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AN ELEMENTARY PROOF OF NOBLE´S THEOREM ON NORMALITY OF POWERS

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Dedicated to Professor M. Katětov on his seventieth birthday

Abstract: We show in a simple way that if all powers of a space are normal, then the space itself is compact.

Key words: Cartesian product, normality, compactness.

Classification: 54B10, 54D15, 54D30

One of the important results in the theory of normality of Cartesian products, originated in 1948 by M. Katětov and A.H. Stone (see [2] and [6]), is the theorem due to N. Noble [4] which states that if all powers of a space are normal, then the space itself is compact. The theorem has been originally obtained in the frame of a general theory developed by N. Noble, and this prompted several authors to propose simpler and more direct proofs (see [1], [3] and [5]). In all these proofs A.H. Stone´s theorem on the non-normality of $N^{\aleph_1}$ is applied and, together with a conveniently chosen rather strong topological result, yields Noble´s theorem.

It turns out that the Noble theorem can also be established in an elementary way by a variant of the argument A.H. Stone used to prove the non-normality of $N^{\aleph_1}$.

We shall show that if for a topological space X the power X^m is normal for every m, then X is compact.

Suppose that X is not compact and consider a family $\{F_s\}_{s\in S}$ of closed subsets of X which has the finite intersection property and an empty intersection; denote by m the cardinality of S. The set $F=\prod_{s\in S} F_s \subset X^m=\prod_{s\in S} X_s$, where $X_s=X$ for $s\in S$, is closed and disjoint from the diagonal $\Delta\subset X^m$. Consider an open set U containing F.

Let x_1 be an arbitrary point in F. There exists a finite set $S_1\subset S$ such that $p_{S_1}^{-1}\, p_{S_1}(x_1) \subset U$. Define a point $x_2 \in F$ by letting $p_s(x_2)=a_1$ for $s \in S_1$, where a_1 is an arbitrary point in $\bigcap_{s\in S_1} F_1$, and $p_s(x_2)=p_s(x_1)$ for $s \notin S_1$, and enlarge S_1 to a finite set $S_2 \subset S$ such that $p_{S_2}^{-1}\, p_{S_2}(x_2) \subset U$. By induction we can define points $x_1,x_2,x_3,\ldots$ in F, finite sets $S_1 \subset S_2 \subset S_3 \subset \ldots \subset S$ and points $a_1,a_2,a_3,\ldots$ in X such that

$$p_s(x_n)=a_{n-1} \text{ for } s \in S_{n-1} \text{ and } p_{S_n}^{-1}\, p_{S_n}(x_n) \subset U.$$

Since, by A.H. Stone's theorem, X does not contain a closed copy of N, there exists a point $a_0 \in X$ every neighbourhood of which contains infinitely many a_n's. The points $y_1,y_2,y_3,\ldots$ of X^m defined by

$$p_s(y_n)=p_s(x_n) \text{ for } s \in S_n \text{ and } p_s(y_n)=a_0 \text{ for } s \notin S_n$$

belong to U and - as one easily sees - every neighbourhood of the point $y \in \Delta$ all of whose coordinates are equal to a_0, contains a y_n. Thus $\Delta \cap \bar{U} \neq \emptyset$; since this is in contradiction with the normality of X^m, it follows that X^m is compact.

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