Christos G. Philos

Some results on the oscillatory and asymptotic behavior of the solutions of differential equations with deviating arguments

Archivum Mathematicum, Vol. 15 (1979), No. 1, 19--36

Persistent URL: `http://dml.cz/dmlcz/107021`

ARCH. MATH. 1, SCRIPTA FAC. SCI. NAT. UJEP BRUNENSIS
XV: 19—36, 1979

SOME RESULTS ON THE OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS*

CH. G. PHILOS, Greece
(Received August 1, 1977)

Let r_i $(i = 0, 1, \dots, n)$ be positive continuous functions on the interval $[t_0, \infty)$. For a real-valued function h on $[T, \infty)$, $T \geqq t_0$, and any $\mu = 0, 1, \dots, n$ we define the *μ-th r-derivative* of h by the formula

$$D_r^{(\mu)}h = r_\mu(r_{\mu-1}(\dots(r_1(r_0h)')'\dots)')'.$$

Then we obviously have

$$D_r^{(0)}h = r_0h \qquad \text{and} \qquad D_r^{(i)}h = r_i(D_r^{(i-1)}h)' \qquad (i = 1, 2, \dots, n).$$

If $D_r^{[n]}h$ is defined on $[T, \infty)$, then the function h is said to be n-times *r-differentiable*.

Now, we consider the n-th order $(n > 1)$ differential equation with deviating arguments of the form

$$(E, \delta) \qquad (D_r^{(n)}x)(t) + \delta\{\sum_{i=1}^{\nu} p_i(t) F_i(x\langle g(t)\rangle) + G(t; x\langle\sigma_0(t)\rangle, (D_r^{(1)}x)\langle\sigma_1(t)\rangle, \dots, (D_r^{(n-1)}x)\langle\sigma_{n-1}(t)\rangle)\} = 0,$$

where $\delta = \pm 1$, $r_0 = r_n = 1$ and

$$\begin{cases} x\langle g(t)\rangle = (x[g_1(t)], x[g_2(t)], \dots, x[g_m(t)]), \qquad g = (g_1, g_2, \dots, g_m) \\ (D_r^{(i)}x)\langle\sigma_i(t)\rangle = ((D_r^{(i)}x)[\sigma_{i1}(t)], (D_r^{(i)}x)[\sigma_{i2}(t)], \dots, (D_r^{(i)}x)[\sigma_{im_i}(t)]), \\ \sigma_i = (\sigma_{i1}, \sigma_{i2}, \dots, \sigma_{im_i}) \qquad (i = 0, 1, \dots, n-1). \end{cases}$$

The continuity of the real-valued functions p_i $(i = 1, 2, \dots, \nu)$, g_j $(j = 1, 2, \dots, m)$ and σ_{ij} $(j = 1, 2, \dots, m_i;\ i = 0, 1, \dots, n-1)$ on $[t_0, \infty)$, F_i $(i = 1, 2, \dots, \nu)$ on $\mathbf{R}^m$

* This paper is a part of the author's Doctoral Thesis submitted to the School of Physics and Mathematics of the University of Ioannina.

and G on $[t_0, \infty) \times \mathbf{R}^{m_0} \times \mathbf{R}^{m_1} \times \ldots \times \mathbf{R}^{m_{n-1}}$ as well as sufficient smoothness for the existence of solutions of (E, δ) on an infinite subinterval of $[t_0, \infty)$ will be assumed without mention. In what follows the term "solution" is always used only for such solutions $x(t)$ of (E, δ) which are defined for all large t. The oscillatory character is considered in the-usual sense, i.e. a continuous real-valued function which is defined on an interval of the form $[T_0, \infty)$ is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

Furthermore, the following conditions are assumed to hold throughout the paper:

(i) *The functions* p_i $(i = 1, 2, \ldots, \nu)$ *are nonnegative on the interval* $[t_0, \infty)$.

(ii) *For any* i, $1 \leqq i \leqq \nu$, *the function* F_i *has the following sign property*

$$(\forall j = 1, 2, \ldots, m)\, y_j > 0 \Rightarrow F_i(y_1, y_2, \ldots, y_m) > 0$$

and

$$(\forall j = 1, 2, \ldots, m)\, y_j < 0 \Rightarrow F_i(y_1, y_2, \ldots, y_m) < 0.$$

(iii) *For every* $j = 1, 2, \ldots, m$

$$\lim_{t \to \infty} g_j(t) = \infty.$$

(iv) *For* $i = 0, 1, \ldots, n-1$ *and every* $j = 1, 2, \ldots, m_i$

$$\lim_{t \to \infty} \sigma_{ij}(t) = \infty.$$

(v) *For every* $(t; z_0, z_1, \ldots, z_{n-1}) \in [t_0, \infty) \times \mathbf{R}^{m_0} \times \mathbf{R}^{m_1} \times \ldots \times \mathbf{R}^{m_{n-1}}$

$$(\forall j = 1, 2, \ldots, m_0)\, z_{0j} > 0 \Rightarrow G(t; z_0, z_1, \ldots, z_{n-1}) \geqq 0$$

and

$$(\forall j = 1, 2, \ldots, m_0)\, z_{0j} < 0 \Rightarrow G(t; z_0, z_1, \ldots, z_{n-1}) \leqq 0,$$

where $z_0 = (z_{01}, z_{02}, \ldots, z_{0m_0})$.

Also, we suppose that:

(R) *For every* $i = 1, 2, \ldots, n-1$

$$\int^{\infty} \frac{dt}{r_i(t)} = \infty.$$

For general interest on oscillation results concerning differential equations involving the r-derivatives $D_r^{(i)}x$ $(i = 0, 1, \ldots, n)$ of the unknown function x we choose to refer to the papers [2], [5], [6], [8]÷[14] and [18]÷[20].

The oscillatory character and the asymptotic behavior of the bounded solutions of the differential equation (E, δ) are well described by the following theorem. The proof of this theorem is omitted, since it follows as in [11, Thm 2] (cf. also [8, Thm 2]).

Theorem 0. *Consider the differential equation* (E, δ) *subject to the conditions* (i)÷(v), (R) *and*:

(C_0) *There exists an integer* k *with* $0 \leqq k \leqq n-1$ *and such that*

$$\begin{cases} \int^{\infty} p_{i_0}(t)\,dt = \infty, & \text{if } k = n-1 \\ \int^{\infty} \frac{1}{r_{k+1}(s_{k+1})} \cdots \int_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} p_{i_0}(s)\,ds\,ds_{n-1} \ldots ds_{k+1} = \infty, & \text{if } k < n-1 \end{cases}$$

for some i_0, $1 \leqq i_0 \leqq \nu$.

Then every bounded solution x *of the equation* $(E, +1)$ [*respectively, of the equation* $(E, -1)$] *for* n *even* [*resp. odd*] *is oscillatory, while for* n *odd* [*resp. even*] *is either oscillatory or such that*

$$\lim_{t \to \infty} (D_r^{(i)} x)(t) = 0 \qquad \textit{monotonically } (i = 0, 1, \ldots, n-1).$$

In this paper we study the oscillatory and asymptotic behavior of all solutions of the differential equation (E, δ). More precisely, we give conditions under which every solution x of the equation $(E, +1)$ for n even is oscillatory while for n odd is either oscillatory or such that

$$\lim_{t \to \infty} (D_r^{(i)} x)(t) = 0 \qquad \text{monotonically } (i = 0, 1, \ldots, n-1).$$

Moreover, we classify the solutions of the equation $(E, -1)$ with respect to their oscillatory character and to their behavior at ∞.

For this purpose, we make use of the following two lemmas given by the author in [10]. We note that the first of these lemmas is a natural extension of Lemma 1 in [1], which is a unified adaptation of two well-known lemmas due to Kiguradze [3, 4], while the second one is rather technical.

