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CERTAIN HIGHER MONOTONICITY PROPERTIES OF BESSEL FUNCTIONS

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1. INTRODUCTION AND NOTATION

In my earlier paper [4] there are derived certain higher monotonicity properties of i-th derivatives of solutions of

$$y'' + a(t)\,y' + b(t)\,y = 0 \qquad t \in (0, \infty) \tag{1.1}$$

in the oscilatoric case. This paper contains certain applications of the above-mentioned results to the Bessel equation

$$y'' + \frac{1}{t}y' + \left(1 - \frac{\nu^2}{t^2}\right)y = 0, \qquad t \in (0, \infty). \tag{1.2$_\nu$}$$

By a Bessel function of order ν we mean any nontrivial solution $\mathscr{C}_\nu(t)$ $(t > 0)$ of (1.2$_\nu$). All functions and quantities considered here are real.

Let the functions $a_0(t) \equiv a(t)$, $b_0(t) \equiv b(t) \neq 0$ in (1.1) be continuous and sufficiently smooth on $(0, \infty)$. Let $a_i(t)$, $b_i(t)$ be defined recurrently for $i = 1, 2, 3, \dots$ by formulas

$$\begin{aligned} a_i(t) &:= a_{i-1} - b'_{i-1}/b_{i-1}, \\ b_i(t) &:= b_{i-1} + a'_{i-1} - a_{i-1}b'_{i-1}/b_{i-1}. \end{aligned} \tag{1.3$_i$}$$

Suppose that $b_i(t) \neq 0$ for $t \in (0, \infty)$ and all needed i. Let the function $f_i(t)$ be defined for $i = 0, 1, 2, \dots$ by

$$f_i(t) := b_i - a'_i/2 - a_i^2/4. \tag{1.4$_i$}$$

Consider the sequences $\{R_k^{(i)}\}_{k=0}^{\infty}$, where the quantities $R_k^{(i)}$ are defined for fixed $\lambda > -1$ and any sufficiently monotonic function $W(t)$ by

$$R_k^{(i)} = R_k^{(i)}(W, \lambda) := \int_{t_k^{(i)}}^{t_{k+1}^{(i)}} W(t) \left| \exp\left(\frac{1}{2}\int a_i(t)\,\mathrm{d}t\right) y^{(i)} \right|^{\lambda} \mathrm{d}t, \tag{1.5.}$$

where $y(t)$ is an arbitrary (non-trivial) solution of (1.1), $\{t_k^{(i)}\}_{k=0}^{\infty}$ denotes any sequence of consecutive zeros of i-th ($i = 0, 1, 2, \ldots$) derivative of any solution $z(t)$ of (1.1) which may or may not be linearly independent of $y(t)$. $\int a(t)\,\mathrm{d}t$ denotes any function $A(t)$ satisfying $A'(t) = a(t)$. The condition $\lambda > -1$ is required to assure convergence of the integral (1.5) and the function $W(t)$ is taken subject to the same restriction. By special choice of $W(t)$, λ, i and $z(t)$ in (1.5) we can obtain $R_k^{(i)}(W, \lambda)$ having different geometrical (or other) meaning.

The function $f(t)$ is said to be *n-times monotonic* or *monotonic of order n in* (a, b), if there exist continuous derivatives $f^0 \equiv f, f', f'', \ldots, f^{(n)}$, satisfying

$$(1.6)\qquad (-1)^j f^{(j)}(t) \geqq 0 \qquad \text{for } t \in (a, b);\ j = 0, 1, \ldots, n.$$

For such a function we write $f \in \mathcal{M}_n(a, b)$. In case $n = \infty$ f is said to be *completely monotonic* in (a, b), $\mathcal{M}_n$ denotes $\mathcal{M}_n\ (0, \infty)$. If the strict inequality holds throughout (1.6), we write $f \in \mathcal{M}_n^*(a, b)$ as it was defined in [2].

