Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 30 (1989), No. 2, 347--356

Persistent URL: http://dml.cz/dmlcz/106752

Probability in the alternative set theory

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Abstract. Random variables and random vectors are studied in detail within the framework of AST. Finally, a stochastic process, which can, in the dependence of "the way of measurement", have an arbitrary system of probability distributions, is constructed.

Keywords: Probability distribution, random variables, random vectors, stochastic processes, set trajectories, interval indiscernibility equivalence, Alternative Set Theory.

Classification: Primary 03H20, 60A10, 60G05, Secondary 60A05

This paper is an application of results of [**K**] to the probability theory. It is proved here that if we have any i–dimensional probability distribution, then there exists an i–dimensional random vector having the same probability distribution (or whose probability distribution is indiscernible form the given one in the case of Borel random vectors). A random vector, which can, in the dependence of "the way of measurement" (BAF), have an arbitrary probability distribution, is constructed. If we have a full i–dimensional probability distribution (or a one–dimensional probability distribution), then this i–dimensional random vector (or random variable, respectively) can be chosen to be a set. Last considerations concern stochastic processes. We have proved that for any b–consistent system of finitely–dimensional probability distributions there exists a stochastic process, having set trajectories, whose system of probability distributions is exactly the given one (or whose system of probability distributions is indiscernible from the given one, in the case of Borel stochastic processes). Finally, a stochastic process, which can, in the dependence of BAF, have an arbitrary system of probability distributions, is constructed. The author greatly appreciates the useful discussions with P.Zlatoš.

1. Preliminaries.

1.1. The reader is assumed to be acquainted with [**V1**] and [**K**]. The notions, results and notations (in the modification used in [**K**]) from [**V1**] will be used freely without any referring.

For $0 \neq c \in N, 0 \neq d \in N$ we denote $c^{d\cdot FN} = \{b; \exists i)(b < c^{d\cdot i})\}$.

We remind some notions and results from [**K**].

β will denote the system of all Borel semisets, R will denote the class of all real numbers ($\infty = \{q \in Q; (\forall j \in FN)(q > j)\}$ and $-\infty = \{q \in Q; (\forall j \in FN)(q < -j)\}$ are assumed to be real numbers, too) and a will denote a fixed infinite natural number.

Any map F which assigns to each semiset $A \in \mathcal{B}$ a sequence of natural numbers $\{s_n; n < a\}$ such that $\bigcup_{i\in FN}\bigcap_{j\geq i} s_j = \underline{A}$ and $\bigcap_{i\in FN}\bigcup_{j\geq i} s_j = \overline{A}$ ($\underline{A}$ and $\overline{A}$ denote the lower and upper cut of A, respectively, for the definition and basic properties see

[**K-Z**]; they are also briefly listed in [**K**]), is said to be the Borel approximating function (BAF, to be short).

Let F be a BAF, $\emptyset \neq A \in \mathcal{B}$ and $B \in \mathcal{B}$. Let $F(B) = \{b_n; n < a\}$ and $F(A) = \{c_n; n < a\}$. The semiset B is said to be $F(A)$, F–observable if there exists a real number r and a natural number $d < a, d \notin FN$ such that for each $m < d, m \notin FN$ there holds $b_m/c_m \in r$. The number r is said to be the $F(A)$, F–measure of B, in symbols $m_{F(A)}(B) = r$. Hence $m_{F(A)<F}$ is a measure having as its domain the system of all $F(A)$, F–observable classes.

1.1.1. Theorem. *Let $A \in \mathcal{B}$ have nonadditive cuts and F be a BAF. Then the measure $m_{F(A),F}$ is σ–additive, nonnegative and nondecreasing,* $\mathrm{Dom}(m_{F(A),F}) = \mathcal{B}$ *and if G is any other BAF, then $m_{F(A),F} = m_{G(A),G}$.*

1.1.2. Theorem. *Let $\emptyset \neq A \in \mathcal{B}$ have an additive cut, $\mathcal{O} \subseteq \mathcal{B}$ be any class of semisets such that*

$$(\forall B \in \mathcal{B})(\overline{B} \subsetneqq |A| \vee \underline{B} \supsetneqq |A| \vee B = A) \Rightarrow B \in \mathcal{O}$$

and let $\lambda : \mathcal{O} \to R$ be any nonegative real–valued function such that there holds

$$(\forall B \in \mathcal{O})(\overline{B} \subsetneqq |A| \Rightarrow \lambda(B) = 0) \ \& \ (\underline{B} \supsetneqq |A| \Rightarrow \lambda(B) = \infty) \ \& \\ \& \ (B = A \Rightarrow \lambda(B) = 1).$$

Then there exists a BAF F such that $\mathcal{O}$ is the system of all $F(A)$, F–observable classes and $m_{F(A),F} = \lambda$.

<u>1.2.</u> We state some modifications of the notions and results form Chapter 10.6 in [**V2**].

