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POLARS ON CLOSURE SPACES

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In this paper there is given a generalization of polar theory from lattice ordered groups (l-groups) on sets with closure systems. Basic properties of polars are generalized in § 1, while § 2 contains a generalization of prime subgroups in an l-group and their property that a factorgroup belonging to a prime subgroup is fully ordered. Examples and special cases of a polarity being in connexion with [1], [2], [3], [4], [5], [6] are given in § 3.

Let us introduce the following notation for the whole paper: *A closure space* (S, Ω) is a nonempty set S with a closure system Ω, the closure of a set $A \subseteqq S$ in Ω is $\bar{A}$, $\bar{a} = \{\bar{a}\}$, for all $a \in S$. If S is a partially ordered set, then $a \parallel b$ means that elements a, $b \in S$ are not comparable. We say that *a set* $A \subseteqq S$ *is convex in* S, when $a, b \in A$, $s \in S$, $a \geqq s \geqq b$ implies $s \in A$.

§ 1. DEFINITIONS, NOTATIONS AND BASIC FACTS

1.1. Definition. Let (S, Ω) be a closure space, $C \subseteqq S$. Then let us define *a relation* $\varrho_c(\Omega)$ on S, called *a C-polarity*, in this way: For every elements $a, b \in S$ there is $a\varrho_c(\Omega)\, b$, if $\bar{a} \cap \bar{b} \subseteqq \bar{C}$.

Further, for each set $A \subseteqq S$ let us define sets $p(A, C) = \{s \in S : s\varrho_c(\Omega)\, a$, *for each* $a \in A\}$, $p^{n+1}(A, C) = p[p^n(A, C), C]$, for each positive integer n. A set $A \subseteqq S$ with a property $A = p^2(A, C)$ is called *a C-polar*.

Remarks. 1. A C-polarity is a symmetric and antireflexive relation ($a\varrho_c(\Omega)\, a \Rightarrow$ $\Rightarrow a\varrho_c(\Omega)\, s$ for each $s \in S$).

2. If S is an l-group, Ω is a system of all convex l-subgroups in S, then $p(A, \{0\}) =$ $= A'$ is a usual polar of a set A in an l-group S, introduced by F. Šik – see [5]. Other examples are in § 3.

1.2. Proposition. For every $A, C \subseteqq S$ it holds:
a) $p(A, C) \cap p^2(A, C) = \bar{C}$,
b) $p(A, C) = p(A, \bar{C})$,
c) $p(A, S) = S, p(A, A) = S, p(S, A) = \bar{A}, p(\Phi, A) = S$,
d) A correspondence $A \rightarrow p(A, C)$ forms a Galois connexion.

Proof. a) If $x \in \bar{C}$ is an arbitrary element, then for each $A \subseteq S$ and each $a \in A$ it is $\bar{x} \cap \bar{a} \subseteq \bar{C}$, i.e., $\bar{C} \subseteq p(A, C) \cap p^2(A, C)$. Conversely, for each $x \in p(A, C) \cap \cap p^2(A, C)$ we have $x \in \bar{x} = \bar{x} \cap \bar{x} \subseteq \bar{C}$ and thus $p(A, C) \cap p^2(A, C) \subseteq \bar{C}$.

The definition 1.1 implies b), c) and d).

1.3. Corollary. For every $I \neq \Phi$, $A_i \subseteq S(i \in I)$, $C \subseteq S$ it holds:

$$p(\bigcup_{i \in I} A_i, C) = \bigcap_{i \in I} p(A_i, C), \qquad p[\bigcap_{i \in I} p^2(A_i, C), C] = p^2[\bigcup_{i \in I} p(A], C), C].$$

1.4. Proposition. If $A, B, C \subseteq S$, $C_i \subseteq S$, $(i \in I \neq \Phi)$, then:

a) $B \subseteq C \Rightarrow p(A, B) \subseteq p(A, C)$,

b) $\bigcap_{i \in I} p(A, C_i) = p(A, \bigcap_{i \in I} C_i)$.

