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A posteriori error estimate of approximate solutions to a mildly nonlinear elliptic boundary value problem

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Abstract. The paper deals with a computable a posteriori error estimate of the approximate solution to a mildly nonlinear elliptic boundary value problem with Dirichlet boundary condition. The convergence of the presented error estimate to the true error is proved.

Keywords: a posteriori error estimates, nonlinear elliptic equations

Classification: 65G99, 65N15

INTRODUCTION

This paper deals with an a posteriori error estimate of the error of the approximate solution to a mildly nonlinear elliptic boundary value problem with homogeneous Dirichlet boundary condition

$$-\Delta u + g(u) = f \qquad \text{in } \Omega, \tag{1}$$
$$u_{|\partial\Omega} = 0 .$$

The main idea consists in the construction of convergent lower estimates for the potential of problem (1). A posteriori error estimates for linear problems (cases $g \equiv 0$ resp. $g = \lambda u, \lambda > 0$) have been studied in [HK], [HH], [K] resp. [A], [AB], [V]. A generalization of our approach for problems more general then (1) is sketched in Remark 4.

In the sequel we shall adopt the following notations: $\Omega \subset R^2$ denotes a simply connected, bounded domain with polygonal boundary $\partial\Omega$, V denotes the Sobolev space $W_0^{1,2}(\Omega)$ endowed with the inner product

$$((u,v)) = \int_\Omega \sum_{i=1}^{2} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx \tag{2}$$

and the norm $\|u\| = ((u,u))^{1/2}$. If B is a Banach space B' denotes its dual and $< .,. >_B$ denotes the duality pairing between B' and B. If $\mathcal{B} : B \to \bar{R}$ is a functional then $\mathcal{B}^* : B' \to \bar{R}$ denotes its conjugate functional

$$\mathcal{B}^*(b') = \sup_{b\in B}\{< b', b >_B - \mathcal{B}(b)\}. \tag{3}$$

If B and C are Banach spaces, $L(B,C)$ denotes the space of all linear bounded operators from B to C, and if $A \in L(B,C)$ then $A' \in L(C',B')$ denotes its transpose defined by $< A'c', b >_B = < c', Ab >_C$ for $b \in B, c' \in C'$.

We suppose that $g : R \to R$ is a surjective increasing continuous function satisfying $g(0) = 0$ and that for some $c > 0, \beta > 0, d > 0$ the following inequality holds

$$|\, g(t) \,| \;\leq\; c + d \,|\, t \,|^{\beta} \qquad t \in R. \tag{4}$$

Further let $f \in V'$, $f = f_0 + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, f_i \in L_2(\Omega), i = 0, 1, 2$. ($\frac{\partial f_i}{\partial x_i}$ are distributive derivatives of $f_i, i = 1, 2$.)

Under conditions stated above we can consider (using Sobolev's imbedding theorem) the weak formulation of (1): *Find $u \in V$, such that*

$$\int_\Omega \mathrm{grad}u.\mathrm{grad}v dx + \int_\Omega g(u) v dx \;=\; < f, v > \qquad v \in V \tag{5}$$

and define its potential $\mathcal{F} : V \to R$

$$\mathcal{F}(v) = \frac{1}{2}\|v\|^2 - < f, v >_V + j(v) \tag{6}$$

with convex continuous G-differentiable functional $j : V \to R$

$$j(v) = \int_\Omega \int_0^{v(x)} g(t) dt dx. \tag{7}$$

It is well known (e.g.[**KF**]) that unique solution u to problem (5) exists and that problem (5) is equivalent to problem *Find $u \in V$, such, that*

$$\mathcal{F}(u) = \inf_{v \in V} \mathcal{F}(v). \tag{8}$$

Functional $\mathcal{F}$ can be minimized e.g.by the Ritz method. If some lower estimate d for $\mathcal{F}(u)$ is known, then $\|u - v\|$ can be estimated for arbitrary $v \in V$ using the inequality