Let us fix an infinite set u and a proper Sd^*–class U. (SD^*_V is a fixed revealment of the class of classes Sd_V. For the definition and basic properties of revealments see [**S–V**]. All the properties of the Sd^*_V–classes, we shall utilize, are also briefly listed in [**G–Z**].)

A pair $\langle W, w\rangle$, where $w \subseteq u, W \subseteq U$ and W is an Sd^*–class, is said to be admissible (notation $W \sim w$), if it fulfills the countable system of conditions

$$(|w| \geq n \Leftrightarrow |W| \geq n) \ \& \ (|u \setminus w| \geq n \Leftrightarrow |U \setminus W| \geq n), \quad (n \in FN).$$

A finite system of admissible pairs $\{\langle W_i, w_i\rangle; i \leq k\}$ is said to be a partition if $\cup\{W_i; i \leq k\} = U, \cup\{w_i; i \leq k\} = u$ and for each $i < j \leq k \quad W_i \cap W_j = \emptyset = w_i \cap w_j$.

Let $A = \{\langle W_n, w_n\rangle; n \leq k\}, B = \{\langle S_m, s_m\rangle; m \leq j\}$ be two partitions. By $A \wedge B$ we shall denote the following

$$A \wedge B = \{\langle W_n \cap S_m, w_n \cap s_m\rangle; n \leq k, m \leq j\}.$$

The partitions A, B are said to be compatible (notation $A \| B$) if $A \wedge B$ is partition, too.

For any admissible pair $\langle W, w\rangle$ we define $\rho(W, w) = \{\langle W, w\rangle, \langle U \setminus W, u \setminus w\rangle\}$. Obviously $\rho(W, w)$ is a partition.

C is said to be a cluster of partitions if

1.) $(\forall A \in C)$ (A is a partition)
2.) $(\forall A, B \in C)$ $(A \wedge B \in C)$.

Obviously, if $\mathcal{C}$ is a cluster of partitions, then for each $A, B \in \mathcal{C}$ $A \| B$ holds.
For the proof of the following lemma the author is indebted to P.Zlatoš.

1.2.1. Lemma. *Let $\mathcal{C}$ be a countable cluster of partitions, $X \subseteq U$ be an Sd^*-class and $w \subseteq u$ be a set. Then there exist a set $x \subseteq u$ and an Sd^*-class $W \subseteq U$ such that $x \sim X$ and $W \sim w$, and for each $P \in \mathcal{C}$ $P \| \rho(X, x)$ & $P \| \rho(W, w)$.*

PROOF : Fix a set $w \subseteq u$. We shall prove the existence of an Sd^*-class W having the required properties.

Let $\mathcal{C} = \{P_i; i \in FN\}$ be an enumeration of the given cluster $\mathcal{C}$. Denote $\widetilde{P}_i = P_0 \wedge \cdots \wedge P_i$. Then obviously $\widetilde{P}_i$ is a partition for each i. Let $\widetilde{P}_i = \{\langle Y_{ij}, y_{ij} \rangle; j \leq \widetilde{i} \in FN\}$. Obviously for each $j \leq \widetilde{i}$ there exists an Sd^*-class $\widetilde{W}_{ij} \subseteq Y_{ij}$ such that it fulfils the following countable system of conditions:

$$(|\widetilde{W}_{ij}| \geq n \Leftrightarrow |w \cap y_{ij}| \geq n) \ \& \ (|Y_{ij} \setminus \widetilde{W}_{ij}| \geq n \Leftrightarrow |y_{ij} \setminus w| \geq n), (n \in FN).$$

Denote $W_i = \bigcup\limits_j \widetilde{W}_{ij}$. Then $\rho(W_i, w) \| \widetilde{P}_i$ immediately follows, and hence also $\rho(W_i, w) \| P_k$ for each $k \leq i$. We prolong the sequence $\{W_i; i \in FN\}$. Since for each $i \in FN$ and $k \leq i$ $\rho(W_i, w) \| P_k$ holds, there exists a $\Theta \in N \setminus FN$ such that $\rho(W_\Theta, w) \| P_k$ holds for each $k \in FN$. Now, it is enough to put $W = W_\Theta$.

The existence of a set x, having all the properties required, can be proved similarly. This can also be proved by a simple modification of the proofs of Theorems 10.6.1 e and h in [V2]. ∎

The following lemma is obvious.

