout this section, $\mathcal{K}$ denotes a fixed concrete category with finite concrete products and constant morphisms.

Given objects P and Q of $\mathcal{K}$, for each map $f: X \to \hom(P,Q)$ we define a map $\hat{f}: X \times |P| \longrightarrow |Q|$ by

$$\hat{f}(x,p) = [f(x)](p) \text{ for } x \in X, \ p \in |P|.$$

2.2. Convention. Let $\mathcal{A} = (|A_i| \xrightarrow{a_i} X)_{i \in I}$ be a structured sink. Denote by $\mathcal{A}^{\downarrow}$ the source of all (non-structured!) maps $p: X \longrightarrow \hom(P,Q)$, where $P, Q \in \mathcal{K}$, such that for each

$i \in I$

$$\widehat{p \cdot a_i}: A_i \times P \longrightarrow Q \text{ is a morphism in } \mathcal{K} .$$

2.3. Definition. Let $\mathcal{A}$ be a structured sink on a set X. The power-closure of $\mathcal{A}$ is the sink $\overline{\mathcal{A}}$ of all structured maps $|A| \xrightarrow{a} X$ with the following property: for each $p: X \longrightarrow \hom(P,Q)$ in $\mathcal{A}^{\downarrow}$,

$$\widehat{p \cdot a}: A \times P \longrightarrow Q \text{ is a morphism in } \mathcal{K} .$$

If $\overline{\mathcal{A}} = \mathcal{A}$, we call $\mathcal{A}$ a power-closed sink.

Note that each structured sink $\mathcal{A}$ fulfils $\mathcal{A} \subset \overline{\mathcal{A}} = \overline{\overline{\mathcal{A}}}$. The fact that $\mathcal{A}$ is power-closed means that $\mathcal{A}$ is "determined" by the source $\mathcal{A}^{\downarrow}$.

2.4. Example. For each object B of $\mathcal{K}$ denote by B^o the following structured sink on the set $X = |B|$:

$$B^o = \{ |A| \xrightarrow{a} X \mid a: A \longrightarrow B \text{ is a morphism in } \mathcal{K} \} ;$$

then B^o is power-closed.

Proof. Let $|A| \xrightarrow{a} X$ belong to $\overline{B^o}$; we shall prove that $a: A \longrightarrow B$ is a morphism in $\mathcal{K}$.

We use the map $p: X \longrightarrow \hom(B,B)$, assigning to each $x \in X$ the morphism $p(x): B \longrightarrow B$, which is constant with the value x. Note that $p \in (B^o)^{\downarrow}$ since for each morphism $a_o: A_o \longrightarrow B$ the map $\widehat{p \cdot a_o}: |A_o \times B| \longrightarrow |B|$ is defined by

$$\widehat{p \cdot a_o}(t,x) = [p(a_o(t))]\,(x) = a_o(t).$$

Thus,

$$\widehat{p \cdot a_o} = \pi_1 \cdot (a_o \times 1_B): A_o \times B \longrightarrow B,$$

where $\pi_1: B \times B \longrightarrow B$ is the first projection. Therefore, $\widehat{p \cdot a_o}$: $: A_o \times B \longrightarrow B$ is a morphism in $\mathcal{K}$; i.e., an element of B^o.

Since $a \in \overline{B^o}$ and $p \in (B^o)^{\downarrow}$, by the definition of power-closure $\widehat{p \cdot a}: A \times B \longrightarrow B$ is a morphism of $\mathcal{K}$. It suffices to exhibit a morphism $r: A \longrightarrow A \times B$ such that $a = \widehat{p \cdot a} \cdot r$. For thi choose an arbitrary morphism $r_0: A \longrightarrow B$. This is possible sinc if $X \neq \emptyset$ then r_0 can be any constant map; if $X = \emptyset$ then $|A| = \emptyset$ (because we have a map $a: |A| \longrightarrow X$) and, by the standing hypothesis of 1.1, it follows that $A = B$ and $r_0 = 1_B$. Now, let $r: A \longrightarrow A \times B$ have components 1_A and r_0. Then, for each $t \in |A|$,

$$\widehat{p \cdot a} \cdot r\ (t) = \widehat{p \cdot a}\ (t, r_0(t)) = [p \cdot a(t)]\ (r_0(t)) = a(t).$$

Thus,

$$a = \widehat{p \cdot a} \cdot r : A \longrightarrow B$$

is a morphism of $\mathcal{K}$, which was to be proved.