The sequence $\{t_k\}_{k=0}^{\infty} \equiv \{t_k\}$ is said to be *n-times monotonic* if

$$(1.7)\qquad (-1)^i\, \Delta^i t_k \geqq 0 \qquad (i = 0, 1, \ldots, n;\ k = 0, 1, 2, \ldots).$$

Here $\Delta^0 t_k = t_k$, $\Delta t_k = t_{k+1} - t_k$, $\Delta^n t_k = \Delta(\Delta^{n-1} t_k)$. For such a sequence we write $\{t_k\} \in \mathcal{M}_n$. If strict inequality holds throughout (1.7), we write $\{t_k\} \in \mathcal{M}_n^*$.

2. PRELIMINARY RESULTS

In the paper [4] for the quantities $R_k^{(i)}$, defined by (1.5) and concerning the solutions of (1.1) there are derived the following assertions, presented here as Lemmas 2.1 and 2.2 in rather different formulations. Lemma 2.2 is in certain sense a consequation of Lemma 2.1.

Lemma 2.1. ([4] Theorems 3.1, 5.1). *Let $n \geqq 1$, $i \geqq 0$ be arbitrary but fixed integers and $W(t) > 0$ be any function of class $\mathcal{M}_n$. For the function $f_i(t)$ defined by* (1.4) *suppose that*

$$(2.1)\qquad f_i'(t) \in \mathcal{M}_n^*, \qquad f_i(\infty) = \Theta > 0.$$

Then

$$\{R_k^{(i)}\}_{k=0}^{\infty} \in \mathcal{M}_n.$$

If $W(t) \in \mathcal{M}_n^*$ or $n \geqq 2$, then $\{R_k^{(i)}\}_{k=0}^{\infty} \in \mathcal{M}_n^*$.

Remark 2.1. In the same way as in [4], Remark iii p. 96 we may prove that (2.1) is implied by

$$(2.2)\qquad \begin{gathered} a_{i-1}(t) \in \mathcal{M}_{n+2}, \qquad b_{i-1}'(t) \in \mathcal{M}_{n+2}^*, \qquad b_{i-1}(t) > 0 \quad \text{for } t \in (0, \infty) \\ b_{i-1}(\infty) - a_{i-1}^2(\infty)/4 = \Theta > 0. \end{gathered}$$

Lemma 2.2. ([4] Theorem 7.2). *Let $n \geqq 1$, $i \geqq 1$ be arbitrary but fixed integers and $t \in (0, \infty)$. For the functions $a_{i-1}(t)$, $b_{i-1}(t)$ defined by* (1.3) *suppose that*

$$a_{i-1}(t) \in \mathcal{M}_{n+i} \qquad b_{i-1}(t) > 0, \quad b'_{i-1}(t) \in \mathcal{M}_{n+i} \tag{2.3}$$

and that

$$a_{i-1}(t) > 0 \quad or \quad a_{i-1} \equiv 0 \quad \text{and} \quad b'_{i-1}(t) > 0. \tag{2.5}$$

Suppose also that the second part of (2.2) *holds.*
Then

$$\{[y^{(i-1)}(t_k^{(i)})]^2\}_{k=0}^{\infty} \in \mathcal{M}_n^*, \tag{2.5}$$

where $y(t)$ denotes any solution of (1.1) *and $\{t_k^{(i)}\}$ any sequence of consecutive zeros of its i-th derivative.*

Lemma 2.3. *Let $v \geqq 0$ be any number and let $\varphi_v(t)$ be for $t > v$ defined by*

$$\varphi_v(t) := \frac{1}{t}\left(1 - \frac{2v^2}{t^2 - v^2}\right).$$

Let α_n denote the unique zero of the equation

$$G_n(s) \equiv 3 - \left(\frac{s}{s-1}\right)^{n+1} - \left(\frac{s}{s+1}\right)^{n+1} = 0, \qquad s \in (1, \infty),\ n = 0, 1, 2, \ldots \tag{2.6$_n$}$$