1.2.2. Lemma. *Let $\mathcal{C}$ be a cluster of partitions and P be a partition such that for each $A \in \mathcal{C}$ $A \| P$ holds. Then $\widetilde{\mathcal{C}} = \mathcal{C} \cup \{P\} \cup \{P \wedge A; A \in \mathcal{C}\}$ is a cluster of partitions. If $\mathcal{C}$ is a countable cluster of partitions then $\widetilde{\mathcal{C}}$ is a countable cluster of partitions, too.*

1.2.3. Lemma. *There exists a cluster of partitions $\mathcal{C}$ such that for each Sd^*-class $W \subseteq U$ there exists a set $w \subseteq u$ for which $w \sim W$ and $\rho(W, w) \in \mathcal{C}$, and for each set $x \subseteq u$ there exists an Sd^*-class $X \subseteq U$ for which $x \sim X$ and $\rho(X, x) \in \mathcal{C}$.*

PROOF : Let $\{\langle i_\alpha, W_\alpha \rangle; i_\alpha \in \{0,1\}, \alpha \in \Omega\}$ be an enumeration of the codable system $\{\langle 0, w \rangle; w \subseteq u\} \cup \{\langle 1, W \rangle; W \subseteq U \ \& \ Sd^*_V(W)\}$. Using transfinite induction we shall construct a sequence of countable clusters of partitions such that for each $\alpha \in \Omega$ $\mathcal{C}_\alpha$ will be the first countable cluster of partitions (in a fixed enumeration of all countable clusters of partitions) for which $\mathcal{C}_\alpha \supseteq \bigcup\{\mathcal{C}_\beta; \beta \in \alpha \cap \Omega\}$ and $\rho(W_\alpha, w) \in \mathcal{C}_\alpha$ in the case $i_\alpha = 1 (\rho(W, w_\alpha) \in \mathcal{C}_\alpha$ in the case $i_\alpha = 0)$. w (W, respectively) is the first set (Sd^*-class) in a well order of the class $\{w \subseteq u; w \sim W_\alpha\}$ ($\{W \subseteq U; W \sim w_\alpha \ \& \ Sd^*_V(W)\}$) such that for each $\beta \in \alpha \cap \Omega$ and each $P \in \mathcal{C}_\beta$ $\rho(W_\alpha, w) \| P$ (or $\rho(W, w_\alpha) \| P$, respectively). Such a set w (Sd^*-class W) is guaranteed by Lemma 1.2.1. By Lemma 1.2.2 such a countable cluster $\mathcal{C}_\alpha$ does exist. Put $\mathcal{C} = \bigcup\{\mathcal{C}_\alpha; \alpha \in \Omega\}$. Obviously, since $\{\mathcal{C}_\alpha; \alpha \in \Omega\}$ is a monotone system of clusters, $\mathcal{C}$ is a cluster, too, and it has all the properties required. ∎

1.2.4. Theorem. *There exists a bijection $T : u \approx U$ such that for each $X \subseteq u$ $\mathrm{Set}(X) \Leftrightarrow Sd^*_V(T''X)$ holds.*

PROOF : Let us fix a cluster of partitions $\mathcal{C}$, guaranteed by Lemma 1.2.3. Then for each $x \in U$ there exists an $\widetilde{x} \in U$ such that $\rho(\{\widetilde{x}\}, \{x\}) \in \mathcal{C}$ (and conversely for each $y \in U$ there exists a $\widetilde{y} \in u$ such that $\rho(\{y\}, \{\widetilde{y}\}) \in \mathcal{C}$). Because of the compatibility of all partitions in $\mathcal{C}$ such an $\widetilde{x} \in U$ (or a $\widetilde{y} \in u$) is uniquely given. Define $T = \{\langle \widetilde{x}, x\rangle; x \in u\}$. Then by 1.2.3 T has the required property. ■

Theorem 1.2.4 has the following, maybe surprising, consequence

1.2.5. Corollary. *Denote $\mathcal{B}_V$ the least σ-ring containing all Sd^*-classes and $\mathcal{B}_u$ the least σ-ring containing all subsets of u. Then ${}^*\mathcal{B}_V$ and $\mathcal{B}_u$ are isomorphic.*

2. Probability theory.

Denote $\mathcal{B}_Q$ the least σ-algebra containing all Sd-classes $Y \subseteq Q$. A function $P : \mathcal{B}_Q \to R$ is said to be a one-dimensional probability distribution if it is nonnegative, σ-additive and $P(Q) = 1$.

Let A be a nonempty Borel semiset and F a Borel approximating function. A function $X : A \to Q$ is said to be a random variable with respect to F if for each $B \in \mathcal{B}_Q$ $(X^{-1})''B$ is $F(A), F$-observable and the function $D : \mathcal{B}_Q \to R$ defined by $D(B) = m_{F(A),F}((X^{-1})''B)$ is σ-additive. The function D is said to be the probability distribution of X.

Obviously the probability distribution of each random variable is one-dimensional.

2.1. Proposition. *Let A be a Borel semiset having nonadditive cuts and $X : A \to Q$ be such a function that for each $B \in \mathcal{B}_Q$ $(X^{-1})''B \in \mathcal{B}$ holds. Then X is a random variable with respect to any BAF and its probability distribution does not depend on the choice of the BAF.*

PROOF : follows immediately from Theorem 1.1.1. ■

Particularly, if $u \neq \emptyset$ then any set function $x : u \to Q$ is a random variable with respeet to any BAF and its probability distribution does not depend on the choice of the BAF.

