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HOPF BIFURCATION IN SYMMETRIC SYSTEMS

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Abstract. In this paper we show on some simple examples how Hopf bifurcation can be handled for symmetric systems, whose symmetries prevent some of the hypotheses of Hopf's theorem to be satisfied. Our examples involve rotational symmetries, which force the eigenvalues to be non-simple, and time-reversibility, which forces the eigenvalues to stay on the imaginary axis, so that the transversality condition is not satisfied. After a brief treatment of the classical Hopf bifurcation we show in part I how a heuristic approach gives branches of circular and collinear solutions. In part II we use an equivariant Liapunov–Schmidt reduction to study the full bifurcation problem, which is 4-dimensional and has an $O(2) \times SO(2)$ or an $O(2) \times O(2)$-equivariance. By bringing the bifurcation equations in a normal form we show that generically only circular and collinear solutions bifurcate. If the system is also time-reversible, then some other bifurcations may arise.

Key words. Hopf bifurcation, symmetry, circular solutions, collinear solutions, time-reversibility.

MS Classification. 58 F 14, 34 C 25.

INTRODUCTION

The aim of this paper is to illustrate on a few simple examples how periodic solutions can bifurcate from equilibria in symmetric systems. For systems which do not exhibit any particular symmetry one has the classical Hopf bifurcation theorem; it describes how a periodic orbit bifurcates from an equilibrium when a pair of simple complex conjugate eigenvalues of the linearization at the equilibrium cross the imaginary axis under a parameter change. This result will in general no longer be applicable for symmetric systems: in many cases the symmetries will force the eigenvalues to have higher multiplicities, or to stay on the imaginary axis. The examples discussed in this paper will indicate how Hopf's theorem can be modified for such cases. The particular symmetries which we will consider are rotational symmetries in the plane ($O(2)$ or $SO(2)$), and time-reversibility; our examples will be second order equations in one and two dimensions. The set-up has been chosen so as to keep the technicalities as low-level as possible. This enables us to give a more or less selfcontained treatment which emphasizes the main ideas behind the approach, thus serving the more didactical purpose of this paper. For more general and complete treatments one can consult e.g. the papers [2, 7, 8].

The paper consists of two parts. In part I we start by showing how one proves the Hopf bifurcation theorem via a Liapunov—Schmidt reduction. Although this theory can be found at numerous places we repeat it here because the approach will be used as a guideline for the more difficult examples which we will encounter further on. Then we will introduce time-reversibility and rotational symmetries and obtain some partial results for systems with such symmetries. In part II we will then derive the bifurcation equations which determine the complete bifurcation picture, and use the symmetries of these bifurcation equations to discuss their solution set.

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PART I

1. HOPF BIFURCATION

We start by showing on a simple scalar equation how one obtains Hopf bifurcation using a Liapunov—Schmidt reduction; the main point is that this approach leads to a two-dimensional bifurcation equation having an $SO(2)$-symmetry.

We consider the equation

$$\ddot{x} + g(x, \dot{x}, \lambda)\, \dot{x} + f(x, \dot{x}, \lambda)\, x = 0, \tag{1.1}$$

where $x \in \boldsymbol{R}$, $\lambda \in \boldsymbol{R}$ is a parameter, while f and g are smooth functions of their arguments such that

$$f(0, 0, 0) = 1, \qquad g(0, 0, 0) = 0. \tag{1.2}$$

We can write (1.1) as a first order system, and linearize at the equilibrium $x = 0$; this gives us

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f(0, 0, \lambda) & -g(0, 0, \lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A(\lambda) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.3}$$

The eigenvalues of $A(\lambda)$ are given by the quadratic equation

$$\mu^2 + g(0, 0, \lambda)\, \mu + f(0, 0, \lambda) = 0; \tag{1.4}$$

from (1.2) it follows that for λ sufficiently small this gives

$$\mu = -\frac{1}{2}g(0, 0, \lambda) \pm i[f(0, 0, \lambda) - \frac{1}{4}g^2(0, 0, \lambda)]^{1/2}. \tag{1.5}$$

For $\lambda = 0$ the equation (1.3) becomes a harmonic oscillator; all its solutions are 2π-periodic, corresponding to the fact that the eigenvalues of $A(0)$ are purely imaginary. When we assume that

$$\frac{\partial g}{\partial \lambda}(0, 0, 0) \neq 0 \tag{1.6}$$

then we see from (1.5) that the eigenvalues cross the imaginary axis at $\lambda = 0$ in a transversal way. This also implies that the equilibrium loses its stability as λ crosses 0 (either from negative to positive, or vice versa). Further on we will refer to (1.6) as the *transversality condition.*

The problem which we want to consider now is that of describing, for all λ near 0, all small periodic solutions of (1.1) with a period near 2π. Although the stability properties of such bifurcating periodic solutions play an important role in the analysis, we will not pursue that point here, and restrict ourselves to the existence problem.