Lemma 1. *Let the condition* (*R*) *be satisfied and let* h *be a positive and* n*-times* r*-differentiable function on the interval* $[T, \infty)$, $T \geqq t_0$, *such that* $D_r^{(n)} h$ *is of constant sign on* $[T, \infty)$ *and not identically zero on any interval of the form* $[\tau, \infty)$, $\tau \geqq T$.

Then there exists an integer l, $0 \leqq l \leqq n$, *with* $n + l$ *odd for* $D_r^{(n)} h \leqq 0$ or $n + l$ *even for* $D_r^{(n)} h \geqq 0$ *and such that*

$$\begin{cases} l \leqq n-1 \Rightarrow (-1)^{l+j} (D_r^{(j)} h)(t) > 0 \textit{ for every } t \geqq T \; (j = l, l+1, \ldots, n-1) \\ l > 1 \Rightarrow (D_r^{(i)} h)(t) > 0 \textit{ for all large } t \; (i = 1, 2, \ldots, l-1). \end{cases}$$

Lemma 2. *Suppose that the condition* (*R*) *is satisfied. Let* h *be a function whose the* r*-derivative* $D_r^{(n-1)} h$ *exists on an interval* $[T, \infty)$, $T \geqq t_0$, *and let*

$$R_{n-1}(t) = \int_{t_0}^{t = s_0} \frac{1}{r_1(s_1)} \int_{t_0}^{s_1} \frac{1}{r_2(s_2)} \cdots \int_{t_0}^{s_{n-2}} \frac{1}{r_{n-1}(s_{n-1})}\, ds_{n-1} \ldots ds_2\, ds_1, \qquad t \geqq t_0.$$

If the $\lim_{t \to \infty} (D_r^{(n-1)} h)(t)$ *exists in* $\mathbf{R}^* - \{0\}$ ($\mathbf{R}^* = \mathbf{R} \cup \{-\infty, \infty\}$ *is the extended real line*), *then*

$$\lim_{t \to \infty} \frac{h(t)}{R_{n-1}(t)} = \lim_{t \to \infty} (D_r^{(n-1)} h)(t).$$

For our purpose, for any integer λ with $0 \leqq \lambda \leqq n-1$ we put

$$R_\lambda(v; u) = \begin{cases} 1, & \text{if } \lambda = 0 \\ \int\limits_u^{v=s_0} \frac{1}{r_1(s_1)} \int\limits_u^{s_1} \frac{1}{r_2(s_2)} \cdots \int\limits_u^{s_{\lambda-1}} \frac{1}{r_\lambda(s_\lambda)} \, ds_\lambda \ldots ds_2 \, ds_1, & \text{if } \lambda > 0, \end{cases}$$

where $v \geqq u \geqq t_0$, and in particular

$$R_\lambda(t) = R_\lambda(t; t_0), \qquad t \geqq t_0.$$

Moreover, we consider the function g^* which is defined on the interval $[t_0, \infty)$ as follows

$$g^*(t) = \min \{t, \min_{1 \leqq j \leqq m} \inf_{s \geqq t} g_j(s)\}.$$

Obviously, for every $t \geqq t_0$ we have

$$g^*(t) \leqq t \qquad \text{and} \qquad g^*(t) \leqq g_j(s) \qquad \text{for all } s \geqq t \qquad (j = 1, 2, \ldots, m)$$

and, if (iii) is satisfied, it holds

$$\lim_{t \to \infty} g^*(t) = \infty.$$

Also, we consider the function F defined on $\mathbf{R}^m$ by the formula

$$F(y_1, y_2, \ldots, y_m) = \min_{1 \leqq i \leqq \nu} | F_i(y_1, y_2, \ldots, y_m) |$$

and for any nonnegative numbers α_j $(j = 1, 2, \ldots, m)$ we set

$$S_F[\alpha_1, \alpha_2, \ldots, \alpha_m] =$$

$$= \max \left\{ \limsup_{\substack{y_j \to \infty \\ 1 \leqq j \leqq m}} \frac{y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_m^{\alpha_m}}{F(y_1, y_2, \ldots, y_m)}, \limsup_{\substack{y_j \to -\infty \\ 1 \leqq j \leqq m}} \frac{|y_1|^{\alpha_1} |y_2|^{\alpha_2} \ldots |y_m|^{\alpha_m}}{F(y_1, y_2, \ldots, y_m)} \right\}.$$

Theorem 1. *Consider the differential equation* $(E, +1)$ *subject to the conditions* (i)$\div$(v), (R), (C_0) *and:*

(C_1) *There exist nonnegative numbers* α_j $(j = 1, 2, \ldots, m)$ *with* $\sum_{j=1}^m \alpha_j = 1$ *and* $S_F[\alpha_1, \alpha_2, \ldots, \alpha_m] < \infty$ *and such that for every integer* l *with* $1 \leqq l \leqq n - 1$ and $n + l$ *odd exactly one of the following is satisfied:*

(c_1) *There exists an integer* k, $l \leqq k \leqq n - 1$, *such that for some* i_0, $1 \leqq i_0 \leqq \nu$,

$$\begin{cases} \int\limits^\infty p_{i_0}(t) \prod_{j=1}^m [R_{l-1}(g_j(t))]^{\alpha_j} \, dt = \infty, \qquad \text{if } k = n - 1 \\ \int\limits^\infty \frac{1}{r_{k+1}(s_{k+1})} \cdots \int\limits_{n-2}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int\limits_{s_{n-1}}^\infty p_{i_0}(s) \prod_{j=1}^m [R_{l-1}(g_j(s))]^{\alpha_j} \, ds \, ds_{n-1} \ldots ds_{k+1} = \infty, \end{cases}$$

if $k < n - 1$.

(c_2) *It holds*

$$\begin{cases} \limsup\limits_{t\to\infty} \left[\int\limits_{t_0}^{g^*(t)} \frac{ds}{r_{n-1}(s)}\right] \int\limits_t^{\infty} \sum\limits_{i=1}^{\nu} p_i(s) \prod\limits_{j=1}^{m} [R_{n-2}(g_j(s);\ g^*(t))]^{\alpha_j}\, ds > S_F[\alpha_1, \alpha_2, \ldots, \alpha_m], \\ \text{if } l = n-1 \\ \limsup\limits_{t\to\infty} \left[\int\limits_{t_0}^{g^*(t)} \frac{ds}{r_l(s)}\right] \int\limits_t^{\infty} \frac{1}{r_{l+1}(s_{l+1})} \cdots \int\limits_{s_{n-2}}^{\infty} \frac{1}{r_{n-1}(s_{n-1})} \int\limits_{s_{n-1}}^{\infty} \sum\limits_{i=1}^{\nu} p_i(s) \\ \prod\limits_{j=1}^{m} [R_{l-1}(g_j(s);\ g^*(t))]^{\alpha_j}\, ds\, ds_{n-1} \ldots ds_{l+1} > S_F[\alpha_1, \alpha_2, \ldots, \alpha_m], \qquad \text{if } l < n-1. \end{cases}$$

Then every solution x of the equation $(E, +1)$ for n even is oscillatory, while for n odd is either oscillatory or such that

$$\lim_{t\to\infty} (D_r^{(i)}x)(t) = 0 \qquad \textit{monotonically } (i = 0, 1, \ldots, n-1).$$

Proof. Let x be an unbounded nonoscillatory solution on an interval $[T_0, \infty)$, $T_0 \geqq t_0$, of the equation $(E, +1)$. We assume, without loss of generality, that $x(t) \neq 0$ for all $t \geqq T_0$. Furthermore, we restrict ourselves in the case where x is positive, since the substitution $z = -x$ transforms $(E, +1)$ into an equation of the same form satisfying the assumptions of the theorem.

