Bruno Bassan; Elisabetta Bona

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BRUNO BASSAN, ELISABETTA BONA

Abstract. We provide a general formula to evaluate the moments of those processes which can be written as integrals with respect to a Poisson random measure. This result applies, for example, to discontinuous Lévy processes and shot noise.

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1) Introduction.

A wide class of stochastic processes including Poisson, compound Poisson, discontinuous Lévy processes, shot noise and others can be represented through integrals with respect to a Poisson random measure. This representation permits us a unified approach to a variety of problems and allows many a computational simplification. For a general outline of Poisson random measures see, for example, Ikeda and Watanabe (1981), where their central role in the theory of stochastic differential equations with jumps is carefully explained; see also Itô (1951). Çinlar and Jacod (1981) show how Poisson random measures, together with Brownian motion, underlie all semimartingale Hunt processes.

We recall that a Poisson random measure N on a measurable space $(\mathbb{E}, \mathcal{E})$, defined on a probability space $(\Omega, \mathcal{A}, P)$ and with σ-finite mean measure ν, is a mapping $N : \Omega \times \mathcal{E} \to \mathbb{N}$ such that:

(i) $\omega \mapsto N(\omega, B)$ is a Poisson distributed random variable with parameter $\nu(B)$, for every $B \in \mathcal{E}$;

(ii) $B \mapsto N(\omega, B)$ is a measure on $\mathbb{E}$, for every $\omega \in \Omega$;

(iii) if $B_1, \dots, B_n$ are disjointed sets of $\mathcal{E}$, then $N(B_1), \dots, N(B_n)$ are independent random variables.

As an example of how certain stochastic processes can be represented through integrals with respect to a Poisson random measure, consider a homogeneous Poisson process $\{X(t) \mid t \in \mathbb{R}_+\}$ with parameter λ; let $\mathbb{E} = \mathbb{R}_+$ and let N be a Poisson random measure on $\mathbb{R}_+$ with mean measure $\nu(ds) = \lambda ds$. Then

$$X(\omega, t) = \int_{\mathbb{R}_+} \mathbf{1}_{(0,t)}(s)\, N(\omega, ds)$$

where $\mathbf{1}_A(\cdot)$ denotes the indicator function of the set A.

A further example is given by shot noise random fields in $\mathbb{R}^n$ (see, for example, Daley (1971), Orsingher and Battaglia (1982) or Bassan and Bona (1988)). These fields are defined by the relation

$$W(P) = \sum_{i \in I} Z_i \gamma(P - Q_i) \qquad P, Q_i \in \mathbb{R}^n \tag{1.1}$$

where $\{Q_i \mid i \in I\}$, $I \subset \mathbb{N}$, are the random points of a Poisson field with parameter λ, $\{Z_i \mid i \in I\}$ is a family of i.i.d. random variables ("impulses") and $\gamma : \mathbb{R}^n \to \mathbb{R}$ is the so-called response function. The field (1.1) can be written as

$$W(x^1, \ldots, x^n) = \int_{\mathbb{R}^n \times \mathbb{R}_+} z\gamma\left(x^1 - y^1, \ldots, x^n - y^n\right) N(dy^1, \ldots, dy^n, dz).$$

where N is a Poisson random measure with mean measure $\nu\left(dy^1, \ldots, dy^n, dz\right) = \lambda\, dy^1 \cdots dy^n\, dF(z)$ (F is the common distribution function of the random variables Z_i).

Formulas for mean and variance of processes which can be written as integrals with respect to Poisson random measures are well known; bounds for higher order moments, in settings slightly different from ours, can be found, for example, in Schmidt (1985). In this note we provide a general formula to evaluate the moments, if existing, of those processes which admit the represntation described above.

2) Main result.