First we remark that, for each $\sigma > 0$, $\tilde{x}(t)$ is a $2\pi/\sigma$-periodic solution of (1.1) if and only if $\tilde{\tilde{x}}(t) = \tilde{x}(t/\sigma)$ is a 2π-periodic solution of

$$\sigma^2\ddot{x} + \sigma g(x, \sigma\dot{x}, \lambda)\,\dot{x} + f(x, \sigma\dot{x}, \lambda)\,x = 0. \tag{1.7}$$

So we want to determine all small 2π-periodic solutions of (1.7), for all (λ, σ) near (0, 1). This formulation allows us to rewrite the problem in the form of a nonlinear operator equation, as follows.

Let $Z = C^0_{2\pi}(\boldsymbol{R})$ be the space of all continuous, 2π-periodic functions $z : \boldsymbol{R} \to \boldsymbol{R}$, and $X = C^2_{2\pi}(\boldsymbol{R})$ the subspace of all $x \in Z$ which are of class C^2; both Z and X are Banach spaces when equipped respectively with the C^0 and the C^2-supremum norm. Now define $M : X \times \boldsymbol{R}^2 \to Z$ by

(1.8)

$$M(x, \lambda, \sigma)(t) = \sigma^2\ddot{x}(t) + \sigma g(x(t), \sigma\dot{x}(t), \lambda)\,\dot{x}(t) + f(x(t), \sigma\dot{x}(t), \lambda)\,x(t), \qquad \forall t \in \boldsymbol{R}.$$

Then our problem is that of solving

$$M(x, \lambda, \sigma) = 0 \tag{1.9}$$

for $(x, \lambda, \sigma) \in X \times \boldsymbol{R}^2$ near (0, 0, 1). This formulation also makes precise what we mean by „small periodic solutions": the smallness has to be interpreted in the sense of the $C^2_{2\pi}(\boldsymbol{R})$-norm.

Before we attempt to solve (1.9), let us first write down some of the properties of the mapping M.

(M1) $M(0, \lambda, \sigma) = 0$ for all (λ, σ); also M is smooth.

(M2) If we define $L \in \mathscr{L}(X, Z)$ by $L := D_xM(0, 0, 1)$, then L is explicitly given by

$(Lx)(t) = \ddot{x}(t) + x(t)$, and $U := N(L)$ is two-dimensional, spanned by the functions $u_1(t) = \sqrt{2}\cos t$ and $u_2(t) = \sqrt{2}\sin t$. It will appear further on that it is convenient to use complex notations; therefore we denote by $Z^c = C^0_{2\pi}(\boldsymbol{C})$ the complexification of Z, and introduce in Z^c an inner product given by:

$$(1.10) \qquad \langle u, v\rangle := \frac{1}{2\pi}\int_0^{2\pi} \bar{u}(t)\,v(t)\,\mathrm{d}t, \qquad \forall u, v \in Z^c.$$

If we let $\zeta(t) = u_1(t) + iu_2(t) = \sqrt{2}\,e^{it}$, then we can define $P \in \mathscr{L}(Z)$ by

$$(1.11) \qquad Pz := \mathscr{R}\mathrm{e}\,(\langle\zeta, z\rangle\,\zeta), \qquad \forall z \in Z.$$

It is easy to verify that P is a continuous projection in Z, with $R(P) = N(L) = U$ and $N(P) = R(L)$. So we have the splittings

$$(1.12) \qquad Z = R(P) \oplus N(P), \qquad X = U \oplus V,$$

with $V := N(P) \cap X$.