Proof. a) For each $x \in p(A, B)$ and each $a \in A$ we have $\bar{x} \cap \bar{a} \subseteq \bar{B} \subseteq \bar{C}$, i.e., $x \in p(A, C)$.

b) $\bigcap_{i \in I} p(A, C_i) \supseteq p(A, \bigcap_{i \in I} C_i)$ – see a) and 1.2, b). If $x \in \bigcap_{i \in I} p(A, C_i)$, then $\bar{x} \cap \bar{a} \subseteq \bar{C}_i$, for each $a \in A$ and each $i \in I$, i.e., $\bar{x} \cap \bar{a} \subseteq \bigcap_{i \in I} \bar{C}_i$, $x \in p(A, \bigcap_{i \in I} C_i)$.

1.5. Proposition. If $A, C \subseteq S$, then:

a) $p(A, C) = p(A, \bar{A} \cap \bar{C}) = p(A \cup C, C)$,

b) $A \subseteq \bar{C} \Leftrightarrow A \subseteq p(A, C) \Leftrightarrow p(A, C) = S$.

Proof. *a)* $p(A, \bar{A} \cap \bar{C}) \subseteq p(A, C)$ – see 1.4, a), $p(A \cup C, C) \subseteq p(A, C)$ – see 1.2, d). If $x \in p(A, C)$, then $\bar{x} \cap \bar{a} \subseteq \bar{C} \cap \bar{A}$ for each $a \in A$ and $x \in p(A, \bar{A} \cap \bar{C})$. Further, $\bar{x} \cap \bar{y} \subseteq \bar{C}$, for each $y \in A \cup C$, i.e., $x \in p(A \cup C, C)$.

b) $A \subseteq \bar{C}$ implies $A \subseteq p(A, C)$ – see 1.2, a). Now, if $A \subseteq p(A, C)$, then $\bar{a} \cap \bar{s} \subseteq \bar{C}$ for each $a \in A$, $s \in S$, i.e., $S = p(A, C)$. Finally, $p(A, C) = S$ implies $a \in \bar{a} = \bar{a} \cap \bar{a} \subseteq \subseteq \bar{C}$, for each $a \in A$.

1.6. Proposition. If $A, B \subseteq S$, then:

a) $\bar{A} = \cap\{p(S \setminus A, C) : \bar{C} \supseteq A\} = p(S \setminus A, A)$,

b) $p(A, B) \cap p(S \setminus A, B) = \bar{B}$.

Proof. *a) If* $x \in \cap\{p(S \setminus A, C) : \bar{C} \supseteq A\} \setminus \bar{A}$, then $\bar{x} \cap \bar{s} \subseteq \bar{C}$ for each $s \notin A$ and each $\bar{C} \supseteq A$ and $x \in \bar{x} = \bar{x} \cap \bar{x} \subseteq \bar{C}$, i.e., $x \in \cap\{\bar{C} : \bar{C} \supseteq A\} = \bar{A}$, a contradiction. The second inclusion is clear with regard to 1.2, a). The rest of a) follows from 1.2, d). *b)* If $x \in p(A, B) \cap p(S \setminus A, B)$, then either $x \in A$ and $x \in \bar{x} = \bar{x} \cap \bar{x} \subseteq \bar{B}$ or $x \in S \setminus A$ and again $x \in \bar{x} = \bar{x} \cap \bar{x} \subseteq \bar{B}$. The second inclusion follows from 1.2, a).

1.7. Proposition. If $A, C \subseteq S$, $B \in \Omega$, $B \cap \bar{A} \subseteq \bar{C}$, then $B \subseteq p(A, C)$.

Proof. If $B \in \Omega$, $B \cap \bar{A} \subseteq \bar{C}$, then $\bar{a} \cap \bar{b} \subseteq \bar{A} \cap B \subseteq \bar{C}$ for each $a \in A$, $b \in B$, i.e., $B \subseteq p(A, C)$.