By (iii) and (iv), we choose a $T \geqq T_0$ so that for every $t \geqq T$

$$(1) \qquad \begin{cases} g_j(t) \geqq T_0 & (j = 1, 2, \ldots, m) \\ \sigma_{ij}(t) \geqq T_0 & (j = 1, 2, \ldots, m_i;\ i = 0, 1, \ldots, n-1). \end{cases}$$

Then, in view of (i), (ii) and (v), from equation $(E, +1)$ we obtain that for all $t \geqq T$

$$-(D_r^{(n)}x)(t) = \sum_{i=1}^{\nu} p_i(t)\, F_i(x\langle g(t)\rangle) +$$
$$+ G(t;\ x\langle\sigma_0(t)\rangle, (D_r^{(1)}x)\langle\sigma_1(t)\rangle, \ldots, (D_r^{(n-1)}x)\langle\sigma_{n-1}(t)\rangle) \geqq 0,$$

namely

$$(2) \qquad (D_r^{(n)}x)(t) \leqq 0 \qquad \text{for every } t \geqq T.$$

Moreover, $(D_r^{(n)}x)(t)$ is not identically zero for all large t. In fact, if for some $\tau \geqq T$ we have $D_r^{(n)}x = 0$ on $[\tau, \infty)$, then equation $(E, +1)$, by (i), (ii) and (v), gives that all functions p_i $(i = 1, 2, \ldots, \nu)$ are identically zero on $[\tau, \infty)$, which contradicts (C_1). Next, by applying Lemma 1 and taking into account the fact that x is unbounded, we conclude that there exists an integer l, $1 \leqq l \leqq n-1$, with $n + l$ odd and such that

$$(3) \qquad (-1)^{l+j}(D_r^{(j)}x)(t) > 0 \qquad \text{for every } t \geqq T \qquad (j = l, l+1, \ldots, n-1)$$

and, when $l > 1$, for some $T^* \geqq T$

$$(4) \qquad (D_r^{(i)}x)(t) > 0 \qquad \text{for every } t \geqq T^* \qquad (i = 1, 2, \ldots, l-1).$$

After these, for every $t \geqq T$ it holds

$$(D_r^{(l)}x)(t) \geqq \tag{5}$$

$$\geqq \begin{cases} -\int\limits_t^\infty (D_r^{(n)}x)(s)\,ds, & \text{if } l = n-1 \\ -\int\limits_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int\limits_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int\limits_{s_{n-1}}^\infty (D_r^{(n)}x)(s)\,ds\,ds_{n-1} \ldots ds_{l+1}, & \text{if } l < n-1. \end{cases}$$

Indeed, by (3), for $j = l, l+1, \ldots, n-1$ and every $t \geqq T$ we have

$$\begin{aligned}(-1)^{l+j}(D_r^{(j)}x)(t) &= (-1)^{l+j}(D_r^{(j)}x)(\xi) + (-1)^{l+(j+1)} \int\limits_t^\xi \frac{1}{r_{j+1}(s)} (D_r^{(j+1)}x)(s)\,ds \\ &\geqq (-1)^{l+(j+1)} \int\limits_t^\xi \frac{1}{r_{j+1}(s)} (D_r^{(j+1)}x)(s)\,ds \qquad \text{for all } \xi \geqq t,\end{aligned}$$

where $r_n = 1$, and consequently

$$(-1)^{l+j}(D_r^{(j)}x)(t) \geqq (-1)^{l+(j+1)} \int\limits_t^\infty \frac{1}{r_{j+1}(s)} (D_r^{(j+1)}x)(s)\,ds,$$

from which (5) can be easily derived.

Now, in view of (i), (ii) and (v) and the definition of F, from equation $(E, +1)$ it follows that

$$-(D_r^{(n)}x)(t) \geqq \sum_{i=1}^{\nu} p_i(t)\, F(x\langle g(t)\rangle) \geqq 0, \qquad t \geqq T.$$

Thus, from (5) for all $t \geqq T$ we obtain

$$(D_r^{(l)}x)(t) \geqq$$

$$\geqq \begin{cases} \int\limits_t^\infty \sum\limits_{i=1}^{\nu} p_i(s)\, F(x\langle g(s)\rangle)\,ds, & \text{if } l = n-1 \\ \int\limits_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int\limits_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int\limits_{s_{n-1}}^\infty \sum\limits_{i=1}^{\nu} p_i(s)\, F(x\langle g(s)\rangle)\,ds\,ds_{n-1} \ldots ds_{l+1}, & \\ & \text{if } l < n-1 \end{cases}$$

$$\geqq \left[\inf_{s \geqq t} \frac{F(x\langle g(s)\rangle)}{\prod\limits_{j=1}^{m} x^{\alpha_j}[g_j(s)]}\right] \begin{cases} \int\limits_t^\infty \sum\limits_{i=1}^{\nu} p_i(s) \prod\limits_{j=1}^{m} x^{\alpha_j}[g_j(s)]\,ds, & \text{if } l = n-1 \\ \int\limits_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int\limits_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int\limits_{s_{n-1}}^\infty \sum\limits_{i=1}^{\nu} p_i(s) & \\ \prod\limits_{j=1}^{m} x^{\alpha_j}[g_j(s)]\,ds\,ds_{n-1} \ldots ds_{l+1}, & \text{if } l < n-1. \end{cases}$$

So, for every $t \geqq T$

$$(D_r^{(l)}x)(t) \left[\sup_{s \geqq t} \frac{\prod\limits_{j=1}^{m} x^{\alpha_j}[g_j(s)]}{F(x\langle g(s)\rangle)}\right] \geqq \tag{6}$$

$$\geqq \begin{cases} \int_t^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} x^{\alpha_j}[g_j(s)]\,ds, & \text{if } l = n-1 \\ \int_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} x^{\alpha_j}[g_j(s)]\,ds\,ds_{n-1} \ldots ds_{l+1}, & \text{if } l < n-1. \end{cases}$$

Furthermore, we set

$$\hat{T} = \begin{cases} T, & \text{if } l = 1 \\ T^*, & \text{if } l > 1 \end{cases}$$

and, by (iii), we choose a $T_1 \geqq \hat{T}$ so that

$$g^*(t) \geqq \hat{T} \qquad \text{for every } t \geqq T_1.$$

Then, because of (3) for $l = 1$ or (4) for $l > 1$, x is increasing on the interval $[\hat{T}, \infty)$ and so for any s and t with $s \geqq t \geqq T_1$ we have

$$x[g_j(s)] \geqq x[g^*(t)] \qquad (j = 1, 2, \ldots, m).$$

Thus, (6) gives

$$\text{(7)} \qquad (D_r^{(l)}x)(t)\left[\sup_{\substack{y_j \geqq x[g^*(t)] \\ 1 \leqq j \leqq m}} \frac{y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_m^{\alpha_m}}{F(y_1, y_2, \ldots, y_m)}\right] \geqq$$

$$\geqq \begin{cases} \int_t^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} x^{\alpha_j}[g_j(s)]\,ds, & \text{if } l = n-1 \\ \int_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} x^{\alpha_j}[g_j(s)]\,ds\,ds_{n-1} \ldots ds_{l+1}, & \text{if } l < n-1 \end{cases}$$

for every $t \geqq T_1$.