Let P and D be two one-dimensional probability distributions and S an indiscernibilty equivalence on Q. Denote φ the least σ-algebra containing the system $\{S''u; u \subset Q\}$. P and D are said to be indiscernible with respect to S if for each $B \in \varphi$, $P(B) = D(B)$.

2.2. Lemma. *Let P and D be two one-dimensional probability distributions and S an indiscernibility equivalence on Q. Then the necessary and sufficient condition for P and D to be indiscernible is that for each set $u \subset Q$ $P(S''u) = D(S''u)$ holds.*

PROOF : The system $\{S''u; u \subset Q\}$ generates the σ-algebra φ, hence the σ-additivity and finiteness of P and D imply their indiscernibility. ■

An indiscernibility equivalence S on Q is said to be an interval one if there exists a linear Sd-order $\leq_1$ such that there holds

$$(\forall x, y \in Q)(x \leq_1 y \ \&\ \neg xSy) \Rightarrow (\forall z \in S''\{x\})(z \leq_1 y).$$

The notion of interval indiscernibility equivalences has been firstly introduced in [W].

2.3. Theorem. *Let P be a one-dimensional probability distribution and S an interval indiscernibility equivalence on Q. Then there exists a set random variable x, whose probability distribution is indiscernible from P with respect to S. Moreover, if $v \subset Q$ is such a set that $S''v = Q$, then the random variable x can be constructed in such a way that* $\mathrm{rng}(x) \subseteq v$.

PROOF : follows immediately from Theorem 2.11, stated below. ∎

Since by proving of Theorem 2.11 we shall utilize only Lemma 2.10, in the proofs of 2.5 – 2.8 we shall also refer to 2.3.

In the sequel we shall assume $i \geq 2$.

Denote $\mathcal{B}^i_Q$ the least σ–algebra containing the system $\{E_1 \times \cdots \times E_i; (\forall 1 \leq k \leq i)(E_K \in \mathcal{B}_Q)\}$. For each $i > 1$ and each $1 \leq k \leq i$ define a function $\varphi_{ki} : Q^i \to Q$ by the formula $\varphi_{ki}(q_1, \dots, q_i) = q_k$.

A (not necessarily σ–additive) function P with $\mathrm{Dom}(P) \subseteq \mathcal{B}^i_Q$ and $\mathrm{Rng}(P) \subseteq R$ is said to be an i–dimensional probability distribution if it is nonnegative, for each $1 \leq k \leq i$ and each $B \in \mathcal{B}_Q$ $(\varphi^{-1}_{ki})''B \in \mathrm{Dom}(P)$ and functions $D_k : \mathcal{B}_Q \to R$ defined by $D_k(B) = P((\varphi^{-1}_{ki})''B)$ are one–dimensional probability distributions.

An i–dimensional probability distribution P is said to be full if $\mathrm{Dom}(P) = \mathcal{B}^i_Q$ and if it is σ–additive.

Let A be a nonempty Borel semiset and F be a BAF. A function $X : A \to Q^i$ is said to be an i–dimensional random vector with respect to F if for each $1 \leq k \leq i$ the functions $Y_k : A \to Q$ defined by $Y_k(y) = \varphi_{ki}(X(y))$ are random variables with respect to F. A function D, with $\mathrm{Dom}(D) \subseteq \mathcal{B}^i_Q$ and $\mathrm{Rng}(D) \subseteq R$, such that for each $B \in \mathcal{B}^i_Q$, $B \in \mathrm{Dom}(D)$ if and only if $(X^{-1})''B$ is $F(A), F$ – observable and for each $B \in \mathrm{Dom}(D)$, $D(B) = m_{F(A),F}((X^{-1})''B)$, is said to be the probability distribution of X.

Obviously, if X is an i–dimensional random vector, then its probability distribution is i–dimensional.

2.4 Remark. The need of the ability of modelling of random vectors, whose probability distributions are not full (so–called vectors of incompatible random variables), follows from some considerations on quantum physics. As well shall see in Theorems 2.5–2.7, in this theory it is possible to model random vectors with arbitrary (not necessarily full) probability distributions. Hence it is an alternative to probability theories utilized in quantum physics (e.g. quantum logic). Moreover, this theory enables us, in a natural way, to create a common look both at the world of the classical (Kolmogorov's) probability theory and at the world of elementary particles.

Let $(S_1, \dots, S_i)$ be any i–tuple of indiscernibility equivalences on Q. By $\times\{S_j; 1 \leq j \leq i\}$ we shall denote the indiscernibility equivalence S on Q^i for which $S''\{(q_1, \dots, q_i)\} = \{(s_1, \dots, s_i) \in Q^i; s_1 S_1 q_1 \ \& \ \cdots \ \& \ s_i S_i q_i\}$ holds for each $(q_1, \dots, q_i) \in Q^i$.