§ 2. PRIME C-SETS

2.1. Definition. Let (S, Ω) be a closure system, $P \in \Omega$, $C \subseteq S$. A set P is called *a prime C-set*, if $p(s, C) \subseteq P$, for each $s \in S \setminus P$. *A prime ω-set*, where $\omega = \cap\{Q: Q \in \Omega\}$, is called *a prime set.*

Remark. For each prime C-set P we have $C \subseteq P$.

2.2. Proposition. If $P \in \Omega$, then following assertions are equivalent:
(I) P is a prime P-set,
(II) $P = A \cap B \Rightarrow P = A$ *or* $P = B$, for each A, $B \in \Omega$,
(III) $P \supseteq A \cap B \Rightarrow P \supseteq A$ or $P \subseteq B$, for each A, $B \in \Omega$,
(IV) $p(A, P) = P$ or $p(A, P) = S$, for each $A \subseteq S$.

Proof. (I) $\Rightarrow$ (II): If $P = A \cap B$, $P \neq A$, then $a \in A \setminus P$ exists and $P \subseteq p(A, P) \subseteq p(\{a\}, P) \subseteq P$. From this $B \supseteq p(A, P) = P = A \cap B \subseteq B$, i.e., $B = P$.
(II) $\Rightarrow$ (III): If $P \supseteq A \cap B$, P *non* $\subseteq A$, then $a \in A \setminus P$ exists and $P = p(\{a\}, P) \cap p^2(\{a\}, P)$, $q \in p^2(\{a\}, P) \neq P$. Hence $B \subseteq p(A, P) \subseteq p(\{a\}, P) = P$.
(III) $\Rightarrow$ (IV): If $P \neq p(A, P)$, then $P = p(A, P) \cap p^2(A, P)$ implies $p^2(A, P) \subseteq P$, i.e., $p^2(A, P) = P$ and it means that $p(A, P) = S$.
(IV) $\Rightarrow$ (I): If $s \in S \setminus P$, then $p(s, P) = S$ implies $P = p(S, P) \cap p^2(S, P) = p^2(S, P)$ and $s \in P$, a contradiction. Thus $p(s, P) = P$.

2.3. Proposition. If P is a prime C-set in a set (S, Ω), then $p(C, P) = S$ and $p^2(P, C) = S$ or P is a maximal C-polar.

Proof. $p(C, P) = S$ – see 1.5, b) and Remark before 2.2. If $P \neq p^2(P, C)$, then $x \in p^2(P, C) \setminus P$ exists and from this $p(P, C) \subseteq p(\{x\}, C) \subseteq P$, i.e., $p(P, C) = p(P, C) \cap p^2(P, C) = \bar{C}$ and $p^2(P, C) = S$. If P is a C-polar and $p^2(A, C) \supseteq P$ such that an element $s \in p^2(A, C) \setminus P$ exists, then $p(A, C) \subseteq p(\{s\}, C) \subseteq P \subseteq p^2(A, C)$ and $C = p(A, C) \cap p^2(A, C) = p(A, C)$, i.e., $p^2(A, C) = S$.

2.4. Definition. Let P, $Q \in \Omega$, $P \subseteq Q$. Then we say that Q *has a property* $P(\Omega)$ (notation: $Q \in P(\Omega)$) if it holds:

If $A \cap B = Q$, for A, $B \in \Omega$, then A', $B' \in \Omega$ exist such that $A' \cap B' = P$ and $A \subseteq \overline{A' \cup Q}$, $B \subseteq \overline{B' \cup Q}$.

Remark. A prime Q-set is clearly a prime C-set, for each $C \subseteq Q$, $C \in \Omega$.

2.5. Proposition.. Let P, $Q \in \Omega$, $P \subseteq Q$. Then Q is a prime Q-set if and only if $Q \in P(\Omega)$ and Q is a prime P-set.