Next, we prove that

$$\text{(8)} \qquad x(v) \geqq (D_r^{(l-1)}x)(u)\,R_{l-1}(v; u)$$

for any v, u with $v \geqq u \geqq \hat{T}$. To this end, we easily derive the following generalization of the Taylor formula

$$x(v) = \sum_{i=0}^{l-1} (D_r^{(i)}x)(u)\,R_i(v; u) +$$

$$+ \int_u^{v=s_0} \frac{1}{r_1(s_1)} \int_u^{s_1} \frac{1}{r_2(s_2)} \cdots \int_u^{s_{l-1}} \frac{1}{r_l(s_l)} (D_r^{(l)}x)(s_l)\,ds_l \ldots ds_2\,ds_1,$$

which, in view of (3) and (4), leads to (8).

Consequently, by using the formula (8), for every s, t with $s \geqq t \geqq T_1$ we have

$$x[g_j(s)] \geqq (D_r^{(l-1)}x)\,[g^*(t)]\,R_{l-1}(g_j(s); g^*(t)) \qquad (j = 1, 2, \ldots, m).$$

Then from (7) for every $t \geqq T_1$ we obtain

$$\text{(9)} \qquad \frac{(D_r^{(l)}x)(t)}{(D_r^{(l-1)}x)[g^*(t)]}\left[\sup_{\substack{y_j \geqq x[g^*(t)] \\ 1 \leqq j \leqq m}} \frac{y_1^{\alpha_1} y_2^{\alpha_2} \dots y_m^{\alpha_m}}{F(y_1, y_2, \dots, y_m)}\right] \geqq$$

$$\geqq \begin{cases} \int_t^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [R_{n-2}(g_j(s); g^*(t))]^{\alpha_j} \, ds, & \text{if } l = n-1 \\ \int_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [R_{l-1}(g_j(s); g^*(t))]^{\alpha_j} \, ds \, ds_{n-1} \dots \\ \qquad \dots ds_{l+1}, & \text{if } l < n-1. \end{cases}$$

Because of $S_F[\alpha_1, \alpha_2, \dots, \alpha_m] < \infty$, it holds

$$\sup_{\substack{y_j \geqq x[g^*(T_1)] \\ 1 \leqq j \leqq m}} \frac{y_1^{\alpha_1} y_2^{\alpha_2} \dots y_m^{\alpha_m}}{F(y_1, y_2, \dots, y_m)} < \infty.$$

Hence, from (9) it follows that

$$\begin{cases} \int_{T_1}^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [R_{n-2}(g_j(s); g^*(T_1))]^{\alpha_j} \, ds < \infty, & \text{if } l = n-1 \\ \int_{T_1}^\infty \frac{1}{r_{l+1}(s_{l+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [R_{l-1}(g_j(s); g^*(T_1))]^{\alpha_j} \, ds \, ds_{n-1} \dots \\ \qquad \dots ds_{l+1} < \infty, & \text{if } l < n-1 \end{cases}$$

and consequently

$$\text{(10)} \qquad \begin{cases} \int^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [R_{n-2}(g_j(s))]^{\alpha_j} \, ds < \infty, & \text{if } l = n-1 \\ \int^\infty \frac{1}{r_{l+1}(s_{l+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [R_{l-1}(g_j(s))]^{\alpha_j} \, ds \, ds_{n-1} \dots \\ \qquad \dots ds_{l+1} < \infty, & \text{if } l < n-1. \end{cases}$$

On the other hand, because of (2) for $l = n - 1$ or (3) for $l < n - 1$, the function $D_r^{(l)}x$ is decreasing on $[T, \infty)$ and so (9) gives

$$\text{(11)} \qquad \frac{(D_r^{(l)}x)[g^*(t)]}{(D_r^{(l-1)}x)[g^*(t)]}\left[\int_{t_0}^{g^*(t)} \frac{ds}{r_l(s)}\right]\left[\sup_{\substack{y_j \geqq x[g^*(t)] \\ 1 \leqq j \leqq m}} \frac{y_1^{\alpha_1} y_2^{\alpha_2} \dots y_m^{\alpha_m}}{F(y_1, y_2, \dots, y_m)}\right] \geqq$$

$$\geqq \left[\int_{t_0}^{g^*(t)} \frac{ds}{r_l(s)}\right] \begin{cases} \int_t^\infty \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [R_{n-2}(g_j(s); g^*(t))]^{\alpha_j} \, ds, & \text{if } l = n-1 \\ \int_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{v} p_i(s) \\ \prod_{j=1}^{m} [R_{l-1}(g_j(s); g^*(t))]^{\alpha_j} \, ds \, ds_{n-1} \dots ds_{l+1}, & \text{if } l < n-1 \end{cases}$$

for all $t \geqq T_1$.

Furthermore, we shall prove that

$$\limsup_{t\to\infty} \frac{(D_r^{(l)}x)[g^*(t)]}{(D_r^{(l-1)}x)[g^*(t)]} \int_{t_0}^{g^*(t)} \frac{ds}{r_l(s)} \leqq 1. \tag{12}$$

To this end, taking into account the fact that $D_r^{(l)}x$ is decreasing on $[T, \infty)$, for every $t \geqq \hat{T}$ we have

$$\frac{(D_r^{(l)}x)(t)}{(D_r^{(l-1)}x)(t)} \int_{\hat{T}}^{t} \frac{ds}{r_l(s)} \leqq \frac{1}{(D_r^{(l-1)}x)(t)} \int_{\hat{T}}^{t} \frac{1}{r_l(s)} (D_r^{(l)}x)(s)\, ds = 1 - \frac{(D_r^{(l-1)}x)(\hat{T})}{(D_r^{(l-1)}x)(t)} \leqq 1.$$

Moreover, it is obvious that

$$\lim_{t\to\infty} \frac{\int_{\hat{T}}^{t} \frac{ds}{r_l(s)}}{\int_{t_0}^{t} \frac{ds}{r_l(s)}} = 1.$$

So, we have

$$\limsup_{t\to\infty} \frac{(D_r^{(l)}x)(t)}{(D_r^{(l-1)}x)(t)} \int_{t_0}^{t} \frac{ds}{r_l(s)} \leqq 1,$$

which implies (12), since $\lim_{t\to\infty} g^*(t) = \infty$.

Now, since x is increasing and unbounded, we have $\lim_{t\to\infty} x(t) = \infty$ and therefore

$$\lim_{t\to\infty}\left[\sup_{\substack{y_j \geqq x[q^*(t)] \\ 1 \leqq j \leqq m}} \frac{y_1^{\alpha_1} y_2^{\alpha_2} \dots y_m^{\alpha_m}}{F(y_1, y_2, \dots, y_m)}\right] = \limsup_{\substack{y_j \to \infty \\ 1 \leqq j \leqq m}} \frac{y_1^{\alpha_1} y_2^{\alpha_2} \dots y_m^{\alpha_m}}{F(y_1, y_2, \dots, y_m)} \leqq S_F[\alpha_1, \alpha_2, \dots, \alpha_m].$$

In view of this and (12), from (11) we obtain

$$S_F[\alpha_1, \alpha_2, \dots, \alpha_m] \geqq \tag{13}$$

$$\geqq \limsup_{t\to\infty}\left[\int_{t_0}^{g^*(t)} \frac{ds}{r_l(s)}\right] \begin{cases} \int_t^\infty \sum_{i=1}^{\nu} p_i(s) \prod_{j=1}^{m} [R_{n-2}(g_j(s); g^*(t))]^{\alpha_j}\, ds, & \text{if } l = n-1 \\ \int_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \dots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{\nu} p_i(s) & \\ \prod_{j=1}^{m} [R_{l-1}(g_j(s); g^*(t))]^{\alpha_j}\, ds\, ds_{n-1} \dots ds_{l+1}, & \text{if } l < n-1. \end{cases}$$

We remark that (10) and (13) contradicts the condition (C_1).

Finally, the proof of our theorem can be completed by applying Theorem 0.