(M3) For each $\varphi \in \boldsymbol{R}$ we can define a *shift operator* $S_+(\varphi) \in \mathscr{L}(Z)$ by

$$(1.13) \qquad (S_+(\varphi)\,z)(t) := z(t + \varphi), \qquad \forall t \in \boldsymbol{R},\ \forall z \in Z.$$

Since $S_+(2\pi) = I_Z$ it follows that $\{S_+(\varphi) \mid \varphi \in \boldsymbol{R}\}$ forms a group, isomorphic to the rotation group $SO(2)$; moreover, it is easily checked from the definitions that

$$(1.14) \qquad M(S_+(\varphi)\,x, \lambda, \sigma) = S_+(\varphi)\,M(x, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R};$$

i.e. M is *$SO(2)$-equivariant.* This equivariance is a translation of the fact that (1.1) is autonomous. The reason for the $+$-index in $S_+(\varphi)$ will become clear in the next section. Finally we remark that also the projection P is $SO(2)$-equivariant:

$$(1.15) \qquad PS_+(\varphi) = S_+(\varphi)\,P, \qquad \forall \varphi \in \boldsymbol{R}.$$

Using the properties (M1)—(M3) we can now perform an equivariant Liapunov—Schmidt reduction on (1.9). We write $x \in X$ as $x = Px + (I - P)\,x =:$ $=: u + v$, with $u \in U$ and $v \in V$, and split the equation (1.9) accordingly. This gives:

$$(1.16a) \qquad (I - P)\,M(u + v, \lambda, \sigma) = 0$$

$$(1.16b) \qquad PM(u + v, \lambda, \sigma) = 0.$$

Using the implicit function theorem we can, for $(u + v, \lambda, \sigma)$ near $(0, 0, 1)$, solve (1.16a) for $v = v^*(u, \lambda, \sigma)$. The mapping $v^* : U \times \boldsymbol{R}^2 \to V$ is smooth near $(0, 0, 1)$, and has the following properties:

(i) $v^*(0, \lambda, \sigma) = 0, \quad \forall \lambda, \sigma$;

(ii) $D_u v^*(0, 0, 1) = 0$;

(iii) $v^*(S_+(\varphi)\,u, \lambda, \sigma) = S_+(\varphi)\,v^*(u, \lambda, \sigma), \quad \forall \varphi \in \boldsymbol{R}$;

the equivariance property (iii) follows from (M3) and the uniqueness of $v^*(u, \lambda, \sigma)$ as given by the implicit function theorem.

Bringing the solution of (1.16a) into (1.16b) gives the *bifurcation equation:*

$$G(u, \lambda, \sigma) := PM(u + v^*(u, \lambda, \sigma), \lambda, \sigma) = 0. \tag{1.17}$$

The bifurcation mapping $G : U \times \boldsymbol{R}^2 \to U$ has properties similar to those of v^*:

$$\begin{aligned} &\text{(i) } G(0, \lambda, \sigma) = 0, \qquad \forall \lambda, \sigma; \\ &\text{(ii) } D_u G(0, 0, 1) = 0; \\ &\text{(iii) } G(S_+(\varphi)\, u, \lambda, \sigma) = S_+(\varphi)\, G(u, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R}. \end{aligned} \tag{1.18}$$

In particular the $SO(2)$-equivariance (1.18.iii) of G will be important for the further analysis. To exploite this equivariance we will use complex coordinates in U, as follows.

We identify U with the complex plane $\boldsymbol{C}$, considered as a *real* vectorspace, via the linear isomorphism:

$$\chi : \boldsymbol{C} \to U, \qquad z \mapsto \chi(z) := \mathcal{R}e\,(z\zeta). \tag{1.19}$$

This brings the bifurcation equation (1.17) in the form

$$F(z, \lambda, \sigma) := \chi^{-1} G(\chi(z), \lambda, \sigma) = 0; \tag{1.20}$$

that is, $F : \boldsymbol{C} \times \boldsymbol{R}^2 \to \boldsymbol{C}$ is given by

$$F(z, \lambda, \sigma) = \langle \zeta, M(\chi(z) + v^*(\chi(z), \lambda, \sigma), \lambda, \sigma)\rangle. \tag{1.21}$$

Now $S_+(\varphi)\, \zeta = e^{i\varphi}\zeta$; therefore

$$(\chi^{-1} S_+(\varphi)\, \chi)\,(z) = e^{i\varphi} z, \qquad \forall \varphi \in \boldsymbol{R}, \forall z \in \boldsymbol{C}, \tag{1.22}$$

and the equivariance of the bifurcation mapping takes the form

$$F(e^{i\varphi} z, \lambda, \sigma) = e^{i\varphi} F(z, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R}. \tag{1.23}$$

The next lemma describes a general form for mappings F satisfying (1.23). We remark again that we consider $\boldsymbol{C}$ as a real vectorspace, and the smoothness refered to in the lemma corresponds to smoothness for mappings between real vectorspaces.

