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Global solution to the ideal compressible heat conductive multipolar fluid

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Dedicated to the memory of Svatopluk Fučík

Abstract. Global existence of solutions to the initial boundary value problem in two or three dimensional bounded domains to the system of partial differential equations for ideal viscous compressible heat conductive multipolar fluid for the polarity $k \geq 4$ is proved. Some other properties as cavitation of density and uniqueness are discussed.

Keywords: viscous multipolar fluids, compressible fluids, initial boundary value problem, existence, uniqueness

Classification: 35Q20, 76N10

I. Introduction.

Presented paper is closely related to [4], [5] where we have investigated the global solvability to the isothermal and barotropic multipolar gas. In this paper we will deal with the ideal heat conductive viscous compressible multipolar flow. The physical backround to the multipolar fluids is studied in [6]; here we recall only some important results. We suppose the polarity $k \geq 4$, a.i. besides the usual stress tensor τ_{ij}^V we consider stress tensors $\tau_{ii_1j}^V, \dots, \tau_{ii_1}^V \dots i_kj$. Navier–Stokes equations will be of $2k$–th order. We assume the linear relation between stress tensors and derivatives of the velocity field. The condition $k \geq 4$ enables us to prove the global solvability to the evolution problem. We get some regularity properties of density (noncavitation included) and regularity of velocity and temperature up to the strong solution. We prove also uniqueness.

II. Formulation to the problem.

The state equation to ideal gas reads

$$p = R\rho\theta, \tag{2.1}$$

where p is the pressure, R universal gas constant, ρ density and θ temperature. We suppose the standard stress tensor

$$\tau_{ij} = -p\delta_{ij} + \tau_{ij}^V \tag{2.2}$$

and higher order stress tensors

$$\tau_{ii_1\dots i_mj}^V, \quad 1 \leq m \leq k, \quad k \geq 4, \tag{2.3}$$

where

$$\tau^V_{ii_1\dots i_m j}(v) = a^m_{iji_1\dots i_m rs} e_{rs}(v) + \sum_{l=2}^{2k-1} a^m_{iji_1\dots i_m j_1\dots j_l s} \frac{\partial^l w_s}{\partial x_{j_1}\dots\partial x_{j_l}}$$

for $m = 0,\dots,k-1, e_{ij}(v) = \frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})$; the stress tensor $\tau^V_{ii_1\dots i_k j}$ is equal to zero by definition. The coefficients of linear dependence are real constants such that $\tau^V_{ij}, \frac{\partial}{\partial x_j}\tau^V_{ii_1 j}$ are symmetric and $\tau^V_{ii_1\dots i_m j}$ are symmetric in $i_1,\dots,i_m$. Moreover, due to the Clausius–Duhem inequality we suppose that

$$\tau^V_{ij}(v)e_{ij}(v) + \sum_{m=1}^{k-1} \frac{\partial}{\partial x_j}(\tau^V_{ii_1\dots i_m j}(v)\frac{\partial^m v_i}{\partial x_{i_1}\dots\partial x_{i_m}}) \geq 0. \tag{2.4}$$

We put

$$\langle v, w\rangle = A^1_{iji_1j_1} e_{ii_1}(v)e_{jj_1}(w) + \\ + \sum_{m=2}^{k} A^m_{iji_1\dots i_m j_1\dots j_m} \frac{\partial^m v_i}{\partial x_{i_1}\dots\partial x_{i_m}} \frac{\partial^m w_j}{\partial x_{j_1}\dots\partial x_{j_m}}. \tag{2.5}$$

We suppose that the coefficients $A^m_{iji_1\dots i_m j_1\dots j_m}(m = 1,\dots,k)$ ar symmetric in (i,j) and in $(i_1,\dots,i_m),(j_1,\dots,j_m)$ and

$$\begin{gathered} A^1_{iji_1j_1}\frac{\partial v_i}{\partial x_{j_1}}\frac{\partial v_j}{\partial x_{j_1}} \geq \alpha e_{ij}(v)e_{ij}(v), \\ \sum_{m=2}^{k} A^m_{iji_1\dots i_m j_1\dots j_m} J^i_{i_1\dots i_m} J^j_{j_1\dots j_m} \geq \alpha \sum_{m=2}^{k} J^i_{i_1\dots i_m} J^i_{i_1\dots i_m} \end{gathered} \tag{2.6}$$

for some $\alpha > 0$ and for every set of real vectors $(J^i_{i_1\dots i_m})$, $m = 2,\dots,k$. For every $v, w \in V = W^{1,2}_0(\Omega, R^N) \cap W^{k,2}(\Omega, R^N)$ we define $((v,w)) = \int_\Omega \langle v,w\rangle\, dx$.

The tensors $\tau^V_{ij}, \tau^V_{ii_1j}, \dots, \tau^V_{ii_1\dots i_k j}$ are supposed to be such that for $v, w \in V \cap W^{2k,2}(\Omega, R^N)$, it holds

$$((v,w)) = -\int_\Omega \frac{\partial}{\partial x_j}(\tau^V_{ij}(v))w_j\, dx + [[v,w]], \tag{2.7}$$

$$\text{where } [[v,w]] = \sum_{m=1}^{k-1}\int_{\partial\Omega} \tau^V_{ii_1\dots i_m j}(v)\frac{\partial^m w_i}{\partial x_{i_1}\dots\partial x_{i_m}}\nu_j\, dS.$$