Then

$$\varphi_v(t) \in \mathcal{M}_n^*(v\alpha_n, \infty). \tag{2.7}$$

Proof. The function $\varphi_v(t)$ can be expressed in the form

$$\varphi_v(t) = 3t^{-1} - (t + v)^{-1} - (t - v)^{-1}.$$

Its n-th ($n = 0, 1, 2, \ldots$) derivative has the form

$$\varphi_v^{(n)}(t) = (-1)^n n!\,[3t^{-(n+1)} - (t + v)^{-(n+1)} - (t - {}_1)^{-(n+1)}]. \tag{2.8$_n$}$$

It is evidently seen that for $t > v$ and $n = 0, 1, 2, \ldots$ $\varphi_v^{(n)}(t)$ is continuous. $\varphi_v^{(n)}$ changes therefore the sign only in the zeros of equation $\varphi_v^{(n)}(t) = 0$. Put $t = sv$. The equation $\varphi_v^{(n)}(t) = 0$ is then equivalent to (2.6$_n$) and (2.6$_n$) is independent of v.

The function $G_n(s)$ defined by (2.4$_n$) has the following properties:

$$\lim_{s \to 1+} G_n(s) = -\infty, \qquad \lim_{s \to \infty} G_n(s) = 1 \tag{2.9}$$

$$G'_n(s) = (n + 1)\, s^n[(s - 1)^{-(n+2)} - (s + 1)^{-(n+2)}] > 0 \quad \text{for} \quad s \in (1, \infty). \tag{2.10}$$

(2.9) and (2.10) imply that for $n = 0, 1, 2, \ldots$ (2.6_n) has the unique zero in $(1, \infty)$. Let α_n denote this unique zero so that we have

$$\varphi_\nu^{(n)}(\nu\alpha_n) = 0$$

$$\text{(2.11)} \qquad \operatorname{sign} \varphi_\nu^{(n)}(t) = \text{const} \qquad \textit{for } t \in (\nu\alpha_n, \infty)$$

$$\lim_{t \to \infty} \varphi_\nu^{(n)}(t) = 0.$$

The Rolle's theorem implies that there exists at least one number $\xi \in (\nu\alpha_n, \infty)$ such that

$$\text{(2.12)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \varphi_\nu^{(n)}(t) \,|_{t=\xi} = \varphi_\nu^{(n+1)}(\xi) = 0.$$

On the other hand since (2.6_{n+1}) has in $(1, \infty)$ the unique zero α_{n+1}, $\varphi_\nu^{(n+1)}(t) = 0$ has in (ν, ∞) the unique zero $\nu\alpha_{n+1}$. So $\xi = \nu\alpha_{n+1}$ and we have

$$\text{(2.13)} \qquad \alpha_{n+1} \geqq \alpha_n \qquad \text{for } n = 0, 1, 2, \ldots$$

The assertion (2.7) follows directly from (2.8_n), (2.11) (2.12) and (2.13).

Remark 2.2. The zeros α_n of (2.6) for $n = 0, 1, 2, \ldots, 40$ were computed by the student of J. E. Purkyně University J. Tryhuk. Their values for $n = 0, 1, \ldots, 29$ are as follows:

n	α_n	n	α_n
0	1,732051	15	17,296313
1	2,757816	16	18,335307
2	3,793285	17	19,374306
3	4,830752	18	20,413311
4	5,868924	19	21,452319
5	6,907419	20	22,491331
6	7,946078	21	23,530347
7	8,984858	22	24,569364
8	10,023695	23	25,608384
9	11,062576	24	26,647406
10	12,101489	25	27,686429
11	13,140426	26	28,725455
12	14,179380	27	29,764481
13	15,218347	28	30,803508
14	16,257326	29	31,842537

3. HIGHER MONOTONICITY PROPERTIES OF $R'_{\nu k}$

As mentioned above, $\mathscr{C}_\nu(t)$ denote any Bessel (cylinder) function of order ν, i.e. any nontrivial solution of the Bessel equation (1.2_ν). For simplicity it seems to be usefull to consider $\nu \geqq 0$. But this condition is not essential, it has the formal character only as it follows from the familiar properties of Bessel equation as well as from the analytic theory of linear differential equations in general.