2.5. Remark. Each power-closed sink $\mathcal{A}$ on a set X is a sieve in the terminology of P. Antoine [An_2]; i.e.,

(i) $\mathcal{A}$ is closed under composition (in the sense that for each $|A| \xrightarrow{a} X$ in $\mathcal{A}$ and each morphism $f: A' \longrightarrow A$ we have $|A'| \xrightarrow{a \cdot f} X$ in $\mathcal{A}$);

(ii) $\mathcal{A}$ contains all constant structured maps into X.

Proof. (ii) Let $|A| \xrightarrow{a} X$ be constant with a value $x_0 \in X$. For each map $p: X \longrightarrow \hom(P,Q)$ the map $\widehat{p \cdot a}: |A \times P| \longrightarrow |Q|$ is defined by

$$\widehat{p \cdot a}(t,x) = [p(a(t))]\ (x) = [p(x_0)]\ (x).$$

Thus $\widehat{p \cdot a}$ is the composition of the second projection $A \times P \longrightarrow P$ and the morphism $p(x_0): P \longrightarrow Q$. Thus, $\widehat{p \cdot a}$ is a morphism in $\mathcal{K}$.

Since this holds in particular for each $p \in \mathcal{A}^{\downarrow}$, we see that $a \in \overline{\mathcal{A}} = \mathcal{A}$.

(i) Let $p \in \mathcal{A}^{\downarrow}$ be arbitrary. We know that $\widehat{p \cdot a}: A \times$

$\times P \longrightarrow Q$ is a morphism and we are to prove that so is

$$\widehat{p \cdot a \cdot f}: A' \times P \longrightarrow Q.$$

This is a consequence of the fact that $f \times 1_P: A' \times P \longrightarrow A \times P$ is a morphism and the following lemma.

2.6. Lemma. Given objects P and Q and maps

$$Y \xrightarrow{f} X \xrightarrow{g} \hom(P,Q)$$

then

$$\widehat{g \cdot f} = \hat{g} \cdot (f \times 1_{|P|}): Y \times |P| \longrightarrow |Q|.$$

Proof. For arbitrary $y \in Y$, $p \in |P|$ we have

$$\widehat{g \cdot f}\,(y,p) = [g(f(y))]\,(p)$$

as well as

$$\hat{g} \cdot (f \times 1_{|P|})(y,p) = \hat{g}(f(y),p) = [g(f(y))]\,(p).$$

2.7. Proposition. Let $\mathcal{K}$ be a strictly small-fibred category. Then for each set X the conglomerate of all power-closed sinks on X is small.

Proof. Since the equivalence $\approx_X$ of Definition 1.11 has a set of representatives (say, of cardinality α), it suffices to prove that each power-closed sink $\mathcal{A}$ is closed under this equivalence, i.e.,

$|A| \xrightarrow{a} X$ in $\mathcal{A}$ implies $|A'| \xrightarrow{a'} X$ is in $\mathcal{A}$ whenever

$$(A,a) \approx_X (A',a').$$

Then power-closed sinks can be indexed by sets of representatives of $\approx_X$; hence, the number of all power-closed sinks on the set X cannot exceed 2^{α}.

Let $\mathcal{A}$ be power-closed and let $(A,a) \approx_X (A',a')$. Assuming that $|A| \xrightarrow{a} X$ is in $\mathcal{A}$, we are to show that $|A'| \xrightarrow{a'} X$ is in $\mathcal{A}$. Given $p: X \longrightarrow \hom(P,Q)$ in $\mathcal{A}^{\downarrow}$, we know that

$\widehat{p \cdot a}: A \times P \longrightarrow Q$ is a $\mathcal{K}$-morphism. By Lemma 2.6, this means that

$$\hat{p} \cdot (a \times 1_{|P|}): A \times P \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism.}$$

Since $(A,a) \approx_X (A',a')$, we have

$$\hat{p} \cdot (a' \times 1_{|P|}) = \widehat{p \cdot a'}: A' \times P \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism.}$$

Thus, since $\mathcal{A}$ is power-closed, (A',a') must belong to it.