Using Green formula, one gets from (2.5), (2.7)

$$\frac{\partial}{\partial x_j}(\tau^V_{ij}(v)) = \\ = \sum_{m=1}^{k}(-1)^{m+1} A^m_{iji_1\dots i_m j_1\dots j_m} \frac{\partial^{2m} v_j}{\partial x_{i_1}\dots\partial x_{i_m}\partial x_{j_1}\dots\partial x_{j_m}} \tag{2.8}$$

and

$$(2.9)\quad [[v,w]] = \sum_{m=1}^{k}\sum_{s=1}^{m-1}(-1)^{s+1}\int_{\partial\Omega} A^m_{iji_1\dots i_m j_1\dots j_m}\frac{\partial^{m+s}v_i}{\partial x_{i_1}\dots\partial x_{i_m}\partial x_{j_1}\dots\partial x_{j_s}}\frac{\partial^{m-s-1}w_j}{\partial x_{j_{s+2}}\dots\partial x_{j_m}}\nu_{j_{s+1}}\,dS,$$

which has to be satisfied for every $w \in V \cap W^{2k,2}(\Omega, R^N)$. Further we put

$$\langle\langle v,w\rangle\rangle = \tau^V_{ij}(v)e_{ij}(w) + \sum_{m=1}^{k-1}\frac{\partial}{\partial x_j}(\tau^V_{ii_1\dots i_m j}(v)\frac{\partial^m w_i}{\partial x_{i_1}\dots\partial x_{i_m}}).$$

It follows form (2.4) that the form $\langle\langle v,w\rangle\rangle$ contents the derivatives of v resp. w at most of order k.

Let Ω be a bounded domain in $R^N (N = 2,3)$ with smooth boundary $\partial\Omega$. We denote $I = (0,T)$. We shall study in the time cylinder $Q_T = I \times \Omega$ the following system of partial differential equations for density ρ, temperature θ and velocity v given by continuity equation

$$(2.10)\qquad \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0,$$

momentum equations

$$(2.11)\qquad \frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j + p(\rho,\theta)\delta_{ij} - \tau^V_{ij}(v)) = \rho F_i$$

and energy equation

$$(2.12)\qquad \begin{aligned}&\frac{\partial}{\partial t}(c_v\rho\theta + \frac{1}{2}\rho|v|^2) + \frac{\partial}{\partial x_j}(c_p\rho\theta v_j + \frac{1}{2}\rho|v|^2 v_j)-\\ &-\frac{\partial}{\partial x_j}(\tau^V_{ij}(v)v_i + \sum_{m=1}^{k}\tau^V_{ii_1\dots i_m j}(v)\frac{\partial^m v_i}{\partial x_{i_1}\dots\partial x_{i_m}}) =\\ &= \lambda\frac{\partial^2\theta}{\partial x_j\partial x_j} + \rho F_i v_i\end{aligned}$$

or

$$(2.13)\qquad \frac{\partial}{\partial t}(c_v\rho\theta) + \frac{\partial}{\partial x_j}(c_v\rho\theta v_j) + R\rho\theta\frac{\partial v_j}{\partial x_j} - \lambda\frac{\partial^2\theta}{\partial x_j\partial x_j} = \langle\langle v,v\rangle\rangle.$$

The coeffitients c_v (specific heat at constant volume), c_p (specific heat at constant pressure), λ (heat conductivity) are positive constants.

Besides initial conditions

$$\rho(0) = \rho_0, \quad v(0) = v_0, \quad \theta(0) = \theta_0, \tag{2.14}$$

we consider boundary conditions

$$v = 0, \frac{\partial \theta}{\partial \nu} = 0 \quad \text{on } I \times \partial\Omega \tag{2.15}$$

(ν denotes the outer normal to $\partial\Omega$) and

$$[[v, w]] = 0 \quad \text{on } I \times \partial\Omega \text{ for every } w \in V \cap W^{2k,2}(\Omega, R^N). \tag{2.16}$$

Due to the strong ellipticity of the form $((v, w))$ (see (2.6)) we obtain

$$((w, w)) \geq \beta \|w\|^2_{W^{k,2}(\Omega, R^N)}, \quad \beta > 0; \tag{2.17}$$

hence $((v, w))$ is a V–coercive bilinear form on V. We can give the weak formulation of (2.11), (2.13) as follows

$$\int_0^T \int_\Omega \frac{\partial}{\partial t}(\rho v_i) w_i \, dx\, dt - \int_0^T \int_\Omega (\rho v_i v_j + p(\rho, \theta)\delta_{ij}) \frac{\partial w_i}{\partial x_j}) \, dx\, dt + \\ + \int_0^T ((v, w))\, dt = \int_0^T \int_\Omega \rho F_i w_i \, dx\, dt \tag{2.18}$$

for every $w \in L^2(I, V)$;

$$-c_v \int_0^T \int_\Omega \rho\theta \frac{\partial \eta}{\partial t} \, dx\, dt - c_v \int_\Omega \rho_0 \theta_0 \eta(0) \, dx + \\ + \lambda \int_0^T \int_\Omega \frac{\partial \theta}{\partial x_j} \frac{\partial \eta}{\partial x_j} \, dx\, dt - c_v \int_0^T \int_\Omega \rho\theta v_j \frac{\partial \eta}{\partial x_j} \, dx\, dt + \\ + R \int_0^T \int_\Omega \rho\theta \frac{\partial v_j}{\partial x_j} \eta \, dx\, dt = \int_0^T \int_\Omega \langle\langle v, v\rangle\rangle \eta \, dx\, dt \tag{2.19}$$

for every $\eta \in \mathcal{C}^\infty(\overline{Q}_T), \eta(T) = 0$.

We recall one very useful lemma on regularity to the elliptic systems.

2.20 Lemma. *The problem*

$$((v, w)) = \int_\Omega f_i w_i \, dx \quad \textit{for every } w \in V, f \in L^2(\Omega, R^N)$$

has a solution $v \in W^{2k,2}(\Omega, R^N)$ *such that*

$$\|v\|_{W^{2k,2}(\Omega, R^N)} \leq c\|f\|_{L^2(\Omega, R^N)}, c > 0.$$

For the proof see [4], [7].