Proof. $\Rightarrow$: If $Q = A \cap B$, then $A = Q$ or $B = Q$ – see 2.2. Let us suppose that $A = Q$. Then $A' \cap B' = P$, for $A' = P$, $B' = B$ and $\overline{A' \cup Q} = \overline{Q \cup P} = Q = A$, $\overline{B' \cup Q} = \overline{B \cup Q} = B$. $\Leftarrow$: If $s \in S \setminus Q$, then $Q = p(\{s\}, Q) \cap p^2(\{s\}, Q)$ implies the existence of sets $A', B' \in \Omega$ such that $A' \cap B' = P$, $p(\{s\}, Q) \subseteqq \overline{A' \cup Q}$, $p^2(\{s\}, Q) \subseteqq \subseteqq \overline{B' \cup Q}$. If $A' \subseteqq Q$, then $p(\{s\}, Q) \subseteqq Q$. If A' non $\subseteqq Q$, then $a \in A' \setminus Q$ exists and thus $p(\{a\}, P) \subseteqq Q$, $B' \subseteqq p(A', P) \subseteqq p(\{a\}, P) \subseteqq Q$, because Q is a prime P-set. Finally, $s \in p^2(\{s\}, Q) \subseteqq B' \cup Q = Q$, a contradiction. Finally, Q is a prime Q-set.

2.6. Theorem. If (S, Ω) is a closure space, then for each $P \in \Omega$ the following assertions are equivalent:

(I) The set inclusion is a fully relation on $\Omega_P = \{X \in \Omega : X \supseteqq P\}$ and for each $Q \in \Omega_P$ and each $s \in S \setminus Q$ it is $p(s, Q) \in \Omega$.

(II) Each $Q \in \Omega_P$ is a prime Q-set.

(III) A set $C \in \Omega$, $C \subseteqq P$ exists such that P is a prime C-set and $\Omega_P \subseteqq C(\Omega)$.

Proof. (I) $\Rightarrow$ (II): If $s \in S \setminus Q$, $s \notin p(s, Q)$, then $p(s, Q) \in \Omega_P$, $s \cup Q \in \Omega_P$ and thus $p(s, Q) \subseteqq s \cup Q$, what is a contradiction.
(II) $\Rightarrow$ (I): If $A, B \in \Omega_P$, $A \neq A \cap B$, then $a \in A \setminus A \cap B$ exists and $B \subseteqq p(\{a\}, B \subseteqq B$, $p(\{a\}, A) \supseteqq p(A, A) = S$ (see 1.2, c)). Further, $B = p(\{a\}, B) \cap p(\{a\}, A) = p(\{a\}, A \cap B) \subseteqq A \cap B$, from $A \cap B \in \Omega_P$ and 1.4, b). It implies $B \subseteqq A$.
(II) $\Leftrightarrow$ (III) immediately from 2.5.

2.7. Proposition. If $(G, \geqq)$ is an l-group with a lattice order $\geqq$ and if Ω is a system of all convex l-subgroups in G, then it holds:

1. $(G, \geqq)$ is a fully ordered set if and only if a system Ω is fully ordered by set inclusion.

2. If P is a prime set in G, then each $Q \in \Omega_P$ is a prime Q-set.

3. $\Omega_P \subseteqq C(\Omega)$, for each prime C-set P, $C \in \Omega$.

Proof. 1. $\Rightarrow$: If $A, B \in \Omega$, $A \parallel B$, then $a \in A \setminus B$, $b \in B \setminus A$ exist such that $a \geqq 0$, $b \geqq 0$. If $a \geqq b (b \geqq a)$, then $b \in A$, $(a \in B)$, a contradiction.