Let P and D be two i–dimensional probability distributions and $(S_1, \ldots, S_i)$ be any i–tuple of indiscernibility equivalences on Q. Denote $S = \times\{S_j; 1 \leq j \leq i\}$ and φ the least σ–algebra containing the system $\{S''(u_1, \ldots, u_i); (\forall 1 \leq j \leq i)(u_j \subset Q)\}$. P and D are said to be indiscernible with respect to S if for each $B \in \varphi$ there holds $B \in \mathrm{Dom}(P)$ iff $B \in \mathrm{Dom}(D)$ and if $B \in \mathrm{Dom}(P)$, then $P(B) = D(B)$.

2.5. Theorem. *Let P be an i–dimensional probability distribution, $(S_1, \ldots S_i)$ be an i–tuple of interval indiscernibility equivalences on Q. Then there exists a BAF G and a Borel i–dimensional random vector X with respect to G, whose probability distribution is, with respect to $S = \times\{S_j; 1 \leq j \leq i\}$, indiscernible from P.*

PROOF : For $1 \leq k \leq i$ denote D_k the one–dimensional probability distributions, defined by $D_k(B) = P((\varphi_{ki}^{-1})''B)$. Further denote $v_k \subset Q$ such sets that $S_k''v_k = Q$ and put $v = v_1 \times \cdots \times v_i$. Let $|v| = c$. Put $A = \cup\{c^{FN} \times \{n\}; n \in c\}$. Obviously A is a Borel semiset, having an additive cut. Let $\{y_n; n \in c\}$ be an enumeration of v. We will define a function $X : A \to v$ by $X^{-1}(y_n) = c^{FN} \times \{n\}$ for any $n \in c$. Hence X is a Borel function, and for each $B \in \mathcal{B}_Q^i, (X^{-1})''B \in \mathcal{B}$ and $|(X^{-1})''B| = c^{FN}$. An easy computation gives $|A| = c^{FN}$, too. By Theorem 2.3 there exist such random variables x_k with $\mathrm{rng}(x_k) \subseteq v_k$ that their probability distributions P_k are indiscernible, with respect to S_k, from D_k.

By Theorem 1.1.2 there exists a BAF G such that for each $B \in \mathcal{B}_Q$ and each $1 \leq k \leq i$ $(X^{-1})''((\varphi_{ki}^{-1})''B)$ is $G(A), G$ - observable, $P_k(B) = m_{G(A),G}((X^{-1})''((\varphi_{ki}^{-1})''B))$, and for each $B \in \varphi$, where φ is the σ–algebra corresponding to S, the class $(X^{-1})''B$ is $G(A), G$ - observable if and only if $B \in \mathrm{Dom}(P)$ and if $B \in \mathrm{Dom}(P)$, then $P(B) = m_{G(A),G}((X^{-1})''B)$. Hence X is a Borel i–dimensional random vector with respect to G and its probability distribution is indiscernible from P with respect to $\times\{S_j; 1 \leq j \leq i\}$. ∎

As one can easily check, using the construction of the proof of Theorem 2.5, we can prove the following

2.6. Theorem. *Let $(S_1, \ldots, S_i)$ be any i–tuple of interval indiscernibility equivalences on Q. Then there exists a Borel function X such that for each i–dimensional probability distribution P there exists a BAF F, for which there holds that X is an i–dimensional random vector with respect to F and its probability distribution is indiscernible form P with respect to $\times\{S_j; 1 \leq j \leq i\}$.*

2.7. Theorem. *There exists a function X such that for each i–dimensional probability distribution P there exists a BAF F such that X is an i–dimensional random vector with respect to F, whose probability distribution is P.*

PROOF : Again, we can use the construction of the proof of Theorem 2.5. We get a function Y with $\mathrm{Rng}(Y) = v \subset Q^i$ such that for each $B \in \mathcal{B}_Q^i, (Y^{-1})''B \in \mathcal{B}$ and $|(Y^{-1})''B| = A$, (A being an additive σ–cut). By Theorem 1.2.4 there exists a bijection $T : v \approx Q^i$ such that for each $U \subseteq v$, $\mathrm{Set}(U) \Leftrightarrow Sd_V^*(T''U)$. Hence, since $Sd_V \subset Sd_V^*$ (see [S–V] p. 115), for each $B \in \mathcal{B}_Q^i, (T^{-1})''B \in \mathcal{B}_Q^i$. Hence it is enough to put $X = T \circ Y$ and the assertion follows.

Theorems 2.6 and 2.7 have the following consequence for the one–dimensional case. ∎

2.8. Corollary. *(a) Let S be an interval indiscernibility equivalence on Q. Then there exists a Borel function X such that for each one-dimensional probability distribution P there exists a BAF F such that X is a random variable with respect to F, whose probability distribution is, with respect to S, indiscernible from P.*

(b) There exists a function X such that for each one-dimensional probability distribution P there exists a BAF F such that X is a random variable with respect to F, whose probability distribution is P.