Remark 1. By a light modification of the proof of Theorem 1 we verify that in this theorem the condition (C_1) can be replaced by the following one:

(C_2^*) *It holds*

$$S_F^* = \max\left\{\limsup_{\substack{y_j\to\infty\\1\leqq j\leqq m}}\frac{y_1+y_2+\dots+y_m}{F(y_1,y_2,\dots,y_m)},\ \limsup_{\substack{y_j\to-\infty\\1\leqq j\leqq m}}\frac{|y_1|+|y_2|+\dots+|y_m|}{F(y_1,y_2,\dots,y_m)}\right\}<\infty$$

and for every integer l with $1 \leqq l \leqq n-1$ and $n+l$ odd exactly one of the following is satisfied:

(c_1^*) *There exists an integer k, $l \leqq k \leqq n-1$, such that*

$$\begin{cases}\int^{\infty} p_{i_0}(t)\,R_{l-1}(g_{j_0}(t))\,dt=\infty, & \text{if } k=n-1\\ \int^{\infty}\frac{1}{r_{k+1}(s_{k+1})}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}p_{i_0}(s)\,R_{l-1}(g_{j_0}(s))\,ds\,ds_{n-1}\dots ds_{k+1}=\infty, & \text{if } k<n-1\end{cases}$$

for some i_0, $1 \leqq i_0 \leqq \nu$, and j_0, $1 \leqq j_0 \leqq m$.

(c_1^*) *It holds*

$$\begin{cases}\limsup_{t\to\infty}\left[\int_{t_0}^{g^*(t)}\frac{ds}{r_{n-1}(s)}\right]\sum_{j=1}^{m}\int_t^{\infty}\sum_{i=1}^{\nu}p_i(s)\,R_{n-2}(g_j(s);g^*(t))\,ds>S_F^*, & \text{if } l=n-1\\ \limsup_{t\to\infty}\left[\int_{t_0}^{g^*(t)}\frac{ds}{r_l(s)}\right]\sum_{j=1}^{m}\int_t^{\infty}\frac{1}{r_{l+1}(s_{l+1})}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}\sum_{i=1}^{\nu}p_i(s) & \\ \qquad R_{l-1}(g_j(s);g^*(t))\,ds\,ds_{n-1}\dots ds_{l+1}>S_F^*, & \text{if } l<n-1.\end{cases}$$

Theorem 2. *Consider the differential equation $(E, -1)$ subject to the conditions* (i)$\div$(v), (R), (C_0) *and:*

(C_2) *For some i_0, $1 \leqq i_0 \leqq \nu$, the function F_{i_0} is increasing on $\mathbf{R}^m$ and such that for every nonzero constant c*

$$\int^{\infty} p_{i_0}(t)\,|F_{i_0}(cR_{n-1}[g_1(t)], cR_{n-1}[g_2(t)],\dots,cR_{n-1}[g_m(t)])|\,dt=\infty.$$

(C_3) *If $n>2$, then there exist nonnegative numbers α_j $(j=1,2,\dots,m)$ with $\sum_{j=1}^m \alpha_j = 1$ and $S_F[\alpha_1,\alpha_2,\dots,\alpha_m]<\infty$ and such that for every integer l with $1 \leqq l \leqq n-2$ and $n+l$ even either* (c_1) *or* (c_3) *below is satisfied:*

(c_3) *It holds*

$$\limsup_{t\to\infty}\left[\int_{t_0}^{g^*(t)}\frac{ds}{r_l(s)}\right]\int_t^{\infty}\frac{1}{r_{l+1}(s_{l+1})}\cdots\int_{s_{n-2}}^{\infty}\frac{1}{r_{n-1}(s_{n-1})}\int_{s_{n-1}}^{\infty}\sum_{i=1}^{\nu}p_i(s)$$
$$\prod_{j=1}^{m}[R_{l-1}(g_j(s);g^*(t))]^{\alpha_j}ds\,ds_{n-1}\dots ds_{l+1}>S_F[\alpha_1,\alpha_2,\dots,\alpha_m].$$

Then every solution x of the equation $(E, -1)$ satisfies exactly one of the following:

(I) *x is oscillatory.*

(II) *x is such that*

$$\lim_{t\to\infty}(D_r^{(i)}x)(t)=0 \qquad \textit{monotonically } (i=0,1,\dots,n-1).$$

(III) *It holds*

$$\lim_{t\to\infty} (D_r^{(i)}x)(t) = \infty \qquad \textit{for all } i = 0, 1, \ldots, n-1$$

or

$$\lim_{t\to\infty} (D_r^{(i)}x)(t) = -\infty \qquad \textit{for all } i = 0, 1, \ldots, n-1.$$

Moreover, (II) *occurs only in the case of even n. Also, every solution x of* $(E, -1)$ *with* $x(t) = O(R_{n-1}(t))$ *as* $t \to \infty$ *for n odd is oscillatory while for n even is oscillatory or satisfies* (II).

Proof. Let x be an unbounded nonoscillatory solution on an interval $[T_0, \infty)$, $T_0 > t_0$, of the equation $(E, -1)$. We suppose, without loss of generality, that $x(t) \neq 0$ for all $t \geqq T_0$ and, furthermore, since the substitution $z = -x$ transforms $(E, -1)$ into an equation of the same form satisfying the assumptions of the theorem, we restrict ourselves in the case of positive x. Next, by (iii) and (iv), we choose a $T \geqq T_0$ so that, for every $t \geqq T$, (1) holds. Then, in view of (i), (ii) and (v), from equation $(E, -1)$ it follows that

$$(14) \qquad (D_r^{(n)}x)(t) \geqq 0 \qquad \text{for every } t \geqq T.$$

Moreover, if $D_r^{(n)}x = 0$ on $[\tau, \infty)$, $\tau \geqq T$, then, in view again of (i), (ii) and (v), all functions p_i $(i = 1, 2, \ldots, \nu)$ are identically zero on $[\tau, \infty)$. This contradicts (C_2) and hence $(D_r^{(n)}x)(t)$ is not identically zero for all large t. Consequently, by taking into account the fact that x is unbounded, Lemma 1 implies the existence of an integer l, $1 \leqq l \leqq n$, with $n + l$ even and such that, if $l \leqq n - 1$, (3) holds and, when $l > 1$, (4) is satisfied for some $T^* \geqq T$. Since $n + l$ is even, we have $l \neq n - 1$. So, we consider the following two cases:

Case 1. $l = n$. In this case, because of (4) and (14), it holds

$$\lim_{t\to\infty} (D_r^{(n-1)}x)(t) > 0$$

and hence, in view of Lemma 2,

$$\lim_{t\to\infty} \frac{x(t)}{R_{n-1}(t)} > 0.$$

Therefore, there exists a positive constant c such that

$$x(t) \geqq cR_{n-1}(t) \qquad \text{for every } t \geqq T_0$$

and consequently, since, by (C_2), the function F_{i_0} is increasing on $\mathbf{R}^m$,

$$F_{i_0}(x\langle g(t)\rangle) \geqq F_{i_0}(cR_{n-1}[g_1(t)], cR_{n-1}[g_2(t)], \ldots, cR_{n-1}[g_m(t)]), \quad t \geqq T.$$

Thus, by virtue of (i), (ii) and (v), we obtain for $t \geqq T$

$$(D_r^{(n)}x)(t) \geqq p_{i_0}(t)\, F_{i_0}(cR_{n-1}[g_1(t)], cR_{n-1}[g_2(t)], \ldots, cR_{n-1}[g_m(t)]) \geqq 0$$

and, furthermore, for every $t \geqq T$ we have

$$(D_r^{(n-1)}x)(t) \geqq$$
$$\geqq (D_r^{(n-1)}x)(T) + \int_T^t p_{i_0}(s) F_{i_0}(cR_{n-1}[g_1(s)], cR_{n-1}[g_2(s)], \ldots, cR_{n-1}[g_m(s)]) \, ds.$$

This, because of condition (C_2), gives

$$\lim_{t \to \infty} (D_r^{(n-1)}x)(t) = \infty.$$

Therefore, it is easy to derive that

$$\lim_{t \to \infty} (D_r^{(i)}x)(t) = \infty \qquad (i = 0, 1, \ldots, n-1)$$

and hence x satisfies (III). Moreover, by Lemma 2, we have $\lim_{t \to \infty} \dfrac{x(t)}{R_{n-1}(t)} = \infty$ and so the solution x has not the property $x(t) = 0(R_{n-1}(t))$ as $t \to \infty$.