Let $\{c_{\nu k}^{(i)}\}_{k=1}^{\infty}$ denote the sequence of consecutive positive zeros of the i-th derivative ($i = 0, 1, 2, \ldots$) of some Bessel function $\mathscr{C}_\nu(t)$ and let $\{d_{\nu k}^{(i)}\}$ denote the analogous sequence of any Bessel function $\bar{\mathscr{C}}_\nu(t)$ of order ν, possibly $\mathscr{C}_\nu(t)$ again.

Theorem 3.1. *Let $n \geqq 0$ be an integer and $\nu \geqq 0$ an arbitrary number. Let $W(t) > 0$ denote any function of class $\mathscr{M}_n(\delta, \infty)$ and let $R'_{\nu k}$ be defined for $t > \nu$ and $\lambda > -1$ by*

$$R'_{\nu k} = R'_{\nu k}(W, \lambda) := \int_{d'_{\nu,k}}^{d'_{\nu,k+1}} W(t)\,|\,t^{3/2}(t^2 - \nu^2)^{-1/2}\mathscr{C}'_\nu(t)\,|^\lambda \,\mathrm{d}t. \tag{3.1}$$

Let $m = \max(\delta, \nu)$ and p be the smallest integer satisfying $m \leqq d'_{\nu p}$. Then

$$\{R'_{\nu k}\}_{k=p}^{\infty} \in \mathscr{M}_\infty^* . \tag{3.2}$$

Proof. In case of Bessel equation (1.2_ν) the coefficients a_0 and b_0 have the form

$$a_0 = a_{0\nu}(t) = t^{-1} \qquad b_0 = b_{0\nu}(t) = 1 - \nu^2 t^{-2}.$$

Let $t > \nu$. The formulas (1.3) yield after a little calculation

$$a_1 = a_{\nu 1}(t) = \frac{1}{t} - \frac{2\nu^2}{t(t^2 - \nu^2)}$$

$$b_1 = b_{\nu 1}(t) = 1 - \frac{1 + \nu^2}{t^2} - \frac{2\nu^2}{t^2(t^2 - \nu^2)}$$

$$f_1 = f_{\nu 1}(t) = 1 - \frac{\nu^2 - 1/4}{t^2} - \frac{1}{t^2 - \nu^2} - \frac{3\nu^2}{(t^2 - \nu^2)^2} =$$

$$= 1 - \frac{\nu^2 + 3/4}{t^2} - \frac{\nu^2}{t^2(t^2 - \nu^2)} - \frac{3\nu^2}{(t^2 - \nu^2)^2},$$

We prove at first that $f'_{\nu 1}(t) \in \mathscr{M}_\infty^*(\nu, \infty)$. Since $t^{-2} \in \mathscr{M}_\infty^*$, $(t - \nu)^{-1} \in \mathscr{M}_\infty^*(\nu, \infty)$, $(t + \nu)^{-1} \in \mathscr{M}_\infty^*(-\nu, \infty)$ the general rules for calculation with higher monotonic functions (for such a rules see e.g. [4] p. 91) give

$$(t - \nu)^{-1} . (t + \nu)^{-1} = (t^2 - \nu^2)^{-1} \in M_\infty^*(\nu, \infty),$$
$$(t^2 - \nu^2)^{-1} . (t^2 - \nu^2)^{-1} = (t^2 - \nu^2)^{-2} \in M_\infty^*(\nu, \infty),$$
$$t^{-2}(t^2 - \nu^2)^{-1} \in \mathscr{M}_\infty^*(1, \infty).$$