2.21 Example. We give one example of viscous multipolar fluid compatible with thermodynamics, which satisfies (2.4), (2.7). We consider ideal gas, e.g. the state equation (2.1) holds. We introduce internal energy and entropy per unit volume

$$e(\rho,\theta) = c_v\rho\theta \tag{2.22}$$

$$\kappa(\rho,\theta) = c_v ln\theta - Rln\rho \tag{2.23}$$

and stress tensors

$$\tau_{ii_1\dots i_m j} = \tau^E_{ii_1\dots i_m j} + \tau^V_{ii_1\dots i_m j}; \quad m = 0,\dots,k, \tag{2.24}$$

where

$$\tau^E_{ij}(\rho,\theta) = -R\rho\theta\delta_{ij}, \quad \tau^E_{ii_1\dots i_m j} = 0 \quad \text{for } 1 \le m \le k \tag{2.25}$$

and

$$\begin{gathered}\tau^V_{iji_1\dots i_m j}(v) = \sum_{r=m}^{k-1}(-1)^{r+m}\Delta^{r-m}\frac{\partial^m q^r_{ij}}{\partial x_{i_1}\dots\partial x_{i_m}},\\ 0 \le m \le k-1, \quad \tau^V_{iji_1\dots i_k} = 0,\end{gathered} \tag{2.26}$$

where $q^r_{ij} = \kappa_r \frac{\partial v_l}{\partial x_l}\delta_{ij} + 2\mu_r e_{ij}(v)$; $\kappa_s, \mu_s (s = 0,\dots,k-1)$ are constants such that $\kappa_r \ge -\frac{2}{3}\mu_r, \mu_r > 0 (r = 0,\dots,k-2)$ and $\kappa_{k-1} > -\frac{2}{3}\mu_{k-1}, \mu_{k-1} > 0$.

It is only a subject of standard computation to verify that the fluid described by the constitution laws (2.1), (2.22)-(2.26) satisfies (2.4), (2.7) with

$$((v,w)) = \int_\Omega \sum_{m=0}^{k-1} \frac{\partial^m q^m_{ij}(v)}{\partial x_{i_1}\dots\partial x_{i_m}} \frac{\partial^m e_{ij}(w)}{\partial x_{i_1}\dots\partial x_{i_m}};$$

consequently Clausius–Duhem inequality is satisfied (see [6]).

III. Modified Galerkin method.

Let $\{w^r\}_{r=1}^{+\infty}$ be an orthonormal system in V given by solving the following eigenvalue problem

$$((v,w^r)) = \lambda_r \int_\Omega v_i w^r_i \, dx \quad \text{for every } v \in V \quad (\lambda_1 < \lambda_2 < \dots). \tag{3.1}$$

From the regularity to the elliptic systems (see 2.20)

$$w^r \in C^\infty(\overline{\Omega}, R^N) \tag{3.2}$$

Let $\gamma = (\gamma_1, \dots, \gamma_n) \in C^1(\overline{I})\,(n = 1, 2, \dots)$ and put $v^n(t) = \sum_{r=1}^{n} \gamma_r(t) w^r$. Let

$$\rho_0 \in C^1(\overline{\Omega}), \rho_0 > 0 \text{ in } \overline{\Omega}. \tag{3.3}$$

Let $\rho^n \in C^1(\overline{Q}_T)$ be a solution to the equation

$$\frac{\partial \rho^n}{\partial t} + \frac{\partial}{\partial x_i}(\rho^n v_i^n) = 0 \tag{3.4}$$

with initial condition (3.3). It reads

$$\rho^n(t, x) = \rho_0(y) \exp\Big(-\int_0^t \frac{\partial}{\partial x_j} v_j^n(\tau, x^n(\tau))\, d\tau\Big), \tag{3.5}$$

where $y = x^n(0), x = x^n(t)$; the characteristics $x^n(t)$ are solutions to the problem

$$x^n(t) = v^n(t, x^n(t)),\ t \in [0, T),\ x^n(0) = y \in \overline{\Omega}. \tag{3.6}$$

For every $t \in \overline{I}, y \to x^n(t)$ is a diffeomorphism of $\overline{\Omega}$ onto $\overline{\Omega}$.

Let

$$\theta_0 \in L^2(\Omega),\ \theta_0 \geq 0 \text{ a.e. in } \Omega \tag{3.7}$$

and $\theta_0^n \in C^2(\overline{\Omega})$, $\dfrac{\partial \theta_0^n}{\partial \nu} = 0$ on $\partial\Omega$, $\theta_0^n > 0$ in $\overline{\Omega}, \theta_0^n \to \theta_0$ strongly in $L^2(\Omega)$. We look for $\theta^n \in C^1(\overline{Q}_T) \cap C^0(\overline{I}, C^2(\overline{\Omega}))$ the classical solution to the following parabolic problem

$$\frac{\partial}{\partial t}(c_v \rho^n \theta^n) + \frac{\partial}{\partial x_j}(c_v \rho^n \theta^n v_j^n) + R\rho^n \theta^n \frac{\partial v_j^n}{\partial x_j} - \lambda \frac{\partial^2 \theta^n}{\partial x_j \partial x_j} = \langle\langle v^n, v^n \rangle\rangle, \tag{3.8}$$

$$\theta^n(0) = \theta_0^n,\ \frac{\partial \theta^n}{\partial \nu} = 0 \quad \text{in } I \times \partial\Omega. \tag{3.9}$$

Due to Ladyzhenskaya [2] such solution exists.