$\Leftarrow$: If $a, b \in G$, $a \parallel b$, then $c \wedge d = 0$, for $c = a - (a \wedge b)$, $d = b - (a \wedge b)$, $c, d \in G \setminus \{0\}$. It means that $p^2(\{c\}, \{0\}) \neq \{0\} \neq p^2(\{d\}, \{0\})$, $p^2(\{c\}, \{0\}), p^2(\{d\}, \{0\}) \in \in \Omega$, $p^2(\{c\}, \{0\}) \cap p^2(\{d\}, \{0\}) = \{0\}$, a contradiction.

2. If P is a prime set in G, then a right decomposition G/P is a fully ordered set. Then for every $A, B \in \Omega_P$, $A \parallel B$ there exist elements $a \in A \setminus B$, $b \in B \setminus A$, $a \geqq 0$, $b \geqq 0$. The right classes $a + P$, $b + P$ are comparable. If $a + P \geqq b + P$, then $a + P \subseteqq A$, $b \in A$, a contradiction. From this Ω_P is fully ordered by set inclusion. The rest follows from Theorem 2.6.

3. P is a prime convex l-subgroup in $G (s \in S \setminus P \Rightarrow p(\{s\}, \{0\}) = p(\{s\}, \omega) \subseteqq \subseteqq p(\{s\}, C) \subseteqq P)$ and the right decomposition G/P is a fully ordered set. Then for every $A, B \in \Omega_P$, $A \parallel B$ there exist elements $a \in A \setminus B$, $b \in B \setminus A$, $a \geqq 0$, $b \geqq 0$. Right classes $a + P$, $b + P$ are comparable. If $a + P \geqq b + P (a + P \leqq b + P)$, then $a + P \subseteqq A (b + P \subseteqq B)$ and $b \in A (a \in B)$, a contradiction. Finally, Ω_P is fully ordered by set inclusion.

Now, for each $Q \in \Omega_P$, $Q = A \cap B$, $A, B \in \Omega_P$ implies $Q = A$ (or $Q = B$). If we choose $A' = C$, $B' = B$ ($A' = A$, $B' = C$), then $A' \cap B' = C$ and $\overline{A' \cup Q} = Q = A$, $\overline{B' \cup Q} = B$ ($\overline{A' \cup Q} = A$, $B' \cup Q = Q = B$) and thus $Q \in C(\Omega)$.

§ 3. EXAMPLES

I. R. D. Byrd in [1] defines a C-polarity on an 1-group $(G, \geqq)$ with a closure system Ω of all convex 1-subgroups in G for each $C \subseteqq G$ in the following way:

$$a\beta b \Leftrightarrow |a| \wedge |b| \in \bar{C}, \text{ for } a, b \in G \text{ (where } |a| = a \vee -a).$$

3.1. Lemma. If $a, b, c \in G$, $a \geqq 0, b \geqq 0, c \geqq 0$, $C \in \Omega$, $a \wedge b \in C$, then $[a \wedge \wedge(b + c)] - (a \wedge c) \in C$ and $(ma \wedge nb) - (a \wedge b) \in C$, for every positive integer m, n.

Proof. $a \wedge c \leqq a \wedge (b + c) = a \wedge (b + c) \wedge (a + c) = a \wedge [(b \wedge a) + c] \leqq \leqq [(b \wedge a) + a] \wedge [(b \wedge a) + c] = (b \wedge a) + (a \wedge c)$ implies $0 \leqq a \wedge (b + c) - - (a \wedge c) \leqq b \wedge a = a \wedge b$. The rest follows from convexity of G.

3.2. Proposition. If $(G, \geqq)$ is an 1-group and Ω is a closure system of all convex 1-subgroups in G, $C \subseteqq G$, then C-polarity β is C-polarity $\varrho_c(\Omega)$.

Proof. If $a\varrho_c(\Omega)\, b$, $a, b \in G$, then $\bar{a} \cap \bar{b} \subseteqq \bar{C}$ and $|a| \wedge |b| \in \bar{a} \cap \bar{b} \subseteqq \bar{C}$, i.e., $a\beta b$.