Case 2. $1 \leqq l \leqq n - 2$. An argument similar to that used in the proof of Theorem 1 gives

$$\int^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{\nu} p_i(s) \prod_{j=1}^{m} [R_{l-1}(g_j(s))]^{\alpha_j} \, ds \, ds_{n-1} \ldots ds_{l+1} < \infty$$

and

$$\limsup_{t \to \infty} \left[\int_{t_0}^{g^*(t)} \frac{ds}{r_l(s)} \right] \int_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{\nu} p_i(s)$$
$$\prod_{j=1}^{m} [R_{l-1}(g_j(s); g^*(t))]^{\alpha_j} \, ds \, ds_{n-1} \ldots ds_{l+1} \leqq S_F[\alpha_1, \alpha_2, \ldots, \alpha_m],$$

which contradicts the condition (C_3).

Finally, Theorem 0 completes the proof of the theorem.

Remark 2. In Theorem 2 the condition (C_3) can be replaced by the following one:

(C_3^*) *If $n > 2$, then $S_F^* < \infty$ and for every integer l with $1 \leqq l \leqq n - 2$ and $n + l$ even either* (c_1^*) *or* (c_3^*) *below is satisfied:*

(c_3^*) *It holds*

$$\limsup_{t \to \infty} \left[\int_{t_0}^{g^*(t)} \frac{ds}{r_l(s)} \right] \sum_{j=1}^{m} \int_t^\infty \frac{1}{r_{l+1}(s_{l+1})} \cdots \int_{s_{n-2}}^\infty \frac{1}{r_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty \sum_{i=1}^{\nu} p_i(s)$$
$$R_{l-1}(g_j(s); g^*(t)) \, ds \, ds_{n-1} \ldots ds_{l+1} > S_F^*.$$

Remark 3. In the proof of Theorem 2 the condition (C_2) is used only in Case 1 where we have

$$\lim_{t \to \infty} \frac{x(t)}{R_{n-1}(t)} > 0,$$

which is a contradiction, when the solution x has the asymptotic property $x(t) = o(R_{n-1}(t))$ as $t \to \infty$. Thus, we derive the following result:

If the conditions (i)÷(v), (R), (C_0) *and* (C_3) *are satisfied, then every solution x with* $x(t) = o(R_{n-1}(t))$ *as* $t \to \infty$ *of the differential equation* $(E, -1)$ *for n odd is oscillatory while for n even is either oscillatory or such that*

$$\lim_{t\to\infty} (D_r^{(i)}x)(t) = 0 \qquad \textit{monotonically } (i = 0, 1, \ldots, n-1).$$

Consider now the special case where for some integer N with $1 \leqq N \leqq n-1$ we have

$$r_i = 1 \quad \text{for } i \neq n - N \quad \text{and} \quad r_{n-N} = r.$$

In this case the differential equation (E, δ) takes the form

$$(\hat{E}, \delta) \qquad [r(t)\, x^{(n-N)}(t)]^{(N)} + \delta\{\sum_{i=1}^{\nu} p_i(t)\, F_i(x\langle g(t)\rangle) + \\ + G(t; x\langle\sigma_0(t)\rangle, x'\langle\sigma_1(t)\rangle, \ldots, x^{(n-N-1)}\langle\sigma_{n-N-1}(t)\rangle, \\ [rx^{(n-N)}]\langle\sigma_{n-N}(t)\rangle, [rx^{(n-N)}]'\langle\sigma_{n-N+1}(t)\rangle, \ldots, [rx^{(n-N)}]^{(N-1)}\langle\sigma_{n-1}(t)\rangle)\} = 0$$

and the condition (R) becomes:

$$(\hat{\mathrm{R}}) \qquad \int^{\infty} \frac{dt}{r(t)} = \infty.$$

We shall apply our main results in the considered special case. For this purpose, for any integer λ with $n - N \leqq \lambda \leqq n - 1$ we define

$$P_\lambda(v; u) = \int_u^v \frac{(v-s)^{n-N-1}(s-u)^{\lambda-(n-N)}}{r(s)}\, ds, \; v \geqq u \geqq t_0$$

and in particular

$$P_\lambda(t) = P_\lambda(t; t_0), \qquad t \geqq t_0.$$

Corollaries 1 and 2 below are new and follow from Theorems 1 and 2 respectively.

Corollary 1. *Consider the differential equation* $(\hat{E}, +1)$ *subject to the conditions* (i)÷(v), ($\hat{\mathrm{R}}$) *and:*

($\hat{C}_0$) *For some* i_0, $1 \leqq i_0 \leqq \nu$, *either*

$$\int^{\infty} t^{N-1} p_{i_0}(t)\, dt = \infty$$

or

$$\int^{\infty} \frac{t^{n-N-1}}{r(t)} \int_t^{\infty} (s-t)^{N-1}\, p_{i_0}(s)\, ds\, dt = \infty.$$

($\hat{C}_1$) *There exist nonnegative numbers* α_j $(j = 1, 2, \ldots, m)$ *with* $\sum_{j=1}^{m} \alpha_j = 1$ *and* $S_F[\alpha_1, \alpha_2, \ldots, \alpha_m] < \infty$ *and such that:*

(A_1^o) *If* $N < n - 1$ *and* $I_o = \{l : l$ *is integer with* $1 \leqq l \leqq n - N - 1$ *and* $n + l$ *odd*$\} \neq$ $\neq \emptyset$, *then for every integer* $l \in I_o$ *exactly one of the following is satisfied:*

(a_1) *For some* i_0, $1 \leqq i_0 \leqq v$, *either*

$$\int^{\infty} t^{N-1} p_{i_0}(t) \prod_{j=1}^{m} [g_j(t)]^{(l-1)\alpha_j} \, dt = \infty$$

or

$$\int^{\infty} \frac{t^{n-N-l-1}}{r(t)} \int_t^{\infty} (s - t)^{N-1} p_{i_0}(s) \prod_{j=1}^{m} [g_j(s)]^{(l-1)\alpha_j} \, ds \, dt = \infty .$$

(a_2) *It holds*

$$\limsup_{t \to \infty} g^*(t) \int_t^{\infty} \frac{(s - t)^{n-N-l-1}}{r(s)} \int_s^{\infty} (u - s)^{N-1} \sum_{i=1}^{v} p_i(u) \prod_{j=1}^{m} [g_j(u) - g^*(t)]^{(l-1)\alpha_j}$$
$$du \, ds > (n - N - l - 1)! \, (N - 1)! \, (l - 1)! \, S_F[\alpha_1, \alpha_2, \ldots, \alpha_m].$$

(A_2^o) *If N is odd, then exactly one of the following is satisfied:*

(a_3) *For some* i_0, $1 \leqq i_0 \leqq v$,

$$\int^{\infty} t^{N-1} p_{i_0}(t) \prod_{j=1}^{m} [g_j(t)]^{(n-N-1)\alpha_j} \, dt = \infty .$$