Let $\overline{v}^n$ be the Galerkin solution to the problem

$$\begin{aligned} &\int_\Omega \Big(\rho^n \frac{\partial \overline{v}_i^n}{\partial t} + \rho^n v_j^n \frac{\partial \overline{v}_i^n}{\partial x_j} + R\frac{\partial}{\partial x_j}(\rho^n \theta^n) - \rho^n F_i\Big) w_i^r\, dx = \\ &= -((\overline{v}^n, w^r)); \quad r = 1, \dots, n; \qquad \overline{v}^n(t) = \sum_{r=1}^{n} \overline{\gamma}_r(t) w^r. \end{aligned} \tag{3.10}$$

We suppose

$$v_0 \in L^2(\Omega, R^N), \tag{3.11}$$

$$F \in L^\infty(Q_T, R^N), \tag{3.12}$$

$$\int_\Omega \overline{v}_i^n(0) w_i^r\, dx = \int_\Omega v_{0i} w_i^r\, dx. \tag{3.13}$$

If we start in the sphere

$$\max_{[0,\sigma]} |\gamma_r(t) - \gamma_r(0)| \le 1, \quad r = 1, \dots, n, \tag{3.14}$$

we get

$$\max_{[0,\sigma]} |\overline{\gamma}_r(t) - \gamma_r(0)| \le 1, \quad r = 1, \dots, n, \tag{3.15}$$

$$\max_{[0,\sigma]} \left|\frac{d}{dt}\dot{\overline{\gamma}}_r(t)\right| \le K(\sigma), K(\sigma) > 0 \tag{3.16}$$

provided $\sigma > 0$ is sufficiently small. The mapping $\overline{\Gamma} = (\overline{\gamma}_1, \dots, \overline{\gamma}_n) \to \Gamma = (\gamma_1, \dots, \gamma_n)$ is continuous and compact in $C^0([0,\sigma], R^n)$; hence according to Schauder fixed point theorem one gets $\overline{\Gamma} = \Gamma$. For more detailed proof see [8].

Lemma 3.17. *For* $0 \le t < \sigma$

$$\frac{1}{2}\int_{\Omega_t} \rho^n |v^n|^2 \, dx - \frac{1}{2}\int_{\Omega} \rho_0 |v_0|^2 \, dx + \tag{3.18}$$
$$+ c_v \int_{\Omega_t} \rho^n \theta^n \, dx - c_v \int_{\Omega} \rho_0 \theta_0^n \, dx = \int_0^t \int_{\Omega} \rho^n F_i v_i^n \, dx \, d\tau,$$

$$\frac{1}{2}\int_{\Omega_t} \rho^n |v^n|^2 \, dx - \frac{1}{2}\int_{\Omega} \rho_0 |v_0|^2 \, dx + \int_0^t ((v^n, v^n)) \, d\tau = \tag{3.19}$$
$$= \int_0^t \int_{\Omega} \rho^n F_i v_i^n \, dx \, d\tau + \int_0^t \int_{\Omega} R \rho^n \theta^n \frac{\partial v_j^n}{\partial x_j} \, dx \, d\tau,$$

$$\int_{\Omega_t} \rho^n \, dx = \int_{\Omega} \rho_0 \, dx. \tag{3.20}$$

3.21 Consequence. The solution (ρ^n, v^n, θ^n) is uniquely defined in $\overline{I} \times \overline{\Omega}$ and 3.17 holds for every $t \in [0, T)$.

3.22 Lemma. $\theta^n \ge 0$ *in* $\overline{Q}_T$.

PROOF : Put $\widehat{\theta}^n(t,x) = (\exp(-\Lambda t))\theta^n(t,x), \Lambda > 0$. Let us suppose that $\widehat{\theta}^n$ is minimal in $(t_0, x_0) \in \overline{Q}_T$, hence $\frac{\partial \widehat{\theta}^n}{\partial x_j}(t_0, x_0) = 0$, $\frac{\partial^2 \widehat{\theta}^n}{\partial x_j \partial x_j}(t_0, x_0) \ge 0$, $\frac{\partial \widehat{\theta}^n}{\partial t}(t_0, x_0) \le \le 0$. From (3.8) we get $\rho^n \widehat{\theta}^n (\Lambda c_v + R\frac{\partial v_j^n}{\partial x_j})(t_0, x_0) \ge 0$, hence $\theta^n(t_0, x_0) \ge 0$. ■

3.23 Lemma. *Let* $k \ge 4, \rho_0 \in C^{k-3}(\overline{\Omega}), \rho_0 > 0$ *in* $\overline{\Omega}$. *Then*

$$(\min_{x \in \overline{\Omega}} \rho_0(x)) \exp(-c_1 t^{\frac{1}{2}}) \le \rho^n(t,x) \le \tag{3.24}$$
$$\le (\min_{x \in \overline{\Omega}} \rho_0(x)) \exp(+c_1 t^{\frac{1}{2}}), c_1 > 0, (t,x) \in \overline{Q}_T,$$

$$\left\| \frac{\partial^s \rho^n}{\partial x_1^{s_1} \dots \partial x_N^{s_N}} \right\|_{L^\infty(Q_T)} \le c_2, c_2 > 0 \quad \textit{for } 0 \le s \le k-3, \tag{3.25}$$

and any multiindex $(s^1, \ldots, s^N)$ *such that* $s = s^1 + \cdots + s^N$,

$$\|\frac{\partial^{s+1}\rho^n}{\partial t \partial x_1^{s^1}\ldots\partial x_N^{s^N}}\|_{L^2(I,L^\infty(\Omega))} \le c_2 \quad \textit{for } 0 \le s \le k-4. \tag{3.26}$$

PROOF : The assertions (3.24)-(3.26) follow from (3.5), 3.17. For details see the paper by Nečas, Novotný, Šilhavý [4]. ∎