$$\begin{gathered}
\mathcal{F}(v) - d \geq \mathcal{F}(v) - \mathcal{F}(u) = \\
\frac{1}{2}\|v\|^2 - < f, v >_V + j(v) - \frac{1}{2}\|u\|^2 + < f, u >_V - j(u) \geq \\
\frac{1}{2}\|v\|^2 + < f, u - v >_V - \frac{1}{2}\|u\|^2 + \int_\Omega g(u)(v - u) dx = \\
\frac{1}{2}\|v\|^2 - \frac{1}{2}\|u\|^2 + ((u, u - v)) = \frac{1}{2}\|v - u\|^2,
\end{gathered}$$

which follows from (5) and from properties of j. It is clear that if $u_n, n = 1, 2, \ldots$ is a minimizing sequence for $\mathcal{F}$ and if we can construct a sequence $d_n, n = 1, 2, \ldots$ of real numbers, which satisfies $d_n \leq \mathcal{F}(u), n = 1, 2, \ldots d_n \to \mathcal{F}(u)$ then

$$\frac{1}{2}\|u_n - u\|^2 \leq \mathcal{F}(u_n) - d_n \to 0.$$

In what follows, such d_n's will be constructed.

DUAL PROBLEM FOR (5)

Using duality theory [**ET**, Chapter III] we shall construct functional $\mathcal{L}$ which satisfies

$$\sup \mathcal{L} = \inf \mathcal{F}. \tag{9}$$

Values of this functional can be used as lower estimates of $\mathcal{F}(u)$. Setting $F : V \to R$, $F(v) = \frac{1}{2}\|v\|^2 - < f, v >, H = L_2(\Omega)$ with usual inner product, $G : H \to \bar{R}$, $G(p) = \int_\Omega \int_0^{p(x)} g(t)dtdx, \Lambda \in L(V, H), \Lambda v = v$, functional $\mathcal{F}$ can be written in the form

$$\mathcal{F}(v) = F(v) + G(\Lambda v).$$

From [**ET**, Chapter III] it follows that for functional $\mathcal{L} : H' \to \bar{R}$

$$\mathcal{L}(p') = -F^*(\Lambda' p') - G^*(-p')$$

holds

$$\sup_{p' \in H'} \mathcal{L}(p') \leq \inf_{v \in V} \mathcal{F}(v).$$

Later we shall see that (9) holds. $\mathcal{L}$ will be called dual functional to $\mathcal{F}$ and problem

$$\mathcal{L}(q') = \sup_{p' \in H'} \mathcal{L}(p')$$

will be called dual problem to (6).

In what follows we shall identify Hilbert space H with its dual using Riesz representation. Thus $\mathcal{L}$ will be considered as $\mathcal{L} : H \to \bar{R}$

$$\mathcal{L}(p) = -F^*(\Lambda' p) - G^*(-p).$$

Let us compute F^*, G^*. If we denote $Z : V' \to V$ the (Green's) operator defined by

$$((Zv', v)) = < v', v >_V \qquad v \in V, \tag{10}$$

then we can compute

$$F^*(v') = \sup_{v \in V}\{< v', v >_V - F(v)\} = \sup_{v \in V} -\frac{1}{2}\|v - Z(f + v')\|^2 + \frac{1}{2}\|Z(f + v')\|^2$$

$$F^*(v') = \frac{1}{2}\|Z(f + v')\|^2 \tag{11}$$

From [**GGZ**, Theorem III.4.8] follows that conjugate function to $r : R \to R, r(s) = \int_0^s g(t)dt$ is $r^*(s) = \int_0^s g^{-1}(t)dt$ and from [**ET**, Theorem IX.2.1]we have $G^* : L_2 \to \bar{R}$