Let $(\mathbb{E}, \mathcal{E})$ be a measurable space and let N be a Poisson random measure on $\mathbb{E}$ with σ-finite mean measure ν. If $f : \mathbb{E} \to \mathbb{R}_+$ is a measurable function, we write

$$Nf(\omega) = \int_{\mathbb{E}} f(x)\, N(\omega, dx)$$

and

$$\nu f = \int_{\mathbb{E}} f(x)\, \nu(dx).$$

For the moments of a stochastic process which can be represented through integrals with respect to a Poisson random measure, the following result holds.

Theorem. *Let $\{X_t \mid t \in T\}$, $(T \subset \mathbb{R}^d, d \geq 1)$ be a stochastic process with values in $\mathbb{R}_+$, and let $\{f_t \mid t \in T\}$ be a family of positive measurable functions defined on $\mathbb{E}$ such that, for every $t \in T$, we can write*

$$X_t = Nf_t.$$

Then, the n-th moment of X_t, if existing, is given by

$$E(X_t)^n = \sum_{(r_1, \ldots, r_n) \in I(n)} K_n(r_1, \ldots, r_n) \cdot (\nu f_t)^{r_1} \cdot \left(\nu f_t^2\right)^{r_2} \cdot \ldots \cdot (\nu f_t^n)^{r_n} \tag{2.1}$$

where

$$I(n) = \left\{ (r_1, \dots, r_n) \in \mathbb{N} \times \mathbb{N} \times \dots \times \mathbb{N} \;\middle|\; \sum_{i=1}^{n} i r_i = n \right\}$$

and

$$K_n(r_1, \dots, r_n) = \frac{n!}{1!^{r_1} \cdot 2!^{r_2} \cdot \ldots \cdot [(n-1)!]^{r_{n-1}} \cdot [n!]^{r_n} \cdot r_1! \cdot \ldots \cdot r_n!} \, .$$

PROOF : It is known that the Laplace functional ϕ_{X_t} of the process $X_t = Nf_t$ is given by

$$\begin{aligned}\phi_{X_t}(\alpha) &= E\left[\exp\left(-\alpha N f_t\right)\right] \\ &= \exp\left\{\nu\left(e^{-\alpha f_t} - 1\right)\right\} = \exp\left\{\int_{\mathbb{E}} \left[e^{-\alpha f_t(x)} - 1\right] \nu(dx)\right\} \, .\end{aligned}$$

For fixed t, we write, for brevity, $\phi(\alpha)$ instead of $\phi_{X_t}(\alpha)$. Let also $\gamma(\alpha) = \log \phi(\alpha)$. We shall use induction to prove that, for all $n \in \mathbb{N}$, the general form of the n-th derivative of $\phi(\alpha)$ is:

(2.2)

$$\phi^{(n)}(\alpha) = \phi(\alpha) \sum_{(r_1, \dots, r_n) \in I(n)} K_n(r_1, \dots, r_n) \left[\gamma'(\alpha)\right]^{r_1} \left[\gamma''(\alpha)\right]^{r_2} \dots \left[\gamma^{(n)}(\alpha)\right]^{r_n}.$$

It is obvious that (2.2) holds for $n = 1$, since $I(1) = \{1\}, K(1) = 1$ and, clearly, $\phi'(\alpha) = \phi(\alpha)\gamma'(\alpha)$.

Suppose now that (2.2) holds for a given $n > 1$; we want to show that

(2.3)

$$\begin{aligned}\phi^{(n+1)}(\alpha) = \phi(\alpha) \sum_{(p_1, \dots, p_n, p_{n+1}) \in I(n+1)} & K_{(n+1)}(p_1, \dots, p_n, p_{n+1}) \cdot \\ & \cdot \left[\gamma'(\alpha)\right]^{p_1} \cdot \ldots \cdot \left[\gamma^{(n)}\right]^{p_n} \left[\gamma^{(n+1)}\right]^{p_{n+1}}.\end{aligned}$$

Let $\widehat{I}(n) = I(n) \setminus \{(0, \dots, 0, 1)\}$. Differentiating (2.2), we have:

(2.4)

$$\begin{aligned}\phi^{(n+1)}(\alpha) = \gamma'(\alpha)\phi^{(n)}(\alpha) + \phi(\alpha) \Bigg\{ \gamma^{(n+1)}(\alpha) + & \sum_{(r_1, \dots, r_n) \in \widehat{I}(n)} K_n(r_1, \dots, r_n) \cdot \\ \cdot \sum_{j=1}^{n-1} r_j \left[\gamma'(\alpha)\right]^{r_1} \dots \left[\gamma^{(j)}\right]^{r_j - 1} & \left[\gamma^{(j+1)}\right]^{r_{j+1}+1} \dots \left[\gamma^{(n)}\right]^{r_n} \Bigg\}.\end{aligned}$$

This can be rewritten, substituting (2.2) for $\phi^{(n)}(\alpha)$ as:

$$\begin{aligned}\frac{\phi^{(n+1)}(\alpha)}{\phi(\alpha)} &= \gamma^{(n+1)}(\alpha) + \gamma^{(n)}(\alpha)\gamma'(\alpha) \\ &+ \sum_{(r_1, \dots, r_n) \in \widehat{I}(n)} K_n(r_1, \dots, r_n) \Bigg\{ \left[\gamma'(\alpha)\right]^{r_1+1} \left[\gamma''(\alpha)\right]^{r_2} \dots \left[\gamma^{(n)}\right]^{r_n} \\ &+ \sum_{j=1}^{n-1} r_j \left[\gamma'(\alpha)\right]^{r_1} \dots \left[\gamma^{(j)}\right]^{r_j - 1} \left[\gamma^{(j+1)}\right]^{r_{j+1}+1} \dots \left[\gamma^{(n)}\right]^{r_n} \Bigg\}.\end{aligned} \tag{2.5}$$

Now, we can observe that, if $(r_1,\dots,r_n) \in \widehat{I}(n)$, then the n-tuples $(r_1+1,r_2,\dots,0)$ and $(r_1,\dots,r_j-1,r_{j+1}+1,\dots,r_n)$, $\forall j \in \{1,\dots,n-1\}$, are in the set $\{(p_1,\dots,p_n)\in \mathbb{N}^n \mid \sum_{i=1}^n ip_i = n+1\}$. Hence, we can rearrange (2.5) as

$$\begin{aligned}\frac{\phi^{(n+1)}(\alpha)}{\phi(\alpha)} &= \gamma^{(n+1)}(\alpha)+\gamma^{(n)}(\alpha)\gamma'(\alpha)\\ &+\sum_{\{(p_1,\dots,p_n)\in\mathbb{N}^n \mid \sum_{i=1}^n ip_i=n+1\}}\Big\{K_n(p_1-1,p_2,\dots,0)\\ &+\sum_{j=1}^{n-1}(p_j+1)K_n(p_1,\dots,p_j+1,p_{j+1}-1,\dots,p_n)\Big\}\cdot\\ &\cdot[\gamma'(\alpha)]^{p_1}\dots\left[\gamma^{(n)}\right]^{p_n}\end{aligned}$$

where the value of K_n is taken to be zero if any of its arguments is negative.

Let us consider now the n-tuple $(p_1,\dots,p_n)=(1,0,\dots,0,1)$, which is the only one with $p_n\neq 0$. The corresponding term is:

$$(p_{n-1}+1)K_n(1,0,\dots,0,1,0)\gamma'(\alpha)\gamma^{(n)}(\alpha)=n\gamma'(\alpha)\gamma^{(n)}(\alpha).$$

since all the terms with $j\neq n-1$ in the sum between braces vanish. Thus, we can write:

$$\begin{aligned}\frac{\phi^{(n+1)}(\alpha)}{\phi(\alpha)} &= \gamma^{(n+1)}(\alpha)+(n+1)\gamma^{(n)}(\alpha)\gamma'(\alpha)\\ &+\sum_{\{(p_1,\dots,p_n)\in\mathbb{N}^n \mid \sum_{i=1}^n ip_i=n+1,p_n=0\}}\Big\{K_n(p_1-1,p_2,\dots p_n)\\ &+\sum_{j=1}^{n-2}(p_j+1)K_n(p_1,\dots,p_j+1,p_{j+1}-1,\dots,p_n)\Big\}\cdot\\ &\cdot[\gamma'(\alpha)]^{p_1}\dots\left[\gamma^{(n)}\right]^{p_n}.\end{aligned}\tag{2.6}$$