3.27 Remark. It is a consequence of 3.17 and 3.23 that

$$\|\rho^n|v^n|^2\|_{L^\infty(I,L^1(\Omega))} \le K, \quad K > 0, \tag{3.28}$$

$$\|\rho^n\theta^n\|_{L^\infty(I,L^1(\Omega))} \le K, \tag{3.29}$$

$$\|v^n\|_{L^2(I,W^{k,2}(\Omega,R^N))} \le K, \tag{3.30}$$

$$\|v^n\|_{L^\infty(I,L^2(\Omega,R^N))} \le K. \tag{3.31}$$

3.32 Lemma. *Let* $v_0 \in W^{k,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$. *Then*

$$\begin{aligned} &\|\frac{\partial v^n}{\partial t}\|_{L^2(Q_T,R^N)} + \sup_{[0,T]}((v^n, v^n)) \le \\ &\le c_3(1 + \|\theta^n\|^2_{L^2(I,W^{1,2}(\Omega))}),\ c_3 > 0, \end{aligned} \tag{3.33}$$

$$\|v^n\|_{L^2(I,W^{2k,2}(\Omega,R^N))} \le c_4(1 + \|\theta^n\|^2_{L^2(I,W^{1,2}(\Omega))}), c_4 > 0. \tag{3.34}$$

PROOF : We use in (3.10) the test function $\frac{\partial v^n}{\partial t}$; after some computation applying 3.23, (3.30), (3.31) we get (3.33) . Due to 2.20 one gets (3.34), too. ∎

3.35 Lemma.

$$\|\theta^n\|_{L^2(I,W^{1,2}(\Omega,R^N))} + \|\theta^n\|_{L^\infty(I,L^2(\Omega))} \le c_5, \quad c_5 > 0.$$

PROOF : We multiply (3.8) by θ^n and integrate over Ω

$$\begin{aligned} &\frac{\partial}{\partial t}(\frac{1}{2}c_v \int_\Omega \rho^n(\theta^n)^2\,dx) + \lambda\int_\Omega \frac{\partial\theta^n}{\partial x_j}\frac{\partial\theta^n}{\partial x_j}\,dx = \\ &= -R\int_\Omega \rho^n\frac{\partial v_j^n}{\partial x_j}(\theta^n)^2\,dx + \int_\Omega ((v^n, v^n))\theta^n\,dx. \end{aligned}$$

R.h.s. of this equation is bounded by

$$\begin{aligned} &a_1(t)\|\theta^n(t)\|^2_{L^2(\Omega)} + (\int_{\Omega_t} ((v^n, v^n))^2\,dx)^{\frac{1}{2}}\|\theta^n(t)\|_{L^2(\Omega)}, \\ &\text{where } a_1(t) = R\|\rho^n\|_{L^\infty(Q_t)}\|v^n(t)\|_{W^{k,2}(\Omega,R^N)}. \end{aligned} \tag{3.36}$$

Due to

$$\|v^n\|_{C^k(\overline{\Omega},R^N)} \leq \varepsilon_1 \|v^n\|_{W^{2k,2}(\Omega,R^N)} + K_1(\varepsilon_1)\|v^n\|_{L^2(\Omega,R^N)} \tag{3.37}$$

for every $\varepsilon_1 > 0$, $K_1(\varepsilon_1) > 0$ (see e.g. Lions [3])

$$\begin{aligned}(\int_{\Omega_t} \langle\langle v^n, v^n\rangle\rangle^2 \, dx)^{\frac{1}{2}} &\leq c_6\varepsilon_1 \|v^n(t)\|_{W^{k,2}(\Omega,R^N)} \|v^n(t)\|_{W^{2k,2}(\Omega,R^N)} + \\ &+ K_2(\varepsilon_1)\|v^n(t)\|_{W^{k,2}(\Omega,R^N)}, c_6, K_2(\varepsilon_1) > 0;\end{aligned}$$

hence according to 3.32 and Young inequality

$$\begin{aligned}&(\int_{\Omega_t} \langle\langle v^n, v^n\rangle\rangle^2 \, dx)^{\frac{1}{2}} \|\theta^n(t)\|_{L^2(\Omega)} \leq \\ &\leq c_7 \|v^n(t)\|^2_{W^{k,2}(\Omega,R^N)} \|\theta^n(t)\|^2_{L^2(\Omega)} + \varepsilon_1 \|\theta^n(t)\|^2_{W^{1,2}(\Omega)} + \\ &\qquad + K_3(\varepsilon_1), \quad c_7, K_3(\varepsilon_1) > 0.\end{aligned} \tag{3.38}$$

Of course $(a_1 + c_7\|v^n(t)\|^2_{W^{k,2}(\Omega,R^N)}) \in L^1(I)$; thus 3.35 holds by Grönwall lemma. ∎

3.39 Lemma.

$$\|\frac{\partial}{\partial t}(\rho^n \theta^n)\|_{L^2(I,W^{-1,2}(\Omega))} \leq c_8, \tag{3.40}$$

$$\|\rho^n \theta^n\|_{L^2(I,W^{1,2}(\Omega))} \leq c_8, \quad c_8 > 0. \tag{3.41}$$

PROOF : (3.40) follows from (3.8). (3.41) is consequence of (3.25), 3.35. ∎

IV. Limit process.

4.1 Lemma. *Let the assumptions 3.23, 3.32, (3.7) be satisfied. Then one can choose a subsequence of* $\{(\rho^n, v^n, \theta^n)\}_{n=1}^{+\infty}$ *(denoted* $\{(\rho^n, v^n, \theta^n)\}_{n=1}^{+\infty}$ *again) such that*

(i)
$$\rho^n \to \rho \quad \textit{strongly in } L^p(Q_T), 1 < p < +\infty,$$
$$\rho > \varepsilon > 0 \quad \textit{a.e.in } Q_T;$$