If $a\beta b$ and $x \in \bar{a} \cap \bar{b}$, then positive integers m, n exist such that $|x| \leqq n|a|$, $|x| \leqq m|b|$, i.e., $|x| \leqq n|a| \wedge m|b|$. Lemma 3.1 implies $n|a| \wedge m|b| \in \in (|a| \wedge |b|) + \bar{C} = \bar{C}$ and $x \in \bar{C}$ from convexity $\bar{C}$, i.e., $\bar{a} \cap \bar{b} \subseteqq \bar{C}$, $a\varrho_c(\Omega)\, b$.

II. Let L be a lattice and I be an ideal in a lattice L ($x, y \in I$, $z \in L$, $z \leqq x \Rightarrow \Rightarrow x \vee y \in I$, $z \in I$). Then a set Ω of all ideals of a lattice L is a closure system and we define a relation γ_C, for each $C \in \Omega$, in the following way:

$$y\gamma_c y \Leftrightarrow x \wedge y \in C, \qquad \text{for } x, y \in L.$$

3.3. Proposition. If L is a lattice with a closure system Ω of all ideals in L, then γ_c is a C-polarity $\varrho_c(\Omega)$, for each $C \in \Omega$.

Proof. $\Leftarrow$: $x\varrho_c(\Omega)\, y \Rightarrow \bar{x} \cap \bar{y} \subseteqq C \Rightarrow x \wedge y \in \bar{x} \cap \bar{y} \subseteqq C \Rightarrow x\gamma_c y$.

$\Rightarrow$: If $m \in \bar{x} \cap \bar{y} = \{l \in L : l \leqq x\} \cap \{l \in L : l \leqq y\} = \{l \in L : l \leqq x \wedge y\}$, then $m \leqq x \wedge y$. If $x\gamma_c y$, then $x \wedge y \in C$ and $m \in C$, $\bar{x} \cap \bar{y} \subseteqq C$, i.e., $x\varrho_c(\Omega)\, y$.

III. Let M be a partially ordered set, $\bar{N} = \{m \in M : m \leqq n,$ for each $n \in N\}$, $N \subseteqq M$, $\Omega = \{\bar{N} : N \subseteqq M\}$. Then Ω is a closure system in M and we define a relation μ_c, for each $C \in \Omega$, in the following way:

$$x\mu_c y \Leftrightarrow \{z \in M, z \leqq x, z \leqq y \Rightarrow z \in C\}, \qquad \text{for } x, y \in M.$$

3.4. Proposition. A relation μ_C is a C-polarity $\mu_c(\Omega)$, for each $C \in \Omega$.

Proof. $\Leftarrow$: If $z \leqq x$, $z \leqq y$, then $z \in \bar{x} \cap \bar{y}$ and if $x\sigma_c(\Omega)\, y$, then $z \in \bar{x} \cap \bar{y} \subseteqq C$ and $x\mu_c y$.

$\Rightarrow$: If $x\mu_c y$, $z \in \bar{x} \cap \bar{y}$, then $z \leqq x$, $z \leqq y$ and thus $z \in C$, i.e., $\bar{x} \cap \bar{y} \in C$, $x\varrho_c(\Omega)\, y$.

IV. A. W. Glass defines in [2] C-polars on an interpolation partially ordered group $G(s, t, u, v \in G,\ s, t \leqq u,\ v \Rightarrow x \in G$ exists such that $s, t \leqq x \leqq u, v)$ with a closure system Ω generated by the set $C(G)$ of all dc-subgroups (directed convex subgroups) in G.

A notation $p_G(A, C)$ will be used for C-polars of Glass. If $C(G)$ is the set of all dc-subgroups in G and $C(G) = \Omega$, then G is called a strong interpolation group (see [2]).