This implies

$$(-f_{\nu 1}(t) + 1) \in \mathcal{M}^*_\infty(\nu, \infty)$$

and

$$f'_{\nu 1}(t) \in \mathcal{M}^*_\infty(\nu, \infty).$$

In the case of Bessel equation we have

$$\exp\left\{\frac{1}{2}\int a_{\nu 1}(t)\,\mathrm{d}t\right\} = \exp\left\{\frac{1}{2}\int [a_{\nu 0}(t) - b'_{\nu 0}(t)/b_{\nu 0}(t)]\,\mathrm{d}t\right\} =$$

$$= [b_{\nu 0}(t)]^{-1/2} \exp\left\{\frac{1}{2}\int a_{\nu 0}(t)\,\mathrm{d}t\right\} = t^{3/2}(t^2 - \nu^2)^{-1/2}.$$

The expression R'_k defined in (1.5) is therefore of the form (3.1). Since $f_{\nu 1}(\infty) = 1 > 0$, it is evidently seen that the conditions of modified form of Lemma 2.1 are satisfied for any $n \geqq 2$ if the interval $(0, \infty)$ is replaced by (m, ∞). So, the assertion (3.2) follows immediately from Lemma 2.1.

It remains to prove the validity of (3.2) for $n = 1$, since the case $n = 0$ is obvious. In case $W'(t) > 0$ for $t \in (\delta, \infty)$, i.e. $W(t) \in \mathcal{M}^*_1(\delta, \infty)$, (3.2) follows from Lemma 2.1, too. If $W(t) \equiv 1$, Lemma 2.1 gives the unsharpened inequality $\Delta R'_{\nu k} \leqq 0$ only. The proof of nonpossibility $\Delta R'_{\nu k} = 0$ is similar to that in [2] p. 352. This completes the proof of Theorem 3.1.

Remark 3.1. It is not necessary to calculate the explicit form of $f_{\nu 1}(t)$. The complete monotonicity of $f'_{\nu 1}(t)$ follows directly from the conditions

$$a_{\nu 0}(t) = t^{-1} \in \mathcal{M}^*_\infty,$$

$$b'_{\nu 0}(t) = 2\nu^2 t^{-3} \in \mathcal{M}^*_\infty, \qquad b_{\nu 0}(\infty) > 0 \quad \text{for } t > \nu,$$

due to Remark 2.1 as well as

$$f_{\nu 1}(\infty) = b_{\nu 0}(\infty) - \frac{1}{4} a^2_{\nu 0}(\infty) = 1 > 0$$

is implied by the same Remark.

Remark 3.2. As a direct conclusion of Theorem 3.1 we receive

$$\{\Delta(c'_{\nu k})^\alpha\}_{k=p}^\infty \in \mathcal{M}^*_\infty \qquad 0 < \alpha \leqq 1 \tag{3.4}$$

$$\{\lg(c'_{\nu,k+1}/c'_{\nu k})\}_{k=p}^\infty \in \mathcal{M}^*_\infty \tag{3.5}$$

$$\{C^2_\nu(c'_{\nu k})\}_{k=p}^\infty \in \mathcal{M}^2_\infty. \tag{3.6}$$

To prove (3.4) it suffices to put $\bar{\mathscr{C}}_\nu(t) \equiv \mathscr{C}_\nu(t)$, i.e. $c'_{\nu k} \equiv d'_{\nu k}$, $\lambda = 0$ and $W(t) = \alpha t^{\alpha - 1}$ in (3.1).

(3.5) follows from (3.4) by using l'Hospital's rule (see [3] p. 364), or directly from Theorem 3.1 if $\bar{\mathscr{C}}_\nu(t) = \mathscr{C}_\nu(t)$, $\lambda = 0$ and $W(t) = t^{-1}$.