2.8. Corollary. For each strictly small-fibred category. the conglomerate of all power-closed sinks is legitimate, i.e., is isomorphic to a class (in the Bernays-Gödel terminology).

Even if all power-closed sinks form a legitimate conglomerate, we cannot, strictly speaking, work with the "class of all power-closed sinks". (Since a power-closed sink, which is itself a proper class, cannot be a member of a class.) Nevertheless, we shall disregard this difficulty which is, evidently, only formal: instead of the "class" of power-closed sinks, considered below, we would formally work with an arbitrary class isomorphic to it.

2.9. Definition. Let $\mathcal{K}$ be a strictly small-fibred category. Then its category of power-closed sinks is the following concrete category, denoted by $PCS(\mathcal{K})$:

Objects are all power-closed sinks.

Morphisms from a sink $\mathcal{A} = (A_i \xrightarrow{a_i} X)_I$ to a sink $\mathcal{B} = (B_j \xrightarrow{b_j} Y)_J$ are maps $f: X \longrightarrow Y$ such that for each $i \in I$ there exists $j \in J$ with $A_i = B_j$ and $f \cdot a_i = b_j$;

the forgetful functor is defined by $|(A_i \xrightarrow{a_i} X)| = X$.

$$\begin{tikzcd} A_i \arrow[d, "a_i"'] & & B_j \arrow[d, "b_j"'] \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

It is easy to see that PCS($\mathcal{X}$) is indeed a correctly defined concrete category (up to the tolerance mentioned in 2.8).

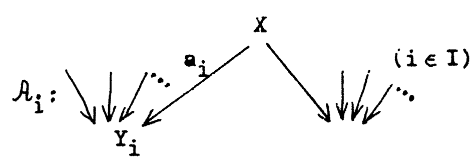
Identifying $\mathcal{X}$-objects B with the power-closed sinks B^o of 2.4, the category $\mathcal{X}$ becomes a full, concrete subcategory of PCS ($\mathcal{X}$). Indeed, each $\mathcal{X}$-morphism $f: B \to C$ is clearly a sink-morphism $f: B^o \to C^o$. Conversely, if $f: B^o \to C^o$ is a sink-morphism then

$$|B| \xrightarrow{1_B} |B| \text{ in } B^o \text{ implies } |B| \xrightarrow{f \cdot 1_B} |C| \text{ in } C^o,$$

hence, $f: B \to C$ is a $\mathcal{X}$-morphism.

2.10. Proposition. Let $\mathcal{X}$ be a strictly small-fibred category with constant morphisms. Then PCS ($\mathcal{X}$) is a topological category.

Proof. (i) PCS ($\mathcal{X}$) is initially complete. To prove this, consider a structured source $(X \xrightarrow{a_i} |\mathcal{A}_i|)_I$



to power-closed sinks $\mathcal{A}_i$. Define a structured sink $\mathcal{C}$ on X to consist of precisely those structured maps $|C| \xrightarrow{c} X$ which satisfy, for each $i \in I$:

$$|C| \xrightarrow{a_i \cdot c} Y_i \text{ is in } \mathcal{A}_i.$$

A) $\mathcal{C}$ is power-closed. Let $|D| \xrightarrow{d} X$ be a structured map in $\overline{\mathcal{C}}$. We are to show that $(D,d) \in \mathcal{C}$.

$$\begin{tikzcd} {|D|} \arrow[r, "d"] & X \arrow[d, "a_i"] & \\ & Y_i \arrow[r, "p"] & \hom(P,G). \end{tikzcd}$$

For each $i \in I$ the structured map $|D| \xrightarrow{a_i \cdot d} Y_i$ is in $\overline{\mathcal{A}}_i$. (Indeed, let $p : Y_i \longrightarrow \hom(P,Q)$ be an element of $\mathcal{A}^{\downarrow}_i$; then $p \cdot a_i : X \longrightarrow \hom(P,Q)$ is easily seen to be an element of $\mathcal{C}^{\downarrow}$, Hence, by the hypothesis on d,

$$\widehat{p \cdot a_i \cdot d} : D \times P \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism.})$$

Since $\mathcal{A}_i$ is a power-closed sink, it follows that $a_i \cdot d$ belongs to it (for each $i \in I$); in other words, d is an element of $\mathcal{C}$.