3.5. Proposition. If G is a strong interpolation group (interpolation group) with a closure system Ω, then $p(\bar{A}, C) = p_G(A, C) = p_G(\langle A\rangle, C$ $(p_G(\langle A\rangle, C) \subseteqq p(\bar{A}, C))$, where $\langle A\rangle$ is the smallest dc-subgroup in G containing A.

Proof. If $k \in p_G(\langle A\rangle, C)$, then $\bar{A} \cap \bar{k} \subseteqq \bar{k} \subseteqq \langle k\rangle$ and because $p_G(\langle A\rangle, C)$ is a dc-subgroup in G (see [2], after 3.2), there is $\bar{k} \cap \bar{A} \subseteqq p_G(\langle A\rangle, C)$. Further, [2], Remark before L.9 and L.8, (i) implies $\bar{k} \cap \bar{A} \subseteqq \bar{A} \subseteqq \langle A\rangle \subseteqq p_G^2(\langle A\rangle, C)$ and $\bar{k} \cap \bar{A} \subseteqq$ $\subseteqq p_G(\langle A\rangle, C) \cap p_G^2(\langle A\rangle, C) = C = C$ – see L.9, (i). Finally, $k \in p(\bar{A}, C)$ and $p_G(\langle A\rangle, C) \subseteqq p(\bar{A}, C)$.

In case that G is a strong interpolation group and $k \in p_G(A, C)$, then $\bar{k} \cap \bar{A} \subseteqq$ $\subseteqq \langle k\rangle \subseteqq p_G(A, C)$, because $p_G(A, C)$ is a dc-subgroup in G. Further, $\bar{k} \cap \bar{A} \subseteqq$ $\subseteqq p_G^2(A, C)$, see [2], L.8, (ii) and thus $\bar{k} \cap \bar{A} \subseteqq p_G(A, C) \cap p_G^2(A, C) = C$. Finally, $p_G(A, C) \subseteqq p(\bar{A}, C)$.

For the converse, if $k \in p(\bar{A}, C)$, then $\bar{k} \cap \bar{A} \subseteqq C \cap \bar{A} = C \cap \langle A\rangle$. Further, [2], L.7, (iv) implies $p_G(\langle A\rangle, C) = p_G(\langle A\rangle, C \cap \langle A\rangle)$ and [2], L.9, (ii) and Remark before implies $\langle A\rangle \cap p_G(\langle A\rangle, C) = \langle A\rangle \cap p_G(\langle A\rangle, C \cap \langle A\rangle) = C \cap \langle A\rangle$. Hence and from [2], L.9, (iv) we have $k \in \langle k\rangle \subseteqq p_G(\langle A\rangle, C \cap \langle A\rangle) = p_G(\langle A\rangle, C)$, i.e., $p(\bar{A}, C) \subseteqq p_G(\langle A\rangle, C) \subseteqq p_G(A, C) \subseteqq p(\bar{A}, C)$ – see [2], L.5, (ii).

V. J. Rachůnek in [3] defines on a po-group G a polarity $\delta : x, y \in G$, $x\delta y \Leftrightarrow$ $\Leftrightarrow a, b \in G$ exist, such that $a \geqq 0$, $b \geqq 0$, $a \in |x|$, $b \in |y|$, $a \wedge b = 0$, where $|x| =$ $= \{g \in G : g \geqq x,\ g \geqq -x\}$ for each $x \in G$.

Po-group G is called 2-isolated, when:

$$a \in G, \quad a \geqq -a \Rightarrow a \geqq 0.$$

3.6. Proposition. Let G be a 2-isolated po-group, Ω be the smallest closure system containing a set $C(G)$ of all dc-subgroups in G and

(I) $|x| \neq \Phi$, for each $x \in G$,

(II) $x \vee -x$ exists for each $x \in G$.

Then a polarity δ is $\varrho_{\{0\}}(\Omega)$.