(a_4) *It holds*

$$\limsup_{t \to \infty} \left[\int_{t_0}^{g^*(t)} \frac{ds}{r(s)} \right] \int_t^{\infty} (s - t)^{N-1} \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [g_j(s) - g^*(t)]^{(n-N-1)\alpha_j} \, ds >$$
$$> (N - 1)! \, (n - N - 1)! \, S_F[\alpha_1, \alpha_2, \ldots, \alpha_m].$$

(A_3^o) *If* $N > 1$, *then for every integer* l *with* $n - N + 1 \leqq l \leqq n - 1$ *and* $n + l$ *odd exactly one of the following is satisfied:*

(a_5) *For some* i_0, $1 \leqq i_0 \leqq v$,

$$\int^{\infty} t^{n-l-1} p_{i_0}(t) \prod_{j=1}^{m} [P_{l-1}(g_j(t))]^{\alpha_j} \, dt = \infty .$$

(a_6) *It holds*

$$\limsup_{t \to \infty} g^*(t) \int_t^{\infty} (s - t)^{n-l-1} \sum_{i=1}^{v} p_i(s) \prod_{j=1}^{m} [P_{l-1}(g_j(s); g^*(t))]^{\alpha_j} \, ds >$$
$$> (n - l - 1)! \, (n - N - 1)! \, [l - 1 - (n - N)]! \, S_F[\alpha_1, \alpha_2, \ldots, \alpha_m].$$

Then every solution x of the equation ($\hat{E}$, $+1$) *for n even is oscillatory, while for n odd is either oscillatory or such that*

$$\begin{cases} \lim_{t \to \infty} x^{(i)}(t) = 0 \ \textit{monotonically} \ (i = 0, 1, \ldots, n - N - 1) \\ \lim_{t \to \infty} [\![r(t) x^{(n-N)}(t)]\!]^{(j)} = 0 \ \textit{monotonically} \ (j = 0, 1, \ldots, N - 1). \end{cases}$$

Corollary 2. *Consider the differential equation* $(\hat{E}, -1)$ *subject to the conditions* (i) ÷ (v), $(\hat{\mathrm{R}})$, $(\hat{\mathrm{C}}_0)$ *and:*

$(\hat{\mathrm{C}}_2)$ *For some* i_0, $1 \leqq i_0 \leqq \nu$, *the function* F_{i_0} *is increasing on* $\mathbf{R}^m$ *and such that for every nonzero constant* c

$$\int^{\infty} p_{i_0}(t) \mid F_{i_0}(cP_{n-1}[g_1(t)], cP_{n-1}[g_2(t)], \ldots, cP_{n-1}[g_m(t)]) \mid \mathrm{d}t = \infty.$$

(C_3) *If* $n > 2$, *then there exist nonnegative numbers* α_j $(j = 1, 2, \ldots, m)$ *with* $\sum_{j=1}^{m} \alpha_j = 1$ *and* $S_F[\alpha_1, \alpha_2, \ldots, \alpha_m] < \infty$ *and such that:*

(A_1^e) *If* $N < n - 1$ *and* $I_e = \{l : l$ *is integer with* $1 \leqq l \leqq n - N - 1$ *and* $n + l$ *even*$\} \neq \emptyset$, *then for every integer* $l \in I_e$ *either* (a_1) *or* (a_2) *is satisfied.*

(A_2^e) *If* $N > 1$ *and* N *is even, then either* (a_3) *or* (a_4) *is satisfied.*

(A_3^e) *If* $N > 2$, *then for every integer* l *with* $n - N + 1 \leqq l \leqq n - 2$ *and* $n + l$ *even either* (a_5) *or* (a_6) *is satisfied.*

Then every solution x *of the equation* $(\hat{E}, -1)$ *satisfies exactly one of the following:*

(I)* x *is oscillatory.*

(II)* x *is such that*

$$\begin{cases} \lim_{t \to \infty} x^{(i)}(t) = 0 \text{ monotonically } (i = 0, 1, \ldots, n - N - 1) \\ \lim_{t \to \infty} [r(t)\, x^{(n-N)}(t)]^{(j)} = 0 \text{ monotonically } (j = 0, 1, \ldots, N - 1). \end{cases}$$

(III)* *It holds*

$$\begin{cases} \lim_{t \to \infty} x^{(i)}(t) = \infty \ (i = 0, 1, \ldots, n - N - 1) \\ \lim_{t \to \infty} [r(t)\, x^{(n-N)}(t)]^{(j)} = \infty \ (j = 0, 1, \ldots, N - 1) \end{cases}$$

or

$$\begin{cases} \lim_{t \to \infty} x^{(i)}(t) = -\infty \ (i = 0, 1, \ldots, n - N - 1) \\ \lim_{t \to \infty} [r(t)\, x^{(n-N)}(t)]^{(j)} = -\infty \ (j = 0, 1, \ldots, N - 1). \end{cases}$$

Moreover, (II)* *occurs only in the case of even* n. *Also, every solution* x *of* $(\hat{E}, -1)$ *with* $x(t) = O(P_{n-1}(t))$ *as* $t \to \infty$ *for* n *odd is oscillatory while for* n *even is oscillatory or satisfies* (II)*.

It is easy to verify that in the considered particular case for any integer λ with $0 \leqq \lambda \leqq n - 1$ we have

$$(15) \qquad R_\lambda(v; u) = \begin{cases} \dfrac{1}{\lambda!}(v - u)^\lambda, & \text{if } \lambda < n - N \\ \dfrac{1}{(n - N - 1)!\,[\lambda - (n - N)]!} P_\lambda(v; u), & \text{if } \lambda \geqq n - N \end{cases}$$

for every v, u with $v \geqq u \geqq t_0$ and in particular

$$(16) \qquad R_\lambda(t) = \begin{cases} \dfrac{1}{\lambda!}(t - t_0)^\lambda, & \text{if } \lambda < n - N \\ \dfrac{1}{(n - N - 1)!\,[\lambda - (n - N)]!} P_\lambda(t), & \text{if } \lambda \geqq n - N \end{cases}$$

for all $t \geqq t_0$. On the other hand, we have the formula

$$(17) \qquad \int_\xi^\infty \int_s^\infty (w - s)^\mu q(w)\, dw\, ds = \int_\xi^\infty \frac{(s - \xi)^{\mu+1}}{\mu + 1} q(s)\, ds,$$

where μ is a nonnegative integer and the function q is continuous and nonnegative on $[\xi, \infty)$. By (15), (16) and (17), it is a matter of elementary calculus to see that in the considered case the conditions (C_0), (C_1), (C_2) and (C_3) follow from $(\hat{C}_0)$, $(\hat{C}_1)$, $(\hat{C}_2)$ and $(\hat{C}_3)$ respectively. So, Corollaries 1 and 2 follow from Theorems 1 and 2 respectively.

Remark 4. Corollaries 1 and 2 for $N = 1$ improve two recent results due to Sficas and Stavroulakis [17, Theorems 2 and 4].