B) $\mathcal{C}$ is the initial lift of the given source. Let $\mathcal{B}$ be a power-closed sink on a set Z and let $f : Z \longrightarrow X$ be

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "f"] & \mathcal{C} \arrow[dl, "a_i"'] \arrow[d] \arrow[dr] & \\
\mathcal{A}_i & &
\end{tikzcd}$$

a map such that $a_i \cdot f : \mathcal{B} \longrightarrow \mathcal{A}_i$ is a sink-morphism for each $i \in I$. We are to show that then $f : \mathcal{B} \longrightarrow \mathcal{C}$ is a sink-morphism. Indeed, given $|B| \xrightarrow{b} Z$ in $\mathcal{B}$ we know that

$$|B| \xrightarrow{a_i \cdot f \cdot b} Y_i \text{ is in } \mathcal{A}_i$$

for each $i \in I$. This means that $(B, f \cdot b)$ is in $\mathcal{C}$.

(ii) PCS ($\mathcal{K}$) is small-fibred; see Proposition 2.7.

(iii) PCS ($\mathcal{K}$) has constant morphisms. This follows immediately from part (ii) of Lemma 2.5.

2.11. <u>Proposition</u>. For each strictly small-fibred category with constant morphisms $\mathcal{K}$, the category PCS$'$($\mathcal{K}$) is cartesian closed. The power-object of sinks $\mathcal{A}$ and $\mathcal{B}$ (with $|\mathcal{A}| = X$; $|\mathcal{B}| = Y$) is the sink $\mathcal{B}^{\mathcal{A}}$ of all structured maps $|C| \xrightarrow{c} \hom(\mathcal{A}, \mathcal{B})$ such that

$|A| \xrightarrow{a} X$ in $\mathcal{A}$ implies $|C \times A| \xrightarrow{\hat{c} \cdot (1_C \times a)} Y$ in $\mathcal{B}$.

Proof. A) The above sink $\mathcal{B}^{\mathcal{A}}$ is power-closed. To prove this, consider an arbitrary structured map $|C_0| \xrightarrow{c_0} \mathrm{hom}\ (\mathcal{A}, \mathcal{B})$ in $\overline{\mathcal{B}^{\mathcal{A}}}$. We shall prove then (C_0, c_0) is an element of $\mathcal{B}^{\mathcal{A}}$. Thus, given $|A_0| \xrightarrow{a_0} X$ in $\mathcal{A}$ we are to verify that the structured map

$$|C_0 \times A_0| \xrightarrow{1 \times a_0} |C_0| \times X \xrightarrow{\hat{c}_0} Y$$

is in $\mathcal{B}$. Now, $\mathcal{B}$ is power-closed; therefore it suffices to show that $\hat{c}_0 \cdot (1 \times a_0) \in \overline{\mathcal{B}}$. Hence, for each $p_0 : Y \longrightarrow \mathrm{hom}\ (P, Q)$ in $\mathcal{B}^{\downarrow}$ we shall prove that

$$\widehat{p_0 \cdot \hat{c}_0 \cdot (1 \times a_0)} : C_0 \times A_0 \times P \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism},$$

thus concluding the proof of A).

First, we define a map

$$p : \mathrm{hom}\ (\mathcal{A}, \mathcal{B}) \longrightarrow \mathrm{hom}\ (A_0 \times P, Q),$$

for which we shall verify that it is an element of $(\mathcal{B}^{\mathcal{A}})^{\downarrow}$. Let $h \in \mathrm{hom}\ (\mathcal{A}, \mathcal{B})$ be any sink-morphism; then $|A_0| \xrightarrow{a_0} X$ in $\mathcal{A}$ implies

$$|A_0| \xrightarrow{h \cdot a_0} Y \text{ is in } \mathcal{B} .$$

Since $p_0 \in \mathcal{B}^{\downarrow}$, it follows that

$$\widehat{p_0 \cdot h \cdot a_0} : A_0 \times P \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism.}$$

Put

$$p(h) = \widehat{p_0 \cdot h \cdot a_0} \text{ for each } h \in \mathrm{hom}\ (\mathcal{A}, \mathcal{B}).$$