(ii)
$$v^n \to v \quad \textit{strongly in } L^2(I, W^{2k-1,2}(\Omega, R^N)),$$
$$v^n \to v \quad \textit{strongly in } L^p(I, W^{k-1,2}(\Omega, R^N)), 1 < p < +\infty,$$
$$D^i v^n \to D^i v \quad \textit{weakly in } L^2(Q_T, R^N) \quad (i = 1, \ldots, 2k)$$

(D^i denotes any differentiation of i-th order with respect to the space variables);

(iii)
$$D^i \theta^n \to D^i \theta \quad \textit{weakly in } L^2(Q_T) \quad (i = 0, 1);$$

(iv) $$\rho^n\theta^n \to \rho\theta \quad \textit{strongly in } L^2(Q_T),$$
$$\theta \geq 0 \quad \textit{a.e. in } Q_T;$$

(v) $$\frac{\partial \rho^n}{\partial t} \to \frac{\partial \rho}{\partial t} \quad \textit{weakly in } L^2(Q_T);$$

(vi) $$\frac{\partial v^n}{\partial t} \to \frac{\partial v}{\partial t} \quad \textit{weakly in } L^2(Q_T, R^N);$$

(vii) $$\rho^n v^n \to \rho v \quad \textit{strongly in } L^2(Q_T, R^N);$$

(viii) $$\rho^n v_i^n v_j^n \to \rho v_i v_j \quad \textit{weakly in } L^2(Q_T);$$

(ix) $$\rho^n\theta^n \frac{\partial v_j^n}{\partial x_j} \to \rho\theta \frac{\partial v_j}{\delta x_j} \quad \textit{weakly in } L^2(Q_T);$$

(x) $$\int_0^T \int_\Omega \langle\langle v^n, v^n \rangle\rangle \phi \, dx \, dt = \int_0^T \int_\Omega \langle\langle v, v \rangle\rangle \phi \, dx \, dt \textit{ for every } \phi \in C^\infty(\overline{Q}_T).$$

PROOF : First we recall one assertion called Lions lemma, see e.g. [3]. Let B_0, B, B_1 be Banach spaces, B_0, B_1 reflexive such that $B_0 \subset\subset B \subset B_1$ ($\subset\subset$ is compact imbedding). Let $1 < p_0, p_1 < +\infty$. Then
$L^{p_0}(I, B) \subset\subset \{g; g \in L^{p_0}(I, B_0), \frac{\partial g}{\partial t} \in L^{p_1}(I, B_1)\}$.

i) First assertion follows directly from Lions lemma; we put $B_0 = W^{k-3,p}(\Omega)$, $B = L^p(\Omega), B_1 = L^p(\Omega), 1 < p < +\infty, p_0 = p, p_1 = 2$. Second assertion follows from (3.24).
ii) First two assertions hold due to Lions lemma with $B_0 = W^{2k,2}(\Omega, R^N)$, $B = W^{2k-1,2}(\Omega, R^N), B_1 = L^2(\Omega, R^N), p_0 = p_1 = 2$ (resp. $B_0 = W^{k,2}(\Omega, R^N)$, $B = W^{k-1,2}(\Omega, R^N), B_1 = L^2(\Omega, R^N), p_0 = p, 1 < p < +\infty, p_1 = 2$). Last assertion ii) follows from the boundedness of $\{v^n\}_{n=1}^{+\infty}$ in $L^2(I, W^{2k,2}(\Omega, R^N))$.
iii) holds due to 3.35.
iv) Strong convergence $\rho^n\theta^n \to a$ in $L^2(Q_T)$ is consequence of Lions lemma with $B_0 = W^{1,2}(\Omega), B = L^2(\Omega), B_1 = W^{-1,2}(\Omega), p_0 = p_1 = 2; a = \rho\theta$ due to i), iii). $\theta \geq 0$ a.e. in Q_T due to 3.22.

v) resp. vi) follow directly from (3.26) resp. (3.33) and 3.35. vii), viii) are consequence of i), ii); ix) is consequence of iv), ii) and x) follows from ii). ■

Due to lemma 4.1 we can pass to the limit in (3.4), (3.8), (3.10). We get the following theorem

4.2 Theorem. *Let $k \geq 4, \rho_0 \in C^{k-3}(\overline{\Omega}), \rho_0 > 0$ in $\overline{\Omega}, \theta_0 \in L^2(\Omega), \theta_0 > 0$ a.e. in Ω, $v_0 \in W^{k,2}(\Omega, R^N) \cap W_0^{1,2}(\Omega, R^N)$. Then there exists (ρ, v, θ)*

(4.3) $$\frac{\partial^s \rho}{\partial x_1^{s^1} \dots \partial x_N^{s^N}} \in L^\infty(Q_T) \textit{ for any multiindex } (s^1, \dots, s^N),$$
$$s = s^1 + \dots + s^N, \quad 0 \leq s \leq k-3,$$
$$\rho > \varepsilon \textit{ a.e. in } Q_T \textit{ for some } \varepsilon > 0,$$

$$\frac{\partial^{s+1}\rho^n}{\partial t\partial x_1^{s1}\dots\partial x_N^{sN}} \in L^2(I,L^\infty(\Omega)) \quad for\ 0\le s\le k-4, \tag{4.4}$$

$$v \in L^\infty(I,W^{k,2}(\Omega,R^N)\cap W_0^{1,2}(\Omega,R^N))\cap L^2(I,W^{2k,2}(\Omega,R^N)), \tag{4.5}$$

$$\frac{\partial v}{\partial t}\in L^2(Q_T,R^N), \tag{4.6}$$

$$\theta\in L^\infty(I,L^2(\Omega))\cap L^2(I,W^{1,2}(\Omega)), \theta\ge 0 \ a.e.\ in\ Q_T \tag{4.7}$$

such that (2.10), (2.11) hold a.e. in Q_T and (2.19) is fullfiled.