2.9. Proposition. *Let $A \neq \emptyset$ be a Borel semiset, having nonadditive cuts, and $i \geq 2$. If $X_j : A \to Q$ for $1 \leq j \leq i$ are random variables with respect to a BAF, then the function $X : A \to Q^i$, defined by $X(x) = (X_1(x), \ldots, X_i(x))$ for $x \in A$, is an i-dimensional random vector with respect to any BAF, its probability distribution is full (i.e. σ-additive and having its domain $\mathcal{B}^i_Q$) and does not depend on the choice of the BAF.*

PROOF : Since X_j, for each $1 \leq j \leq i$, is a random variable, for each $B \in \mathcal{B}_Q, (X_j^{-1})''B \in \mathcal{B}$. Hence for each $B \in \mathcal{B}_Q$ and each $1 \leq j \leq i, (X^{-1})''((\varphi_{kj}^{-1})''B) \in \mathcal{B}$. Obviously, since X is a function and classes of the type $(\varphi_{kj}^{-1})''B$ for $B \in \mathcal{B}_Q$ and $1 \leq k \leq i$ generate the whole σ-algebra $\mathcal{B}^i_Q$, it holds that for each $B \in \mathcal{B}^i_Q, (X^{-1})''B \in \mathcal{B}$. Hence Theorem 1.1.1 implies the assertion of this proposition. ∎

The following lemma is a straightforward generalization of Lemma 2.2. Therefore its proof is omitted.

2.10. Lemma. *Let P and D be two full i-dimensional probability distributions and $(S_1, \ldots, S_i)$ be any i-tuple of indiscernibility equivalences on Q. Then the necessary and sufficient condition for P and D to be indiscernible is that for each i-tuple of subsets of Q $(u_1, \ldots, u_i), P(S_1''u_1 \times \cdots \times S_i''u_i) = D(S_1''u_1 \times \cdots \times S_i''u_i)$ holds.*

2.11. Theorem. *Let P be a full i-dimensional probability distribution and $(S_1, \ldots, S_i)$ be an i-tuple of interval indiscernibility equivalences on Q. Then there exists an i-dimensional random vector x, whose probability distribution is indiscernible form P with respect to $\times\{S_j; 1 \leq j \leq i\}$. Moreover, if $(v_1, \ldots, v_i)$ is any i-tuple of subsets of Q such that for each $1 \leq k \leq i$, $S_k''v_k = Q$, then the random vector x can be constructed to have $\mathrm{rng}(x) \subseteq v_1 \times \cdots \times v_i$.*

PROOF : Since $P(Q^i) = 1$ and P is σ-additive, for each $1 \leq k \leq i$ there exists an at most countable class $D_k \subset v_k$ such that

$$(\forall q \in Q)(P((\varphi_{ki}^{-1})''(S_k''\{q\})) > 0 \Rightarrow S_k''\{q\} \cap D_k \neq \emptyset) \ \& \\ \& \ (\forall x, y \in D_k)(x \neq y \Rightarrow \rceil x S_k y).$$

The indiscernibility equivalences S_k, for $1 \leq k \leq i$, are interval ones, hence there exists an i-tuple of linear Sd-orders $(\leq_1, \ldots, \leq_i)$ such that for each $1 \leq k \leq i$ there holds

$$(\forall x, y \in Q)(x \leq_k y \ \& \rceil x S_k y) \Rightarrow (\forall z \in S_k''\{x\})(z \leq_k y).$$

Let $W_k \subset v_k$ be a countable dense class in S_k, containing D_k, for any $1 \leq k \leq i$. For each $1 \leq k \leq i$ denote $\{z_{jk}; j \in FN\}$ a sequence of subsets of W_k such that for each j there holds $z_{jk} \mathrel{\widehat{\approx}} j+1, z_{jk} \subseteq z_{j+k,k}$ and $\bigcup\{z_{jk}; j \in FN\} = W_k$.Denote $z_j = z_{j1} \times \cdots \times z_{ji}$.

Fix a $j \geq 1$. Let $z_{jk} = \{y_{nk}; n \leq j\}$ and assume that for each $m, n \leq j, m \leq n$ implies $y_{mk} \leq_k y_{nk}$. Denote

$$\begin{aligned}(n,k\rangle &= \{q \in Q; (y_{n-1,k} <_k q \leq_k y_{nk} \ \& \ \rceil y_{n-1,k} S_k q) \vee \\ &\qquad \vee (q S_k y_{nk})\} \text{ for } 1 \leq n < j, \\ (0,k\rangle &= \{q \in Q; q \leq_k y_{0k} \vee q S_k y_{0k}\}, \\ (j,k\rangle &= \{q \in Q; y_{j-1,k} <_k q \ \& \ \rceil y_{j-1,k} S_k q\}.\end{aligned}$$

Take an infinite set u and a function $x_j : u \to z_j$ such that $m_{F(u),F}((x_j^{-1})''\{(y_{n_1,1}, \ldots, y_{n_i,i})\}) = P((n_1,1\rangle \times \cdots \times (n_i,i\rangle)$ for any $(n_1, \ldots, n_i) \in \{0,1,\ldots,j\}^i$ and a fixed BAF F. Obviously, since P is full, such s function x_j does exist for each $j \geq 1$.