$$G^*(p) = \int_\Omega \int_0^{p(x)} g^{-1}(t)dtdx.$$

Thus $\mathcal{L}$ can be written in the form $\mathcal{L} : H \to \bar{R}$

$$\mathcal{L}(p) = -\frac{1}{2}\|Z(f + \Lambda' p)\|^2 - G^*(-p).$$

Since we are interested only in $\sup \mathcal{L}$ we shall use from now a slightly modified definition of $\mathcal{L}$

$$\mathcal{L}(p) = -\frac{1}{2}\|Z(f - \Lambda' p\|^2 - G^*(p).$$

Lemma 1. *For $v \in V$ it holds*

$$G(v) + G^*(g(v)) = \int_\Omega v g(v) dx. \tag{13}$$

PROOF : From [**GGZ**, Theorem III.4.8] it follows

$$r(v(x)) + r^*(g(v(x))) = v(x) g(v(x)).$$

The assertion of Lemma 1 follows by integration of this equality in Ω. ■

Functional $\mathcal{L}$ attains its supremum at point

$$q = g(u) \tag{14}$$

because using (5),(10) and Lemma 1 we obtain

$$\mathcal{L}(g(u)) = -\frac{1}{2}\|Z(f - \Lambda' g(u)\|^2 - G^*(g(u)) = -\frac{1}{2}\|u\|^2 - G^*(g(u)) =$$
$$-\frac{1}{2}\|u\|^2 + G(u) - \int_\Omega u g(u) dx = -\frac{1}{2}\|u\|^2 + G(u) - < f, u >_V + \|u\|^2 = \mathcal{F}(u).$$

Thus (9) holds and the maximization of $\mathcal{L}$ can be considered as searching for $g(u)$.

Taking into account (14) we can maximize $\mathcal{L}$ on the set

$$\{p \mid p = g(v) \text{ for some } v \in V\}.$$

Hence instead of maximizing $\mathcal{L}$ over H it suffices to solve the problem:
Find $h \in V$ such that

$$\mathcal{G}(h) = \sup_{v \in V} \mathcal{G}(v) \tag{15}$$

for $\mathcal{G} : V \to R$

$$\mathcal{G}(v) = -\frac{1}{2}\|Z(f - \Lambda' g(v))\|^2 - G^*(g(v)).$$

Assertion 1. *Let $u_n, n = 1, 2, \dots$ be a minimizing sequence for $\mathcal{F}$. Then $u_n, n = 1, 2, \dots$ is a maximizing sequence for $\mathcal{G}$.*

PROOF : $u_n \to u$ in V together with (4) implies $g(u_n) \to g(u)$ in $L_2(\Omega)$. Relation (13) implies $G^*(g(u_n)) \to G^*(g(u))$. The rest follows from the continuity of Z. ■

Remark 1. The sequence $v_n = Z(f - \Lambda' g(u_n))$ is a minimizing sequence for $\mathcal{F}$ too.

Realization

In the definition of the dual problem (12) resp. (15), there appears term of type $-\frac{1}{2}\|Zv'\|$, where Z is defined by (10). These values cannot be computed explicitly (with the exception of very special cases). Problem (15) can be transformed in a saddle point problem

$$\sup_{v\in V}\inf_{w\in V}\mathcal{S}(v,w)$$

for $\mathcal{S}: V\times V\to R$

$$\mathcal{S}(v,w)=\frac{1}{2}\|w\|^2-<f-\Lambda'(g(v)),w>_V-G^*(g(v)).$$

The values of $\mathcal{S}$ can be computed explicitly. However this saddle point formulation cannot be used for our purposes, because usual saddle point methods (e.g. Uzawa type methods) do not produce lower estimates for $\inf\mathcal{F}$ in general. In what follows, the values $-\frac{1}{2}\|Zv'\|^2$ will be approximated from below using the dual formulation of problem (1) for $g\equiv 0$.