The assertion (3.6) follows from Lemma 2.2 for $i = 1$, if the interval $(0, \infty)$ is replaced by (m, ∞). We can receive (3.6) also from Theorem 3.1, if we put $\bar{\mathscr{C}}_\nu(t) \equiv \equiv \mathscr{C}_\nu(t)$, $\lambda = 2$ and $W(t) = 2(t^2 - \nu^2)^{-1}$ in (3.1).

The assertions (3.4) for $\alpha = 1$ and (3.6) were noticed in my preprint [3] in 1972, the assertions (3.4), (3.5) and (3.6) were published independently in [2] in the same year. The result (3.4) is rather unexpected for $\nu \in \langle 0, 1/2 \rangle$ since the sequence $\{c_{\nu k}\}$ for $\nu \in \langle 0, 1/2 \rangle$ satisfies the contrary inequalities

$$\Delta^i c_{\nu k} > 0 \qquad (i = 1, 2) \qquad \nu \in \langle 0, 1/2)$$
$$\Delta c_{1/2,k} = \text{const} = \pi$$

and the sequences $\{c_{\nu k}\}$ and $\{c'_{\nu k}\}$ are interlaced.

Theorem 3.2. *Let $\nu \geqq 0$ be an arbitrary number, let α_n denote the unique zero of* (2.6$_n$) *and $e = e(n)$ the smallest integer satisfying $c'_{\nu,e(n)} > \nu\alpha_n$. Then for $n = 0, 1, 2, 3, \ldots$ there hold*

$$\{|\mathscr{C}_\nu(c'_{\nu,k+1})| + |\mathscr{C}_z(c'_{\nu,k}(|\}_{k=e(n)}^{\infty} \in \mathscr{M}_n^* \tag{3.7}$$

$$\{|\mathscr{C}_\nu(c'_{\nu,2k})|\}_{k=[e(n)/2]}^{\infty} \in \mathscr{M}_n^* \tag{3.8}$$

$$\{|\mathscr{C}_\nu(c'_{\nu,2k+1})|\}_{k=[e(n)+1/2]}^{\infty} \in \mathscr{M}_n^* \tag{3.9}$$

so the sequences (3.7), (3.8) *and* (3.9) *are monotonic of order n, if we omit the members for which $c'_{\nu k}$ is smaller than $\nu\alpha_n$.*

Proof. Theorem 3.2 is a direct corollary of Theorem 3.1. Put $\{c'_{\nu k}\} \equiv \{d'_{\nu k}\}$, $\lambda = 1$ and

$$W(t) = W_1(t) := t^{-3/2}(t^2 - \nu^2)^{1/2} = \exp\left\{-\frac{1}{2}\int [t^{-1} - 2\nu^2(t^2 - \nu^2)^{-1}]\,dt\right\}$$

in (3.1) Lemma 2.3 implies

$$a_{\nu 1}(t) = t^{-1}[1 - 2\nu^2(t^2 - \nu^2)^{-1}] \in \mathscr{M}_n^*(\nu\alpha_n, \infty).$$

Using the general rules for calculation with higher monotonic functions (see e.g. [4] Lemma 2.3), we receive

$$\exp\left\{-\frac{1}{2}\int a_{\nu 1}(t)\,dt\right\} = W_1(t) \in \mathscr{M}_n^*(\nu\alpha_n, \infty).$$

The expression (3.1) is in our case of the form

$$R'_{\nu k}(W_1, 1) = \int_{c'_{\nu,k}}^{c'_{\nu,k+1}} |\mathscr{C}'_\nu(t)|\,dt = |\mathscr{C}_\nu(c'_{\nu,k+1})| + |\mathscr{C}_\nu(c'_{\nu,k})|$$

and (3.6) follows directly from Theorem 3.1. (3.7) and (3.8) follows from (3.6) since

$$\Delta[|\mathscr{C}_\nu(c'_{\nu k+1})| + |\mathscr{C}_\nu(c'_{\nu,k})|] = |\mathscr{C}_\nu(c'_{\nu,k+2})| - |\mathscr{C}_\nu(c'_{\nu,k})| = \Delta|\mathscr{C}(c'_{\nu,2r})|.$$

Here $\Delta t_{2r} = t_{2(r+1)} - t_{2r}$.

