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A note on the almost left and almost right joint spectra of R.Harte

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Abstract. It is proved that a complex unital normed algebra has a nonzero continuous multiplicative linear functional if and only if the almost left [right] joint spectrum $\widetilde{\sigma}_l(a_1,\dots,a_n)$ [$\widetilde{\sigma}_r(a_1,\dots,a_n)$] is non–empty for every finite set of elements $a_1,\dots,a_n$ in the algebra. This is a counterpart of the main result in [1] to the normed algebra case.

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Let A be a complex normed algebra with the unit 1 and let $a_1,\dots,a_n \in A$. The *left spectrum* of $(a_1,\dots,a_n)$ is the set

$$\sigma_l^A(a_1,\dots,a_n) = \left\{(\lambda_1,\dots,\lambda_n) \in \mathbf{C}^n : 1 \notin \sum_{j=1}^n A(a_j - \lambda_1)\right\}$$

(We simply write $a_j - \lambda_j$ instead of $a_j - \lambda_j 1$) and the *almost left spectrum* of $(a_1,\dots,a_n)$ is the set

$$\widetilde{\sigma}_l^A(a_1,\dots,a_n) = \left\{(\lambda_1,\dots,\lambda_n) \in \mathbf{C}^n : 1 \notin \left(\sum_{j=1}^n A(a_j - \lambda_1)\right)^-\right\}.$$

(Here the bar denotes the closure in the norm topology of A.) The definitions of the *right* and *almost right spectra* of $(a_1,\dots,a_n)$ are similar. (See [2], pp. 457-458.) The sets

$$\sigma^A(a_1,\dots,a_n) = \sigma_l^A(a_1,\dots,a_n) \cup \sigma_r^A(a_1,\dots,a_n)$$
$$\text{and } \widetilde{\sigma}^A(a_1,\dots,a_n) = \widetilde{\sigma}_l^A(a_1,\dots,a_n) \cup \widetilde{\sigma}_r^A(a_1,\dots,a_n)$$

are called the *Harte spectrum* and, respectively, the *almost Harte spectrum* of $(a_1,\dots,a_n)$.

It is obvious that always

$$\widetilde{\sigma}_l^A(a_1,\dots,a_n) \subset \sigma_l^A(a_1,\dots,a_n), \quad \widetilde{\sigma}_r^A(a_1,\dots,a_n) \subset \sigma_r^A(a_1,\dots,a_n),$$
$$\text{and } \widetilde{\sigma}^A(a_1,\dots,a_n) \subset \sigma^A(a_1,\dots,a_n).$$

In the algebra A is complete, then it is easy to see that the above inclusions can be replaced by the equalities. In general, we have the following

Lemma. *Let A be a complex unital normed algebra and let $\widehat{A}$ denote its completion. For arbitrary elements $a_1, \ldots, a_n \in A$ the following equalities hold:*

$$\widetilde{\sigma}_l^A(a_1, \ldots, a_n) = \sigma_l^{\widehat{A}}(a_1, \ldots, a_n), \tag{1}$$

$$\widetilde{\sigma}_r^A(a_1, \ldots, a_n) = \sigma_r^{\widehat{A}}(a_1, \ldots, a_n), \tag{2}$$

$$\widetilde{\sigma}^A(a_1, \ldots, a_n) = \sigma^{\widehat{A}}(a_1, \ldots, a_n). \tag{3}$$

PROOF : We shall give the proof of (1). Equality (2) can be shown in a similar manner. It is seen at once that (1) and (2) imply (3).

Let $(a_1, \ldots, a_n) \in A^n$. It is clear that (cf. [2], p. 460)

$$\sigma_l^{\widehat{A}}(a_1, \ldots, a_n) = \widetilde{\sigma}_l^{\widehat{A}}(a_1, \ldots, a_n) \subset \widetilde{\sigma}_l^A(a_1, \ldots, a_n).$$

To prove the converse assume that $(\lambda_1, \ldots, \lambda_n) \notin \sigma_l^{\widehat{A}}(a_1, \ldots, a_n)$. Then there exists $\widehat{b}_1, \ldots, \widehat{b}_n \in \widehat{A}$ such that $\sum_{j=1}^n \widehat{b}_j(a_j - \lambda_j) = 1$. Since A is a dense subset of $\widehat{A}$, we have $c_j^{(k)} \to \widehat{b}_j$ as $k \to \infty$ $(j = 1, \ldots, n)$ for some $c_j^{(k)} \in A$. Then

$$\sum_{j=1}^n c_j^{(k)}(a_j - \lambda_j) \to \sum_{j=1}^n \widehat{b}_j(a_j - \lambda_j) = 1$$

as $k \to \infty$ and so $1 \in \left(\sum_{j=1}^n A(a_j - \lambda_j)\right)^-$ meaning that $(\lambda_1, \ldots, \lambda_n) \notin \widetilde{\sigma}_l^A(a_1, \ldots, a_n)$. ∎

A nonzero complex homomorphism of an algebra A will be shortly called a *multiplicative functional.*

The above lemma has the following obvious

Corollary. *Let A be a commutative complex normed algebra with unit and let $a_1, \ldots, a_n \in A$. Then*

$$\widetilde{\sigma}_l^A(a_1, \ldots, a_n) = \widetilde{\sigma}_r^A(a_1, \ldots, a_n) = \widetilde{\sigma}^A(a_1, \ldots, a_n)$$

$$= \{(\phi(a_1), \ldots, \phi(a_n)) : \phi \textit{ is a continuous multiplicative functional of } A\}.$$