Proof. If $x\delta y$, then $a, b \in G$ exist such that $a, b \geqq 0$, $a \in |x|$, $b \in |y|$, $a \vee b = 0$. [3], Prop. 2.5 implies that $g^\delta = \{x \in G_v\ x\delta g\}$ is a dc-subgroup in G, for each $g \in G$. Hence $\overline{g^\delta} = g^\delta$ and therefore $y \in x^\delta$, $y \in \bar{y} \subseteqq \overline{x^\delta} = x^\delta$, $x \in \bar{x} \subseteqq x^{\delta\delta}$ – see [3]. Definition mentioned after 2.3. Finally, $\bar{x} \cap \bar{y} \subseteqq x^\delta \cap x^{\delta\delta} = \{0\}$ (see [3], Th. 2.6). It means that $x\varrho_{\{0\}}(\Omega)\,y$.

If $x\varrho_{\{0\}}(\Omega)\,y$, then $\bar{x} \cap \bar{y} = \{0\}$ and $\bar{x} = \cap\{Q \in C(G) : x \in Q\}$. Further, if $x \in Q$, $Q \in C(G)$, then $-x \in Q$ and $d \in Q$ exists such that $d \geqq x, -x$. Hence $d \geqq x \vee -x \geqq$ $\geqq x, -x$, i.e., $x \leqq -x \in Q$ and $x \vee -x \in \bar{x}$. Similarly $y \vee -y \in \bar{y}$ and from this $x \vee -x \geqq 0$, $y \vee -y \geqq 0$, $x \vee -x \in |x|$, $y \vee -y \in |y|$, $(x \vee -x) \wedge (y \vee -y) \in$ $\in \bar{x} \cap \bar{y} = \{0\}$, i.e., $x\delta y$.

3.7. Proposition. Let $(G, \geqq)$ be a 2-isolated po-group, Ω be the smallest closure system containing the set $C(G)$ of all dc-subgroups in G. Then the following assertions are equivalent:

(I) $(|G|, \subseteqq)$ is fully ordered, where $|G| = \{|g| : g \in G\}$,

(II) $(C(G), \subseteqq)$ is fully ordered,

(III) $(G^+, \geqq)$ is fully ordered, where $G^+ = \{g \in G : g \geqq 0\}$.

Moreover, in case that $(G, \geqq)$ is directed, then $(C(G), \subseteqq)$ is fully ordered if and only if $(G, \geqq)$ is fully ordered.

Remark. If G is directed interpolation group, then Ω is not fully ordered (see [2], Remark before Th. 23).

Proof. (I) $\Rightarrow$ (II): If $A, B \in C(G)$, $A \parallel B$, then elements $a \in A \setminus B$, $b \in B \setminus A$ exist and $|x| \parallel |a|$ for each $x \in B \setminus A$. Namely, if $|x| \supseteqq |a|$, then $x \in A$, see [3], Lemma after 1.1, a contradiction. For $|x| \subseteqq |a|$ we have $a \in B$, similarly and again a contradiction.

(II) $\Rightarrow$ (III): If $a, b \in G^+$ exist such that $a \parallel b$, then $|a|$ non $\supseteqq |b|$ and $|b|$ non $\supseteqq |a|$. Suppose $\langle |a| \rangle \supseteqq \langle |b| \rangle$. Then for each $x \in |b| \subseteqq \langle |b| \rangle \subseteqq \langle |a| \rangle$ there exist elements $g_i \in |a|$, $i = 1, 2, \ldots, n$ such that for $p = g_1 + g_2 + \ldots + g_n$ there is $|x| \supseteqq |p|$. But $p \geqq g_i \geqq a$, for $i = 1, 2, \ldots, n$ and thus $p \in |a|$, which is in a contradiction with [3], Lemma mentioned after 1.1.

(III) $\Rightarrow$ (I): $a \leqq b$ if and only if $|a| \supseteqq |b|$, for every $a, b \in G^+$. The rest is evident from the fact that $G = G^+ - G^+$ for a directed po-group.

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