Remark 3.2. The assertions (3.8) and (3.9) mean that the sequence of maxima as well as the sequence of absolute value of (negative) minima of any Bessel function are monotonic of an arbitrary order n if we omit a sufficiently great number of the first members.

Remark 3.3. In case of Bessel functions of order, zero $\mathscr{C}_0(t)$ are (3.7), (3.8) and (3.9) valid for all k; in another words (3.7), (3.8) and (3.9) are completely monotonic for all $c'_{0k} > 0$.

Remark 3.4. It seems

$$\{| C_\nu(c'_{\nu k}) |\}_{k=e(n)}^{\infty} \in \mathscr{M}_n^*$$

to be valid, but I did not succeed in deriving it from above relations.

4. HIGHER MONOTONICITY PROPERTIES OF $R''_{\nu k}$

Theorem 4.1. *Let δ, $\nu \geqq 0$ be arbitrary numbers. Let $W(t) > 0$ denote any function of class $M_n(\delta, \infty)$ and α_n the unique zero of (2.4$_n$). Let β_n and $R''_{\nu k}$ be defined by*

$$\beta_n = \beta_n(\nu) := \max \{\delta; \nu\alpha_n; [\nu^2 + 1/2 + (2\nu^2 + 1/4)^{1/2}]^{1/2}\}$$

$$\text{(4.1)} \quad R''_{\nu k} = R''_{\nu k}(W, \lambda) := \int_{d''_{\nu k}}^{d''_{\nu, k+1}} W(t) \, | \, t^3(t^2 - \nu^2)^{1/2}(t^2 - \nu^2 - 1)^{-1}\mathscr{C}''_\nu(t) \, |^\lambda \, dt.$$

Then for $n = 0, 1, 2, \ldots$ there holds

$$\text{(4.2)} \qquad \{R''_{\nu k}\}_{k=r(n)}^{\infty} \in \mathscr{M}_n^*$$

where $r = r(n)$ denotes the smallest integer satisfying $\mathscr{C}''_{\nu r} > \beta_{n+2}(\nu)$.

Proof. The direct calculation gives for $t > \beta_0(\nu)$

$$\exp\left\{\frac{1}{2}\int a_{\nu 2}(t)\,dt\right\} = \exp\left\{\frac{1}{2}\int [a_{\nu 1}(t) - b'_{\nu 1}(t)/b_{\nu 1}(t)]\,dt\right\} =$$

$$= [b_{\nu 1}(t)]^{-1/2} \exp\frac{1}{2}\int a_{\nu 1}(t)\,dt = t^3(t^2 - v^2)^{1/2}(t^2 - \nu^2 - 1)^{-1}.$$

Thus the expression (1.5$_2$) has in the case of Bessel equation the form (4.1). We prove that the conditions of Lemma 2.1 are satisfied for $i = 2$, if the interval $(0, \infty)$ is replaced by (β_{n+2}, ∞).

The explicit form of $a_{\nu 1}(t)$ and $b_{\nu 1}(t)$ is given in (3.3). Lemma 2.3 implies $a_{\nu 1}(t) \in \in \mathscr{M}_n^*(\nu\alpha_n, \infty)$. By the same way as in the proof of Theorem 3.1 we can prove that $b'_{v 1}(t) \in \mathscr{M}_\infty^*(\nu, \infty)$. By a direct calculation we can show that

$$b_{\nu 1}(t) > 0 \qquad \text{for } t > [\nu^2 + 1/2 + (2\nu^2 + 1/4)^{1/2}]^{1/2}.$$