It is well–known that the almost spectra may be empty. Notice, however, that if a normed algebra A has a continuous multiplicative functional ϕ, then

$$(\phi(a_1), \ldots, \phi(a_n)) \in \widetilde{\sigma}_l^A(a_1, \ldots, a_n) \cap \widetilde{\sigma}_r^A(a_1, \ldots, a_n)$$

since

$$\left(\sum_{j=1}^n A(a_j - \phi(a_j))\right)^- \cap \left(\sum_{j=1}^n (a_j - \phi(a_j))A\right)^- \subset \text{kernel of } \phi.$$

Thus in that case $\widetilde{\sigma}_l^A(a_1, \ldots, a_n), \widetilde{\sigma}_r^A(a_1, \ldots, a_n)$, and $\widetilde{\sigma}^A(a_1, \ldots, a_n)$ are always non–empty. Now we show the converse of this fact:

Theorem. *If $\widetilde{\sigma}_l^A(a_1,\dots,a_n)$ [respectively $\widetilde{\sigma}_r^A(a_1,\dots,a_n)$ or $\widetilde{\sigma}^A(a_1,\dots,a_n)$] is non-empty for an arbitrary n-tuple $(a_1,\dots,a_n)$ of elements in the complex unital normed algebra A with $n = 1,2,\dots$, then A has a continuous multiplicative functional.*

PROOF : We shall only give the proof for the almost left spectrum. The other cases can be shown in a similar way.

Assume that $\widetilde{\sigma}_l^A(a_1,\dots,a_n) \neq \emptyset$ for arbitrary $a_1,\dots,a_n \in A$ and every $n = 1,2,\dots$. By the lemma we have

$$\sigma_l^{\widehat{A}}(a_1,\dots,a_n) = \widetilde{\sigma}_l^A(a_1,\dots,a_n) \neq \emptyset.$$

Since A is dense in its completion $\widehat{A}$, the upper semicontinuity of $\sigma_l^{\widehat{A}}$ implies that $\sigma_l^{\widehat{A}}(\widehat{a}_1,\dots,\widehat{a}_n) \neq \emptyset$ for every finite subset $\{a_1,\dots,a_n\}$ of $\widehat{A}$ (cf. [**2**], p. 463). To make the proof self-contained we shall show this fact directly. Take an arbitrary n-tuple $(\widehat{a}_1,\dots,\widehat{a}_n) \in \widehat{A}^n$. Then there exist $(b_1^{(k)},\dots,b_n^{(k)}) \in A^n$ $(k = 1,2,\dots)$ such that $\sum_{j=1}^n \|\widehat{a}_j - b_j^{(k)}\| < \frac{1}{k}$ for all k. Let $(\lambda_1^{(k)},\dots,\lambda_n^{(k)}) \in \sigma_l^{\widehat{A}}(b_1^{(k)},\dots,b_n^{(k)})$. Since

$$\sigma_l^{\widehat{A}}(b_1^{(k)},\dots,b_n^{(k)}) \subset \sigma^{\widehat{A}}(b_1^{(k)}) \times \dots \times \sigma^{\widehat{A}}(b_n^{(k)})$$
$$\subset D(0,\|b_1^{(k)}\|) \times \dots \times D(0,\|b_n^{(k)}\|) \subset D(0,1+\|\widehat{a}_1\|) \times \dots \times D(0,1+\|\widehat{a}_n\|)$$

(where $D(0,r)$ denotes the closed disc in the complex plane centered at zero and with radius r), we may suppose, passing if necessary to a subsequence, that $(\lambda_1^{(k)},\dots,\lambda_n^{(k)}) \to (\lambda_1,\dots,\lambda_n)$ as $k \to \infty$. We claim that $(\lambda_1,\dots,\lambda_n) \in \sigma_l^{\widehat{A}}(\widehat{a}_1,\dots,\widehat{a}_n)$. If, on the contrary, it was not so, then there would exist $\widehat{u}_1,\dots,\widehat{u}_n \in \widehat{A}$ such that $\sum_{j=1}^n \widehat{u}_j(\widehat{a}_j - \lambda_j) = 1$. And further

$$\|1 - \sum_{j=1}^n \widehat{u}_j(b_j^{(k)} - \lambda_j^{(k)})\| \leq \|\sum_{j=1}^n \widehat{u}_j(\widehat{a}_j - b_j^{(k)} + \lambda_j^{(k)} - \lambda_j)\|$$
$$\leq \max_j \|u_j\| \left\{ \sum_{j=1}^n \|\widehat{a}_j - b_j^{(k)}\| + \sum_{j=1}^n |\lambda_j^{(k)} - \lambda_j| \right\}.$$

Thus we would have $\|1 - \sum_{j=1}^n \widehat{u}_j(b_j^{(k)} - \lambda_j^{(k)})\| < 1$ for sufficiently large k and consequently $(\lambda_1^{(k)},\dots,\lambda_n^{(k)} \notin \sigma_l^{\widehat{A}}(b_1^{(k)},\dots,b_n^{(k)})$, which would contradict our assumption.

Now by the theorem of [1] (cf. also [3]) the Banach algebra $\widehat{A}$ has a complex homomorphism. Its restriction to A is the desired continuous multiplicative functional. ■

Let us conclude with the following

Problem. Assume that $\sigma_l^A(a_1,\dots,a_n) \neq \emptyset$ [or $\sigma_r^A(a_1,\dots,a_n) \neq \emptyset$, or $\sigma^A(a_1,\dots,a_n) \neq \emptyset$] for an arbitrary finite subset $\{a_1,\dots,a_n\}$ of a complex unital normed algebra A. Does there exist a multiplicative (not necessarily continuous) functional on A?

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