V. Strong solutions.

First we improve the estimates of ρ^n. Due to 3.32, 3.35, 3.23 we get from (3.4), (3.5)

Lemma 5.1. *Let the assumptions of 3.23, 3.32 be satisfied. Then*

$$\|\frac{\partial^{s+1}\rho^n}{\partial t\partial x_1^{s1}\dots\partial x_N^{sN}}\|_{L^\infty(Q_T)}\le c_9 \ for\ 0\le s\le k-4, \quad c_9>0.$$

Moreover, let $\rho_0\in C^{2k-3}(\overline{\Omega})$. Then

$$\|\frac{\partial^{s}\rho^n}{\partial x_1^{s1}\dots\partial x_N^{sN}}\|_{L^\infty(Q_T)}\le c_9 \quad for\ 0\le s\le 2k-3,$$
$$\|\frac{\partial^{s+1}\rho^n}{\partial t\partial x_1^{s1}\dots\partial x_N^{sN}}\|_{L^2(I,L^\infty(\Omega))}\le c_9 \quad for\ 0\le s\le 2k-4.$$

Multiplying (3.8) by $\frac{\partial\theta^n}{\partial t}$ and integrating over Ω we get

$$\begin{aligned} c_v\int_\Omega \rho^n(\frac{\partial\theta^n}{\partial t})^2\,dx + \lambda\frac{\partial}{\partial t}\int_\Omega \frac{\partial\theta^n}{\partial x_j}\frac{\partial\theta^n}{\partial x_j}\,dx = \\ = -c_v\int_\Omega \rho^n v_j^n\frac{\partial\theta^n}{\partial x_j}\frac{\partial\theta^n}{\partial t}\,dx - \int_\Omega R\rho^n\theta^n\frac{\partial v_j^n}{\partial x_j}\frac{\partial\theta^n}{\partial t}\,dx + \\ + \int_\Omega \langle\langle v^n,v^n\rangle\rangle\frac{\partial\theta^n}{\partial t}\,dx. \end{aligned} \tag{5.2}$$

But r.h.s. of (5.2) is bounded by $c_{10}\|\frac{\partial\theta^n}{\partial t}\|_{L^2(\Omega)}$. Applying Young and Gronwall inequality we verify

5.3 Lemma. *Let $\theta_0\in W^{1,2}(\Omega)$, $\theta_0^n\in C^2(\overline{\Omega})$, $\frac{\partial\theta_0^n}{\partial\nu}=0$ on $\partial\Omega$, $\theta_0^n>0$ in $\overline{\Omega}$, $\theta_0^n\to\theta_0$ strongly in $W^{1,2}(\Omega)$. Then*

$$\|\frac{\partial\theta^n}{\partial t}\|_{L^2(Q_T)} + \|\theta^n\|_{L^\infty(I,W^{1,2}(\Omega))}\le c_{10}, c_{10}>0.$$

We can rewrite (3.8)

$$-\lambda\frac{\partial^2\theta^n}{\partial x_j\partial x_j}=-\frac{\partial}{\partial t}(c_v\rho^n\theta^n)-\frac{\partial}{\partial x_j}(c_v\rho^n\theta^n v_j^n)- \\ -R\rho^n\theta^n\frac{\partial}{\partial x_j}v_j^n+\langle\langle v^n,v^n\rangle\rangle. \tag{5.4}$$

R.h.s. of (5.4) is bounded in $L^2(Q_T)$; hence, due to the regularity to the elliptic systems (Neuman boundary conditions are considered)

$$\|\theta^n\|_{L^2(I,W^{2,2}(\Omega))}\le c_{11},\quad c_{11}>0. \tag{5.5}$$

Thus , following theorem holds

5.6. Theorem.

a) *Let the assumptions of 4.2 be satisfied and let $\theta_0\in W^{1,2}(\Omega)$. Then there exists (ρ,v,θ) satisfying (4.3)-(4.7) and*

$$\frac{\partial^{s+1}\rho}{\partial t\partial x_1^{s1}\dots\partial x_N^{sN}}\in L^\infty(Q_T)\quad for\ 0\le s\le k-4, \tag{5.7}$$

$$\frac{\partial\theta}{\partial t}\in L^2(Q_T),\quad \theta\in L^2(I,W^{2,2}(\Omega))\cap L^\infty(I,W^{1,2}(\Omega)) \tag{5.8}$$

such that the equations (2.10), (2.11), (2.13) are fullfiled a.e. in Q_T

b) *If moreover $\rho_0\in C^{2k-3}(\overline{\Omega})$ then*

$$\frac{\partial^{s}\rho}{\partial x_1^{s1}\dots\partial x_N^{sN}}\in L^\infty(Q_T)\ for\ 0\le s\le 2k-3, \tag{5.9}$$

$$\frac{\partial^{s+1}\rho}{\partial t\partial x_1^{s1}\dots\partial x_N^{sN}}\in L^2(I,L^\infty(\Omega))\quad for\ 0\le s\le 2k-4. \tag{5.10}$$

VI. Uniqueness.

Our aim in this section is to prove the following theorem.