We prove that p is in $(\mathcal{B}^{\mathcal{A}})^{\downarrow}$, i.e., that given

$$|C| \xrightarrow{c} \mathrm{hom}\ (\mathcal{A}, \mathcal{B}) \text{ in } \mathcal{B}^{\mathcal{A}}$$

then

$$\widehat{p \cdot c}: C \times (A_0 \times P) \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism.}$$

Since c is in $\mathcal{B}^{\mathcal{A}}$ and a_0 is in $\mathcal{A}$, we conclude that

$$\hat{c} \cdot (1 \times a_0): |C \times A_0| \longrightarrow Y \text{ is in } \mathcal{B} \ .$$

Thus, $p_0 \in \mathcal{B}^{\downarrow}$ implies that

$$\widehat{p_0 \cdot \hat{c} \cdot (1 \times a_0)}: C \times A_0 \times P \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism}$$

and it suffices to show that

$$\widehat{p \cdot c} = \widehat{p_0 \cdot \hat{c} \cdot (1 \times a_0)}.$$

Indeed, for arbitrary points $z \in |C|$, $x \in |A_0|$ and $t \in |P|$ we have

$$\widehat{p \cdot c}(z,x,t) = [p(c(z))](x,t) = [\widehat{p_0 \cdot c(z) \cdot a_0}](x,t) =$$
$$= [(p_0 \cdot c(z))(a_0(x))](t)$$

as well as

$$\widehat{p_0 \cdot \hat{c} \cdot (1 \times a_0)}(z,x,t) = [(p_0 \cdot \hat{c} \cdot (1 \times a_0))(z,x)](t) =$$
$$= [(p_0 \cdot \hat{c})(z,a_0(x))](t) = [(p_0 \cdot c(z))(a_0(x))](t).$$

Since p is in $(\mathcal{B}^{\mathcal{A}})^{\downarrow}$ and $c_0 \in \overline{\mathcal{B}^{\mathcal{A}}}$ we conclude that

$$\widehat{p \cdot c_0}: C \times A_0 \times P \longrightarrow Q \text{ is a } \mathcal{K}\text{-morphism.}$$

This concludes the proof since, as above, $\widehat{p \cdot c_0} =$
$= \widehat{p_0 \cdot \hat{c}_0 \cdot (1 \times a_0)}$.

B) The category PCS ($\mathcal{K}$) has finite concrete products, since it is topological. It remains to be shown that the sinks $\mathcal{B}^{\mathcal{A}}$ have the required universal property. Indeed, let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be power-closed sinks with underlying sets X, Y and Z respectively; let $f: Z \longrightarrow \hom(\mathcal{A}, \mathcal{B})$ be an arbitrary map. Then

$$f : \mathscr{C} \longrightarrow \mathscr{B}^{\mathscr{A}} \quad \text{is a sink-morphism}$$

iff for arbitrary $|C| \xrightarrow{c} Z$ in $\mathscr{C}$ and $|A| \xrightarrow{a} X$ in $\mathscr{A}$ we have:

$$\widehat{f \cdot c} \cdot (1_C \times a) : |C \times A| \longrightarrow Y \text{ is in } \mathscr{B} \ .$$

On the other hand, the product $\mathscr{C} \times \mathscr{A}$ consists of all structured maps

$$|D| \xrightarrow{(c,a)} Z \times X$$

where (c,a) is the map with components $|D| \xrightarrow{c} Z$ in $\mathscr{C}$ and $|D| \xrightarrow{a} X$ in $\mathscr{A}$. Hence

$$\hat{f} : \mathscr{C} \times \mathscr{A} \longrightarrow \mathscr{B} \quad \text{is a sink-morphism iff}$$

for arbitrary $|C| \xrightarrow{c} Z$ in $\mathscr{C}$ and $|A| \xrightarrow{a} X$ in $\mathscr{A}$ we have:

$$\hat{f} \cdot (c \times a) : |C \times A| \longrightarrow Y \text{ is in } \mathscr{B} \ .$$

But by Lemma 2.6,

$$\hat{f} \cdot (c \times a) = \widehat{f \cdot c} \cdot (1_C \times a),$$

hence the two conditions on f coincide.