In order to express the coefficients between braces in the relation above in terms of K_{n+1}, we shall make use of the following relations, which are easy to check:

$$K_n(p_1,\dots,p_n)=\frac{1}{n+1}K_{n+1}(p_1,\dots,p_n,0)\tag{2.7}$$

$$p_1K_{n+1}(p_1,p_2,\dots,p_n,0)=K_{n+1}(p_1-1,p_2,\dots,p_n,0)\tag{2.8}$$

$$\begin{aligned}&K_{n+1}(p_1,\dots,p_j+1,p_{j+1}-1,\dots,p_n,0)\\ &\qquad=\frac{(j+1)p_{j+1}}{p_j+1}K_{n+1}(p_1\dots,p_j,p_{j+1},\dots,p_n,0).\end{aligned}\tag{2.9}$$

In view of (2.7), the quantity between braces in (2.6) becomes:

$$\frac{1}{n+1}\left\{K_{n+1}(p_1-1,p_2,\dots,p_n,0)\right.$$
$$\left.+\sum_{j=1}^{n-2}(p_j+1)K_{n+1}(p_1,\dots,p_j+1,p_{j+1}-1,\dots,p_n,0)\right\}$$

which in turn can be written, using (2.8) and (2.9), as:

$$\frac{1}{n+1}\left\{K_{n+1}(p_1,p_2,\dots,p_n,0)\left[p_1+\sum_{j=1}^{n-2}(j+1)p_{j+1}\right]\right\}$$
$$=\frac{1}{n+1}\left\{K_{n+1}(p_1,p_2,\dots,p_n,0)\sum_{i=1}^{n-1}ip_i\right\}.$$

Thus, (2.6) becomes

$$\begin{aligned}\frac{\phi^{(n+1)}(\alpha)}{\phi(\alpha)} &= \gamma^{(n+1)}(\alpha)+(n+1)\gamma^{(n)}(\alpha)\gamma'(\alpha)\\ &+\sum_{\{(p_1,\dots,p_n,p_{n+1})\in I(n+1)\ \mid\ p_n=p_{n+1}=0\}} K_{n+1}(p_1,\dots,p_n,p_{n+1})\cdot\\ &\cdot[\gamma'(\alpha)]^{p_1}\dots\left[\gamma^{(n+1)}\right]^{p_{n+1}}\end{aligned}\tag{2.10}$$

The relation (2.3) can be derived immediately from (2.10) simply observing that $K_{n+1}(0,\dots,0,1)=1$ and $K_{n+1}(1,0,\dots,0,1,0)=n+1$; (obviously, these two $(n+1)$-tuples are the only ones in $I(n+1)$ with $p_n>0$ or $p_{n+1}>0$).

Thus, it has been proved that (2.2) holds for every $n\in\mathbb{N}$. Now, the theorem follows easily; in fact, observing that

$$\gamma^{(j)}(0)=(-1)^j\nu f^j=(-1)^j\int_{\mathbb{E}}f^j(x)\,\nu(dx)$$

and recalling that

$$E(X_t)^n=(-1)^n\phi_{X_t}^{(n)}(0),$$

we have

$$\begin{aligned}E(X_t)^n &= (-1)^n\phi_{X_t}(0)\sum_{(r_1,\dots,r_n)\in I(n)}[-\nu f_t]^{r_1}\left[(-1)^2\nu f_t^2\right]^{r_2}\dots\left[(-1)^n\nu f_t^n\right]^{r_n}\\ &= (-1)^{n+\sum_{i=1}^n ir_i}\sum_{(r_1,\dots,r_n)\in I(n)}[\nu f_t]^{r_1}\left[\nu f_t^2\right]^{r_2}\dots\left[\nu f_t^n\right]^{r_n}.\end{aligned}$$

■

Remark. It is immediate to derive from (2.1) the well known formulas $E(Nf) = \nu f$ and $E(Nf)^2 = \nu f^2 + (\nu f)^2$.