6.1 Theorem. *Let the assumptions os theorem 5.6 a) be satisfied. Then in the class (4.3)- (4.7), (5.7), (5.8) there exists at most one solution (ρ,v,θ) satisfying initial and boundary conditions (2.14)-(2.16) and equations (2.10), (2.11), (2.13) a.e. in Q_T.*

PROOF : Let (ρ,v,θ), $(\overline{\rho},\overline{v},\overline{\theta})$ be two solutions with the same initial condition. Then $(\xi,w,\eta)=(\rho-\overline{\rho},v-\overline{v},\theta-\overline{\theta})$ satisfies

$$\frac{\partial\xi}{\partial t}+\frac{\partial}{\partial x_j}(\xi v_j)+\frac{\partial}{\partial x_j}(\overline{\rho}w_j)=0\text{ a.e. in }Q_T, \tag{6.2}$$

$$\overline{\rho}\frac{\partial w_i}{\partial t}+\xi v_j\frac{\partial v_i}{\partial x_j}+\overline{\rho}w_j\frac{\partial v_i}{\partial x_j}+\overline{\rho v}_j\frac{\partial w_i}{\partial x_j}-\frac{\partial}{\partial x_j}(\tau_{ij}^V(w))+ \\ +R\frac{\partial}{\partial x_i}(\xi\theta)+R\frac{\partial}{\partial x_i}(\overline{\rho}\eta)+\xi\frac{\partial v_i}{\partial t}=\xi F_i\quad\text{a.e. in }Q_T, \tag{6.3}$$

$$c_v\bar{\rho}\frac{\partial\eta}{\partial t}+c_v\xi v_j\frac{\partial\theta}{\partial x_j}+c_v\bar{\rho}v_j\frac{\partial\eta}{\partial x_j}+c_v\bar{\rho}\frac{\partial\bar{\theta}}{\partial x_j}w_j-\lambda\frac{\partial^2\eta}{\partial x_j\partial x_j}+ \tag{6.4}$$
$$+R\theta\frac{\partial v_j}{\partial x_j}\xi+R\bar{\rho}\frac{\partial v_j}{\partial x_j}\eta+R\bar{\rho}\bar{\theta}\frac{\partial w_j}{\partial x_j}+c_v\xi\frac{\partial\theta}{\partial t}=$$
$$=\langle\langle v,v\rangle\rangle-\langle\langle\bar{v},\bar{v}\rangle\rangle.$$

From (6.2) one obtains the estimates

$$\|\xi(t)\|^2_{W^{1,2}(\Omega)}\le K_4(\varepsilon_1)\|\xi\|^2_{L^2((0,t),W^{1,2}(\Omega))}+ \tag{6.5}$$
$$+\varepsilon_1\|w\|^2_{L^2((0,t),W^{k,2}(\Omega,R^N))},$$

which holds for every $\varepsilon_1>0$ $(K_4(\varepsilon_1)>0)$. From (6.3) we get

$$\int_{\Omega_t}\bar{\rho}|w|^2\,dx+\|w\|^2_{L^2((0,t),W^{k,2}(\Omega,R^N))}\le \tag{6.6}$$
$$\le K_5(\varepsilon_1)\int_0^t b(\tau)(\|\xi(\tau)\|^2_{W^{1,2}(\Omega))}+\|w(\tau)\|^2_{L^2(\Omega,R^N)}+$$
$$+\|\eta(\tau)\|^2_{L^2(\Omega)})\,d\tau\quad+$$
$$+\varepsilon_1(\|w\|^2_{L^2((0,t),W^{k,2}(\Omega,R^N))}+\|\eta\|^2_{L^2((0,t),W^{1,2}(\Omega))})$$

for a.e. $t\in I$ and every $\varepsilon_1>0$ $(K_5(\varepsilon_1)>0)$, where

$$b=c_{12}(1+\|\frac{\partial\bar{\rho}}{\partial t}\|_{L^\infty(\Omega)}+\|\bar{\rho}\|_{L^\infty(\Omega)}+\sum_{i=1}^N\|\frac{\partial\bar{\rho}}{\partial x_i}\|_{L^\infty(\Omega)})^2$$
$$(1+\|\bar{v}\|_{W^{2k,2}(\Omega,R^N)}+\|v\|_{W^{2k,2}(\Omega,R^N)}+\|\frac{\partial v}{\partial t}\|_{L^2(Q_T,R^N)}+\|\frac{\partial\theta}{\partial t}\|_{L^2(\Omega)})^2$$
$$(1+\|\bar{\theta}\|_{W^{1,2}(\Omega)}+\|\theta\|_{W^{1,2}(\Omega)})^2,$$

$c_{12}>0$, hence $b\in L^1(I)$.

We multiply (6.4) by η and integrate over Ω. Due to (2.4) $\langle\langle v,v\rangle\rangle=\mathcal{L}v\mathcal{L}v$, where $\mathcal{L}v$ is some linear combination of

$$e_{ij}(v),\frac{\partial^2 v_i}{\partial x_{i_1}\partial x_{i_2}},\dots,\frac{\partial^k v_i}{\partial x_{i_1}\dots\partial x_{i_k}};$$

hence r.h.s. of (6.4) is equal to $\mathcal{L}w(\mathcal{L}v+\mathcal{L}\bar{v})$.

After some laborious computation we get the estimate

$$\int_{\Omega_t}\bar{\rho}\eta^2\,dx+\|\eta\|^2_{L^2((0,t),W^{1,2}(\Omega))}\le \tag{6.7}$$
$$\le K_6(\varepsilon_1)\int_0^t b(\gamma)(\|\xi(\tau)\|^2_{W^{1,2}(\Omega))}+\|w(\tau)\|^2_{L^2(\Omega,R^N)}+$$
$$+\|\eta(\tau)\|^2_{L^2(\Omega)})\,d\tau+\varepsilon_1(\|w\|^2_{L^2((0,t),W^{k,2}(\Omega,R^N))}+\|\eta\|^2_{L^2((0,t),W^{1,2}(\Omega))})$$

for a.e. $t\in I$ and every $\varepsilon_1>0$ $(K_6(\varepsilon_1)>0)$.

We take into consideration that $\int_\Omega\bar{\rho}|w|^2\,dx$ resp. $\int_\Omega\bar{\rho}\eta^2\,dx$ are equivalent norms in $L^2(\Omega,R^N)$ resp. $L^2(\Omega)$; hence due to (6.5), (6.6), (6.7) and Gronwall lemma $\xi=0,w=0,\eta=0$ a.e. in I. The proof is finished. ∎

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