Validity of

$$b_{\nu 1}(\infty) - a_{\nu 1}^2(\infty)/4 = 1 > 0$$

is obvious. Since $\alpha_n > 1$, Remark 2.1 implies

$$(4.3) \qquad \begin{aligned} &f'_{\nu 2}(t) \in \mathscr{M}_n^*(\beta_{n+2}, \infty) \\ &f_{\nu 2}(\infty) = 1. \end{aligned}$$

Thus the condition of Lemma 2.1 are satisfied for $i = 2$, if $(0, \infty)$ is replaced by (β_{n+2}, ∞) and (4.2) follows from this Lemma for $n \geqq 2$ as well as for $n = 1$ in case $W'(t) > 0$. For $n = 0$ the validity of (4.2) is obvious. The proof in case $n = 1$, $W(t) \equiv 1$ is similar to that of Theorem 3.1.

Corollary 4.1. *Let γ_n be defined by*

$$\gamma_n = \gamma_n(\nu) := \max\{\nu\alpha_n, [\nu^2 + 1/2 + (2\nu^2 + 1/4)^{1/2}]^{1/2}\}.$$

If $\mathscr{C}_\nu(t)$ *denotes any Bessel function and* $\{c''_{\nu k}\}_{k=0}^{\infty}$ *denotes the sequence of consecutive positive zeros of* $\mathscr{C}''_\nu(t)$, *then for* $n = 0, 1, 2, \ldots$ *there hold*

$$(4.4) \qquad \{\Delta(c''_{\nu k})^\alpha\}_{k=q(n)}^{\infty} \in \mathscr{M}_n^* \qquad 0 < \alpha \leqq 1$$

$$(4.5) \qquad \{(c''_{\nu k})^\alpha\}_{k=q(n)}^{\infty} \in \mathscr{M}_n^* \qquad \alpha < 0$$

$$(4.6) \qquad \{\lg(c''_{\nu,k+1}/c''_{\nu k})\}_{k=q(n)}^{\infty} \in \mathscr{M}_n^*,$$

where $q = q(n)$ *denote the smallest integer satisfying* $c''_{\nu q} > \gamma_{n+2}(\nu)$.

Proof. For $\alpha \neq 0$ and $\bar{\mathscr{C}}_\nu(t) \equiv \mathscr{C}_\nu(t)$ we have

$$R''_{\nu k}(\alpha t^{\alpha-1}, 0) = \int_{c''_{\nu,k}}^{c''_{\nu,k+1}} \alpha t^{\alpha-1}\,dt = \Delta(c''_{\nu k})^\alpha$$

If $\alpha \in (0, 1]$, than $\alpha t^{\alpha-1} \in \mathscr{M}_\infty$ and (4.4) follows directly from Theorem 4.1. If $\alpha < 0$, than $[-\alpha t^{\alpha-1}] \in \mathscr{M}_\infty^*$ and we have $\{-\Delta(c''_{\nu k})^\alpha\} \in \mathscr{M}_\infty^*$. Since

$$-(-1)^m \Delta^m[(c''_{\nu k})^\alpha] = (-1)^{m+1}\Delta^{m+1}(c''_{\nu k})^\alpha$$

and $(c''_{\nu k})^\alpha > 0$, there holds (4.5).

To prove (4.6), it suffices to put $W(t) = t^{-1}$ and $\lambda = 0$.

Theorem 4.2. *Let the conditions of* Corollary 4.1 *are fulfilled. Then*

$$(4.7) \qquad \{[C'_\nu(c''_{\nu k})]^2\}_{k=q(n)}^{\infty} \in \mathscr{M}_n^*,$$

Proof. Theorem 4.2 is the direct consequence of Lemma 2.2 for $i = 2$, if the interval $(0, \infty)$ is replaced by (γ_{n+2}, ∞). The validity of conditions (2.3), (2.4) and the second part of (2.2) were proved in the proof of Theorem 4.1.

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