Prolong the sequence $\{x_j; j \geq 1\}$. Since for each $j \geq 1 \operatorname{rng}(x_j) \subseteq v_1 \times \cdots \times v_i$, there exists a $c \in N \setminus FN$ such that $\operatorname{rng}(x_c) \subseteq v_1 \times \cdots \times v_i$. Moreover, by the construction of $\{x_j; j \geq 1\}$, x_c can be chosen in such a manner that

(1) since $D_k \subseteq \bigcup\{z_{jk}; j \in FN\}$, for each $q \in Q$ and each $1 \leq k \leq i$

$$m_{F(u),F}((x_c^{-1})''((\varphi_{ki}^{-1})''(S_k''\{q\}))) = P((\varphi_{ki}^{-1})''(S_k''\{q\})),$$

(2) since $W_k = \bigcup\{z_{jk}; j \in FN\}$ is dense in S_k, and because of (1), for all i–tuples of rational numbers $(q_1, \ldots, q_i)$ and $(t_1, \ldots, t_i)$ such that $q_k \leq_k t_k$ for each $1 \leq k \leq i$

$$\begin{aligned}m_{F(u),F}((x_c^{-1})''((S_1''\{s \in Q; q_1 \leq_1 s \leq_1 t_1\}) \times \cdots \times \\ \times (S_i''\{s \in Q; q_i \leq_i s \leq_i t_i\}))) = \\ = P((S_1''\{s \in Q; q_1 \leq_1 s \leq_1 t_1\}) \times \cdots \times (S_i''\{s \in Q; q_i \leq_i s \leq_i t_i\})).\end{aligned}$$

By Lemma 2.10, we are to prove that for each i–tuple of subsets of $Q(w_1, \ldots, w_i)$, $P(S_1''w_1 \times \cdots \times S_i''w_i) = m_{F(u),F}((x_c^{-1})''(S_1''w_1 \times \cdots \times S_i''w_i))$ holds. Since each of the indiscernibility equivalence S_k is ordered, $S_k''w_k$ is the union of an, at most countable, system of classes of the type $S_k''\{s \in Q; q_k \leq_k s \leq_k t_k\}$ for $q_k, t_k \in Q$. Hence, because of the σ–additivity of P and (2), $P(S_1''w_1 \times \cdots \times S_1''w_i) = m_{F(u),F}((x_c^{-1})''(S_1''w_1 \times \cdots \times S_i''w_i))$ holds for each i–tuple of subsets of $Q \ (w_1, \ldots, w_i)$. ∎

2.12. Remark. Theorem 2.11, roughly speaking, says that the whole Kolmogorov's probability theory is reduced to Laplace's scheme.

In the remainder of this paper let b be a fixed infinite natural number. Denote I the class of all nonempty finite subsets of b and $a = \{|u| \times \{u\}; u \in I\}$. Define for each $i > 1$ and $1 \leq k \leq i$ a function $\Psi_{ik} : Q^i \to Q^{i-1}$ by $\Psi_{ik}((q_1, \ldots, q_i)) = (q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_i)$. Further take a function $t : \bigcup a \to b$ such that for each $u \in I$ and each $k \in |u|, t(k,u)$ is the k–th element of u in the natural order of N.

For each $u \in I$ let P_u be a $|u|$–dimensional probability distribution. Then the family $\mathcal{P} = \{P_u; u \in I\}$ is said to be a b–consistent system of finitely–dimensional probability distributions if for each $u \in I$ and each $c \in b, c \notin u$, there holds

(1) $B \in \mathrm{Dom}(P_u)$ if and only if $(\Psi^{-1}_{|u|+1,k})''B \in \mathrm{Dom}(P_{u\cup\{c\}})$, where $k \leq |u|$ is such that $t(k, u \cup \{c\}) = c$.

(2) if $B \in \mathrm{Dom}(P_u)$, then $P_u(B) = P_{u\cup\{c\}}((\Psi^{-1}_{|u|+1,k})''B)$.

Let $A \neq \emptyset$ be a Borel semiset and F be a BAF. A function $X : A \times b \to Q$ is said to be a stochastic process with respect to F, if for each $u \in I$ the function $Y_u : A \to Q^{|u|}$, defined by $Y_u(w) = (X(w, t(0, u)), \ldots, X(w, t(|u| - 1, u)))$, is a $|u|$–dimensional random vector with respect to F. The system $\mathcal{P} = \{P_u; u \in I\}$, where P_u is the probability distribution of the corresponding random vector Y_u, is said to be the system od finitely–dimensional probability distributions of X. For each $w \in A$ the function $T : b \to Q$, defined by $T(c) = X(w, c)$, is called the trajectory of X.