Now, from Theorems 1 and 2, by applying them in the usual case where

$$r_1 = r_2 = \ldots = r_{n-1} = 1,$$

we obtain the following Corollaries 1′ and 2′ respectively concerning the differential equation

$$(\tilde{E}, \delta) \qquad x^{(n)}(t) + \delta\{\sum_{i=1}^{v} p_i(t) F_i(x\langle g(t)\rangle) + G(t; x\langle\sigma_0(t)\rangle, x'\langle\sigma_1(t)\rangle, \ldots, x^{(n-1)}\langle\sigma_{n-1}(t)\rangle)\} = 0.$$

Corollary 1′. *Consider the differential equation* $(\tilde{E}, +1)$ *subject to the conditions* (i)÷(v) *and:*

$(\tilde{C}_0)$ *For some* i_0, $1 \leqq i_0 \leqq v$,

$$\int^\infty t^{n-1} P_{i_0}(t)\, dt = \infty.$$

$(\tilde{C}_1)$ *There exist nonnegative numbers* α_j $(j = 1, 2, \ldots, m)$ *with* $\sum_{j=1}^{m} \alpha_j = 1$ *and* $S_F[\alpha_1, \alpha_2, \ldots, \alpha_m] < \infty$ *and such that for every integer* l *with* $1 \leqq l \leqq n - 1$ *and* $n + l$ *odd exactly one of the following is satisfied:*

(d_1) *For some* i_0, $1 \leqq i_0 \leqq v$,

$$\int^\infty t^{n-l-1} p_{i_0}(t) \prod_{j=1}^{m} [g_j(t)]^{(l-1)\alpha_j}\, dt = \infty.$$

(d_2) *It holds*

$$\limsup_{t\to\infty} g^*(t) \int_t^\infty (s-t)^{n-l-1} \sum_{i=1}^{\nu} p_i(s) \prod_{j=1}^{m} [g_j(s) - g^*(t)]^{(l-1)\alpha_j} \, ds >$$
$$> (n-l-1)!\,(l-1)!\, S_F[\alpha_1, \alpha_2, \ldots, \alpha_m].$$

Then every solution of the equation $(\tilde{E}, +1)$ *for n even is oscillatory, while for n odd is either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with its first* $n-1$ *derivatives.*

Corollary 2′. *Consider the differential equation* $(\tilde{E}, -1)$ *subject to the conditions* (i)÷(v), $(\tilde{C}_0)$ *and:*

$(\tilde{C}_2)$ *For some* i_0, $1 \leqq i_0 \leqq \nu$, *the function* F_{i_0} *is increasing on* $\mathbf{R}^m$ *and such that for every nonzero constant c*

$$\int^\infty p_{i_0}(t) \,|\, F_{i_0}(c[g_1(t)]^{n-1}, c[g_2(t)]^{n-1}, \ldots, c[g_m(t)]^{n-1}) \,|\, dt = \infty.$$

$(\tilde{C}_3)$ *If* $n > 2$, *then there exist nonnegative numbers* α_j $(j = 1, 2, \ldots, m)$ *with* $\sum_{j=1}^{m} \alpha_j = 1$ and $S_F[\alpha_1, \alpha_2, \ldots, \alpha_m] < \infty$ *and such that for every integer l with* $1 \leqq l \leqq n-2$ *and* $n + l$ *even either* (d_1) *or* (d_2) *is satisfied.*

Then every solution x of the equation $(\tilde{E}, -1)$ *satisfies exactly one of the following:*

(I)′ *x is oscillatory.*

(II)′ *x and its first* $n-1$ *derivatives tend monotonically to zero as* $t \to \infty$.

(III)′ *It holds*

$$\lim_{t\to\infty} x^{(i)}(t) = \infty \qquad \textit{for all } i = 0, 1, \ldots, n-1$$

or

$$\lim_{t\to\infty} x^{(i)}(t) = -\infty \qquad \textit{for all } i = 0, 1, \ldots, n-1.$$

Moreover, (II)′ *occurs only in the case of even n. Also, every solution x of* $(\tilde{E}, -1)$ *with* $x(t) = O(t^{n-1})$ *as* $t \to \infty$ *for n odd is oscillatory while for n even is either oscillatory or tending monotonically to zero as* $t \to \infty$ *together with its first* $n-1$ *derivatives.*

Remark 5. Corollaries 1′ and 2′ improve two results due to Stavroulakis [21, Theorems 1.2 and 1.3]. For earlier related results concerning particular cases of the differential equation $(\tilde{E}, \delta)$ we refer to Lovelady [7] and Sficas [15, 16].

Acknowledgment. The author would like to thank Professor V. A. Staikos for his helpful suggestions concerning this paper.

REFERENCES

[1] M. K. Grammatikopoulos, Y. G. Sficas and V. A. Staikos: *Oscillatory properties of strongly superlinear differential equations with deviating arguments*, J. Math. Anal. Appl. 67 (1979), 171–187.

[2] A. G. Kartsatos: *Oscillation properties of solutions of even order differential equations,* Bull. Fac. Sci., Ibaraki Univ., Math., 2 (1969), 9—14.

[3] I. T. Kiguradze: *On the oscillation of solutions of the equation* $\frac{d^m u}{dt^m} + \alpha(t)u^n \operatorname{sgn} u = 0$ (Russian), Mat. Sb. 65 (1964), 172—187.

[4] I. T. Kiguradze: *The problem of oscillation of solutions of nonlinear differential equations* (Russian), Differencial'nye Uravnenija 1 (1965), 995—1006.

[5] T. Kusano and H. Onose: *Asymptotic behavior of nonoscillatory solutions of functional equations of arbitrary order,* J. London Math. Soc. 14 (1976), 106—112.

[6] T. Kusano and H. Onose: *Nonoscillation theorems for differential equations with deviating argument,* Pacific J. Math. 63 (1976), 185—192.

[7] D. L. Lovelady: *Oscillation and even order linear differential equations,* Rocky Mountain J. Math. 6 (1976), 299—304.

[8] Ch. G. Philos: *Oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments,* Hiroshima Math. J. 8 (1978), 31—48.

[9] Ch. G. Philos: *An oscillatory and asymptotic classification of the solutions of differential equations with deviating arguments,* Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur., 63 (1977), 195—203.

[10] Ch. G. Philos: *Oscillatory and asymptotic behavior of all solutions of differential equations with deviating arguments, Proc. Roy. Soc. Edinburgh Sect. A 81 (1978), 195—210.*

[11] Ch. G. Philos: *On the oscillatory and asymptotic behavior of the bounded solutions of differential equations with deviating arguments,* Ann. Mat. Pura Appl., *in press.*

[12] Ch. G. Philos, Y. G. Sficas and V. A. Staikos: *Some results on the asymptotic behavior of nonoscillatory solutions of differential equations with deviating arguments,* to appear.

[13] Ch. G. Philos and V. A. Staikos: *Non-slow oscillations with damping,* University of Ioannina, Technical Report No 92, March 1977.

[14] Ch. G. Philos and V. A. Staikos: *Quick oscillations with damping, Math. Nachr., to appear.*

[15] Y. G. Sficas: *Retarded actions on oscillations,* Bull. Soc. Math. Grèce 17 (1976), 1—10.

[16] Y. G. Sficas: *The effect of the delay on the oscillatory and asymptotic behavior of n-th order retarded differential equations,* J. Math. Anal. Appl. 49 (1975), 748—757.

[17] Y. G. Sficas and I. P. Stavroulakis: *On the oscillatory and asymptotic behavior of a class of differential equations with deviating arguments,* SIAM J. Math. Anal., 9 (1978), 956—966.

[18] V. A. Staikos and Ch. G. Philos: *On the asymptotic behavior of nonoscillatory solutions of differential equations with deviating arguments,* Hiroshima Math. J. 7 (1977), 9—31.

[19] V. A. Staikos and Ch. G. Philos: *Asymptotic properties of nonoscillatory solutions of differential equations with deviating argument,* Pacific J. Math., 70 (1977), 221—242.

[20] V. A. Staikos and Ch. G. Philos: *Nonoscillatory phenomena and damped oscillations,* Nonlinear Analysis, Theory, Methods and Applications, 2 (1977), 197—210.

[21] I. P. Stavroulakis: *Differential Equations with Deviating Arguments-Contribution to the study of oscillatory and asymptotic properties of the solutions* (Greek), Ph. D. Thesis, University of Ioannina, 1976.

Ch. G. Philos
Department of Mathematics
University of Ioannina
Ioannina
Greece