Let $\mathbf{H}=(L_2(\Omega))^2$ resp. $U=\{v\in W^{1,2}(\Omega)\mid\int_\Omega v\,dx=0\}$ are endowed with inner products $(\mathbf{u},\mathbf{v})=\int_\Omega \mathbf{u}.\mathbf{v}\,dx$ resp. (2) and corresponding norms denoted by $[.]$ resp. $\|.\|$. $\mathbf{H}'$ will be again identified with $\mathbf{H}$. Let $K\in L(V,\mathbf{H}), L\in L(U,\mathbf{H})$,

$$Kv=(\frac{\partial v}{\partial x_1},\frac{\partial v}{\partial x_2}),\ Lv=(\frac{\partial v}{\partial x_2},-\frac{\partial v}{\partial x_1}).$$

From [H], [HK] follows that $\mathrm{Im}L=\mathrm{Ker}K'$. This and [HH], [K], [HK] implies

$$-\frac{1}{2}\|Zv'\|^2=\sup_{K'\mathbf{z}=v'}-\frac{1}{2}[\mathbf{z}]^2=\sup_{K'(\mathbf{z}-\mathbf{w})=0}-\frac{1}{2}[\mathbf{z}]^2$$

(16) $$-\frac{1}{2}\|Zv'\|^2=\sup_{v\in U}-\frac{1}{2}[\mathbf{w}+Lv]^2$$

where $\mathbf{w}\in\mathbf{H}$ satisfies $K'\mathbf{w}=v'$. Let $Z_1\in L(U',U)$ be defined by

$$((Z_1v',v))=-<v',v>_U\qquad v\in U$$

Then (16) can be rewritten as

(17) $$-\frac{1}{2}\|Zv'\|^2=\frac{1}{2}(\sup_{v\in U}-\|v-Z_1L'\mathbf{w}\|^2+\|Z_1L'\mathbf{w}\|^2-[\mathbf{w}]^2)$$

(18) $$-\frac{1}{2}\|Zv'\|^2=\frac{1}{2}(\|Z_1L'\mathbf{w}\|^2-[\mathbf{w}]^2)$$

This value can be approximated from below by maximizing the quadratic functional $\mathcal{D}: U\to R$

$$\mathcal{D}(v)=-\frac{1}{2}(\|v\|^2+2[\mathbf{w},Lv]+[\mathbf{w}]^2).$$

Let U_k, $k = 1,2\ldots$ be a sequence of (finite dimensional) subspaces of U and P_k, $k = 1,2\ldots$ be the sequence of corresponding orthogonal projectors $P_k : U \to U_k$, satisfying

$$\lim_{k\to\infty} \|v - P_k v\| = 0 \qquad v \in V. \tag{19}$$

Then the Ritz approximation of (18) is (using (17))

$$\frac{1}{2}(-\|P_k Z_1 L'\mathbf{w} - Z_1 L'\mathbf{w}\|^2 + \|Z_1 L'\mathbf{w}\|^2 - [\mathbf{w}]^2).$$

Let us return to problem (15). Let $R : H \to \mathbf{H}$ be the (continuous) operator

$$Rl = (-\int_0^{x_1} \bar{f}_0(t,x_2)dt + \int_0^{x_1} \bar{l}(t,x_2)dt - f_1,\ -f_2),$$

where $\bar{f}_0 = f_0$ in $\Omega, \bar{f}_0 = 0$ in $R^2 - \Omega$. ($\bar{l}$ is defined analogously.) It holds $K'Rl = f - \Lambda' l$. From (19) it follows that for $s : V \to R$

$$s(v) = -\frac{1}{2}\|Z(f - \Lambda' g(v)\|^2 - G^*(g(v)) = \frac{1}{2}\|Z_1 L' Rg(v)\|^2 - \frac{1}{2}[Rg(v)]^2 - G^*(g(v))$$

for its Ritz approximation

$$s_k(v) = -\frac{1}{2}\|P_k Z_1 L' Rg(v) - Z_1 L' Rg(v)\|^2 + s(v)$$

and for arbitrary minimizing sequence $u_n, n = 1,2\ldots$ of $\mathcal{F}$ it holds