2.12. Remark. Particularly, given objects A and B in $\mathscr{K}$, the power-sink $(B^o)^{A^o}$ consists of those structured maps $|C| \xrightarrow{c} \hom(A,B)$ which fulfil:

$$\hat{c} : C \times A \longrightarrow B \text{ is a } \mathscr{K}\text{-morphism.}$$

We denote this sink by B^A.

§ 3. The description of the CCT hull

3.1. Theorem. For each strictly small-fibred category $\mathscr{K}$ with constant morphisms the CCT hull is the category PCS ($\mathscr{K}$) of power-closed sinks.

Proof. We know that PCS ($\mathscr{K}$) is a CCT category (2.10

and 2.11) which contains $\mathcal{K}$ as a full, concrete subcategory (2.9).

(i) $\mathcal{K}$ is finally dense in PCS ($\mathcal{K}$). Indeed, each power-closed sink $\mathcal{A} = (|A_i| \xrightarrow{a_i} X)_I$ is the final lift of itself, i.e., more precisely, of the sink $(|A_i^o| \xrightarrow{a_i} X)_I$.

(ii) The power-objects of $\mathcal{K}$ are initially dense in PCS ($\mathcal{K}$). Let $\mathcal{A} = (|A_i| \xrightarrow{a_i} X)_I$ be a power-closed sink with

$$\mathcal{A}^{\downarrow} = (X \xrightarrow{p_j} \hom(P_j, Q_j))_J.$$

We shall prove that then $\mathcal{A}$ is the initial lift of the source

$$(X \xrightarrow{p_j} |Q_j^{P_j}|)_J$$

(where the sinks $Q_j^{P_j}$ are as described in 2.12).

First, for each $j \in J$,

$$p_j \colon \mathcal{A} \longrightarrow Q_j^{P_j}$$

is a sink-morphism. Indeed, given $|A| \xrightarrow{a} X$ in $\mathcal{A}$ then $p_j \in \mathcal{A}^{\downarrow}$ implies that

$$\widehat{p_j \cdot a} \colon A \times P_j \longrightarrow Q_j \text{ is a } \mathcal{K}\text{-morphism,}$$

in other words that

$$|A| \xrightarrow{p_j \cdot a} \hom(P_j, Q_j) \text{ is an element of } Q_j^{P_j}.$$

Secondly, let $\mathcal{B}$ be a power-closed sink and let $f \colon |\mathcal{B}| \longrightarrow X$ be a map such that $p_j \cdot f \colon \mathcal{B} \longrightarrow Q_j^{P_j}$ are sink-morphisms

$$\begin{tikzcd} \mathcal{B} \arrow[r, "f"] & \mathcal{A} \arrow[dl, "p_j"'] \arrow[d] \arrow[dr] & \\ Q_j^{P_j} & & \ldots \end{tikzcd}$$

for all $j \in J$. We are to show that also $f \colon \mathcal{B} \longrightarrow \mathcal{A}$ is a sink-morphism, i.e., that given $|B| \xrightarrow{b} Y$ in $\mathcal{B}$ then $|B| \xrightarrow{f \cdot b} X$ is in $\overline{\mathcal{A}}$. (Then, of course, $f \cdot b$ is an element of $\mathcal{A}$.) For each p_j in $\mathcal{A}^{\downarrow}$ we know that $p_j \cdot f$ is a **sink-morphism**, thus

$$p_j \cdot f \cdot b: |B| \longrightarrow \hom(P_j, Q_j) \text{ is in } Q_j^{P_j}.$$

This means that

$$\widehat{p_j \cdot f \cdot b}: B \times P_j \longrightarrow Q_j \text{ is a } \mathcal{K}\text{-morphism.}$$

3.2. Corollary. For each category $\mathcal{K}$ with finite concrete products and constant morphisms the following conditions are equivalent:

(i) $\mathcal{K}$ has a finitely productive CCT extension;

(ii) $\mathcal{K}$ has a CCT hull;

(iii) for each set X the conglomerate of all power-closed sinks on X is small;

(iv) PCS ($\mathcal{K}$) is a CCT hull of $\mathcal{K}$;

(v) $\mathcal{K}$ is strictly small-fibred.

Let us remark that the proof of (v) $\Longrightarrow$ (ii), presented in the current paper, is much simpler than the original proof of [AK], where an extension of $\mathcal{K}$ is constructed by a complicated transfinite induction.

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