Example. We shall compute, using (2.1), the third moment of a shot noise process on $\mathbb{R}^2$. The reader can check that this procedure is computationally much easier than the one based on differentiating the Laplace functional.

Let the "impulses" $\{Z_i \mid i \in \mathbb{N}\}$ be exponentially distributed with parameter μ, and let the (radially symmetric) response function γ be given by

$$\gamma\left(x^1 - y^1, x^2 - y^2\right) = \exp\left\{-\frac{\rho^2}{2}\right\} 1_{\{\rho \le R\}}$$

where $\rho = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2}$ and $R > 0$. The corresponding shot noise field, evaluated in $(0,0)$, can be written as:

$$W(0,0) = \int_0^{2\pi} \int_0^R \int_0^\infty z \exp\left\{-\frac{\rho^2}{2}\right\} \lambda \rho \mu \exp\{-\mu z\}\, dz\, d\rho\, d\phi.$$

Notice that the distribution of the field does not depend on the point where it is evaluated.

We can write $W(0,0) = Nf$ with $f(z, \rho, \phi) = z \exp\left\{-\frac{\rho^2}{2}\right\}$. Some calculations yield:

$$\begin{aligned}
\nu f &= \frac{2\pi\lambda}{\mu}\left(1 - \exp\left\{-\frac{R^2}{2}\right\}\right) \\
\nu f^2 &= \frac{2\pi\lambda}{\mu^2}\left[2\pi\lambda\left(1 - \exp\left\{-\frac{R^2}{2}\right\}\right) + 1 - \exp\left\{-R^2\right\}\right] \\
\nu f^3 &= \frac{4\pi\lambda}{\mu^3}\left(1 - \exp\left\{-\frac{3R^2}{2}\right\}\right)
\end{aligned}$$

Since $I(3) = \{(0,0,1), (1,1,0), (3,0,0)\}$, formula (2.1) gives

$$\begin{aligned}
E[W(0,0)]^3 &= \nu f^3 + 3\nu f\, \nu f^2 + (\nu f)^3 \\
&= \frac{4\pi\lambda}{\mu^3}\left(1 - \exp\left\{-\frac{3R^2}{2}\right\}\right) + \frac{8\pi^3\lambda^3}{\mu^3}\left(1 - \exp\left\{-\frac{R^2}{2}\right\}\right)^3 \\
&\quad + \frac{12\pi^2\lambda^2}{\mu^3}\left(1 - \exp\left\{-\frac{R^2}{2}\right\}\right)\left(1 - \exp\left\{-R^2\right\}\right)
\end{aligned}$$

Notice that the evaluation of the Laplace functional of the field permits us to write:

$$E\left[\exp\{-\alpha W(0,0)\}\right] = \left[\frac{\mu + \alpha \exp\left\{-\frac{R^2}{2}\right\}}{\mu + \alpha}\right]^{2\pi\lambda}. \tag{2.11}$$

Notice, incidentally, that $W(0,0)$ is asimptotically distributed as a $\Gamma(2\pi\lambda, \mu)$ when $R \to \infty$. The reader may verify directly that the calculations needed to differentiate (2.11) three times are very lengthy. Furthermore, it must be pointed out that the Laplace functional of a process is, in general, hard if not impossible to compute explicitly.

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Dipartimento di Statistica, Probabilità e Statistiche Applicate, Università di Roma "La Sapienza", P. le Aldo Moro 5, 00185 Roma, Italy

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