$$\lim_{n\to\infty}\lim_{k\to\infty} s_k(u_n) \;=\; s(u).$$

Moreover it holds

Theorem 1. *Let u_n be a minimizing sequence for $\mathcal{F}$, and P_k, Z_1, L, R, s, s_k are as defined above. Then*

$$\lim_{n\to\infty} s_n(u_n) = s(u) = \mathcal{F}(u).$$

PROOF : $u_n \to u$ in V implies $g(u_n) \to g(u), Rg(u_n) \to Rg(u), G^*(g(u_n)) \to G^*(g(u)), Z_1 L' Rg(u_n) \to Z_1 L' Rg(u)$. From (19) an from property $\|P_k\| = 1, k = 1,2,\ldots$ it follows

$$\|P_n Z_1 L' Rg(u_n) - Z_1 L' Rg(u_n)\| \le \|P_n Z_1 L'(Rg(u_n) - Rg(u))\| +$$
$$\|P_n Z_1 L' Rg(u) - Z_1 L' Rg(u)\| + \|Z_1 L' Rg(u) - Z_1 L' Rg(u_n)\| \to 0. \qquad \blacksquare$$

Remark 2. If any tool for minimization of $\mathcal{F}$ is on hand then the Ritz solver of dual problem for (linear) Poisson equation is all what is needed for obtaining convergent a posteriori estimates of $\|u_n - u\|$.

Remark 3. In practice, the convergence can be improved via solving linear problems on finer grids (that is by computing $s_k(u_n)$ for $k > n$).

The use of the Ritz method for maximizing $\mathcal{D}$ (i.e for approximation of $s(u_n)$ from below) is not necessary. In general, arbitrary maximizing sequence of $\mathcal{D}$ can be used. If $r_k(u_n)$ are convergent lower approximations of $s(u_n)$ then $\lim_{n\to\infty}\lim_{k\to\infty} r_k(u_n) = s(u)$. However $r_n(u_n) \to s(u)$ does not hold in general. Sufficient for it is the uniformity (in n) of the convergence $r_k(u_n) \to s(u_n)$. This (rather uncomfortable) condition can be avoided by the following way.

Theorem 2. *Let $a_k(u_n), k = 1,2\dots$ be a sequence of real numbers with the property*

$$\lim_{k\to\infty} a_k(u_n) = \inf_{v\in V} \mathcal{P}_n(v), a_k(u_n) \geq \inf \mathcal{P}_n, k = 1,2\dots$$

for the quadratic functional $\mathcal{P}_n : V \to R$

$$\mathcal{P}_n = \frac{1}{2}\|v\|^2 - < f - \Lambda' g(u_n), v >_V - G^*(g(u_n))$$

for $n = 1,2,\dots.$

Then the sequence of real numbers $d_n, n = 1,2,\dots$ generated by the following procedure tends to $s(u)$ from below.

Step I $\epsilon = 1,\ n = 1$
Step II $k = 1$
Step III *if $a_k(u_n) - r_k(u_n) > \epsilon$ then $k = k+1$ goto Step III*
if $a_k(u_n) - r_k(u_n) \leq \epsilon$ then $d_n = r_k(u_n), n = n+1, \epsilon = \frac{1}{2}\epsilon$ goto Step II.

Proof : For n fixed, Step III will be performed only finite number times, because $a_k(u_n) - r_k(u_n) \to 0$. The algorithm guarantees, that for better approximations $s(u_n)$of $s(u)$, better approximations $r_k(u_n)$ of $s(u_n)$ will be computed. ∎

Remark 4. Results analogous to those obtained in this paper can be obtained for the equation

$$-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{i,j}\frac{\partial u}{\partial x_j}) + g(u) = f,$$

where $a_{i,j} = a_{j,i}, i,j = 1,2$, are bounded measurable functions, the matrix $(a_{i,j})$ is uniformly elliptic in Ω.

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