By an easy consideration we can get that, if X is a stochastic process with respect to a BAF, then its system of finitely–dimensional probability distributions is b–consistent.

Let $\{P_u; u \in I\}$ and $\{D_u; u \in I\}$ be two b–consistent systems of finitely–dimensional probability distributions and $\{S_n; n \in b\}$ be a system of indiscernibility equivalences on Q. $\{P_u; u \in I\}$ and $\{D_u; u \in I\}$ are said to be indiscernible with respect to $\{S_n; n \in b\}$, if for each $u \in I$, P_u and D_u are indiscernible with respect to $\times\{S_n; n \in u\}$.

2.13. Theorem. *Let $\{P_u; u \in I\}$ be a b-consistent system of finitely-dimensional probability distributions and $\{S_n; n \in b\}$ be a system of interval indiscernibility equivalences on Q such that there holds $(\exists w \subset Q)(\forall n \in b)(S''_n w = Q)$. Then there exists a BAF F and a Borel stochastic process X with respect to F, whose system of finitely-dimensional probability distributions is, with respect to $\{S_n; n \in b\}$, indiscernible from $\{P_u, u \in I\}$. Moreover, X can be constructed in such a way that its trajectories are sets.*

PROOF : Denote $c = |w|$ and $A = \{f; f : b \to c^{FN}\}$. An easy computation gives that for each j, each $\{t_k \in b; k \leq j\}$ and $\{q_k \in c^{FN}; k \leq j\}$

$$|A| = c^{b\cdot FN} = |\{f \in A; (\forall k \leq j)(f(t_k) = q_k)\}| \tag{1}$$

holds. Define a function $X : A \times b \to Q$ by $X(f, n) = f(n)$. Obviously X is a Borel function. By Theorem 2.3, for each $n \in b$ there exists a random variable x_n with $\mathrm{rng}(x_n) \subseteq w$, whose probability distribution D_n is, with respect to S_n, indiscernible from $P_{\{n\}}$. By Theorem 1.1.2, (1) implies the existence of a BAF F such that

1.) for each $n \in b$ and each $B \in \mathcal{B}_Q$, $\{f \in A; f(n) \in B\}$ is $F(A), F$ - observable and $m_{F(A),F}(\{f \in A; f(n) \in B\}) = D_n(B)$,

2.) for each $u \in I$ and each $B \in \varphi_u$, where φ_u is the σ–algebra corresponding to $\times\{S_n; n \in u\}$, the class $E = \{f \in A; (f(t(0, u)), \ldots, f(t(|u| - 1, u))) \in B\}$ is $F(A), F$ - observable if and only if $B \in \mathrm{Dom}(P_u)$, and if E is $F(A), F$ - observable, then $m_{F(A),F}(E) = P_u(B)$.

Hence X is the searched Borel stochastic process. ■

As one can easily check, using the construction of the proof of Theorem 2.13, we can prove the following

2.14. Theorem. *Let $\{S_n; n \in b\}$ be a system of interval indiscernibility equivalences on Q such that there holds $(\exists w \subset Q)(\forall n \in b)(S_n'' w = Q)$. Then there exists a Borel function X such that for each b-consistent system of finitely dimensional probability distributions $\mathcal{P}$ there exists a BAF F such that there holds: X is a stochastic process with respect to F, whose system of finitely-dimensional probability distributions is indiscernible from $\mathcal{P}$ with respect to $\{S_n; n \in b\}$ and the trajectories of X are sets.*

2.15. Theorem. *There exists a function X such that for each b-consistent system of finitely-dimensional probability distributions $\mathcal{P}$ there exists a BAF F such that X is a stochastic process with respect to F, whose system of finitely-dimensional probability distributions is $\mathcal{P}$, and whose trajectories are sets.*

PROOF : Denote $U = \{f; f : b \to Q\}$. Obviously U is an Sd-class and by Theorem 1.2.4 there exists a bijective function $T : b \to U$ such that for each $E \subseteq b, \mathrm{Set}(E) \Leftrightarrow Sd_V^*(T''E)$.

Put $A = \bigcup\{b^{FN} \times \{n\}; n \in b\}$ and take a function $Y : A \to b$ such that for each $n \in b, Y^{-1}(n) = b^{FN} \times \{n\}$. Hence $|A| = b^{FN} = |Y^{-1}(n)|$ for each $n \in b$. Define the function $X : A \times b \to Q$ by $X(w,t) = T(Y(w))''\{t\}$ for each $w \in A$ and $t \in b$. Obviously X has the required properties. ■

2.16. Remark. From the physical point of view we can consider a system of probability distributions of a stochastic process to be the result of a measurement, running in time, and the Borel approximating function to be the way of the measurement. Then Theorems 2.14 and 2.15 say that measuring some phenomena, in general, arbitrary results can be obtained in dependence of the way of measurement.

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(*Received* December 22,1988)