Lemma. *Let Λ be a Banach space, and $F : \boldsymbol{C} \times \Lambda \to \boldsymbol{C}$ a smooth mapping such that*

$$F(e^{i\varphi} z, \lambda) = e^{i\varphi} F(z, \lambda) \qquad \forall \varphi \in \boldsymbol{R}. \tag{1.24}$$

Then there exists a unique smooth mapping $h : \boldsymbol{C} \times \Lambda \to \boldsymbol{C}$ such that

$$F(z, \lambda) = h(z, \lambda)\, z, \qquad \forall z, \lambda; \tag{1.25}$$

moreover:

(i) $h(e^{i\varphi} z, \lambda) = h(z, \lambda)$, $\forall \varphi \in \boldsymbol{R}$;
(ii) if $F(\bar{z}, \lambda) = \bar{F}(z, \lambda)$ then $h(z, \lambda)$ is real-valued.

Proof. The lemma follows from a somewhat stronger result which we will prove in part II. Here we give a direct elementary proof.

Assume first that $F(\bar{z}, \lambda) = F(z, \lambda)$ for all z, λ. Then, using (1.24), we see that both $F(z, \lambda)\,\bar{z}$ and its complex conjugate $\overline{F(z, \lambda)}\, z$ equal $F(|z|, \lambda)\,|z|$, which is real by our additional condition. So we have $\operatorname{Im} F(z, \lambda)\,\bar{z} = 0$. Writing $z = a + ib$ and $F = F_1 + iF_2$ this takes the form

$$bF_1(a, b, \lambda) = aF_2(a, b, \lambda). \tag{1.26}$$

This implies $F_1(0, b, \lambda) = 0$ for all (b, λ), and $F_1(a, b, \lambda) = ah(a, b, \lambda)$ for some smooth, real-valued function h; then (1.26) implies that $F_2(a, b, \lambda) = bh(a, b, \lambda)$ which proves (1.25) for this particular case.

For general F we write

$$F(z, \lambda) = \frac{1}{2}[F(z, \lambda) + \overline{F(\bar{z}, \lambda)}] + i\frac{1}{2i}[F(z, \lambda) - \overline{F(\bar{z}, \lambda)}] =$$
$$= \tilde{F}(z, \lambda) + i\tilde{\tilde{F}}(z, \lambda).$$

Now we can apply the first part on both $\tilde{F}$ and $\tilde{\tilde{F}}$; combining the results gives (1.25). For $z \neq 0$ the uniqueness of h follows immediately from (1.25); by continuity we also have uniqueness for $z = 0$. Both (i) and (ii) now follow easily from (1.25).

Applying the lemma to the bifurcation equation (1.20) we see that we can write $F(z, \lambda, \sigma) = h(z, \lambda, \sigma)\, z$, where $h : \boldsymbol{C} \times \boldsymbol{R}^2 \to \boldsymbol{C}$ is smooth and such that

$$h(e^{i\varphi}z, \lambda, \sigma) = h(z, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R}. \tag{1.27}$$

Finding nontrivial solutions of (1.20) then reduces to solving the equation

$$h(z, \lambda, \sigma) = 0. \tag{1.28}$$

We will solve (1.28) for λ and σ, as a function of z, by the implicit function theorem. To see that this is possible we make the following calculation.

For $\varrho \in \boldsymbol{R}$ we have $\chi(\varrho) = \varrho u_1$, and from (1.21) and the definition of h we find:

$$h(\varrho, \lambda, \sigma)\,\varrho = \langle \zeta, M(\varrho u_1 + v^*(\varrho u_1, \lambda, \sigma), \lambda, \sigma)\rangle;$$

differentiation in ϱ at $\varrho = 0$ gives:

$$h(0, \lambda, \sigma) = \langle \zeta, D_x M(0, \lambda, \sigma) \,.\, (u_1 + D_u v^*(0, \lambda, \sigma) \,.\, u_1)\rangle. \tag{1.29}$$

From (1.29) we obtain:

(a) $h(0, 0, 1) = \langle \zeta, Lu_1\rangle = 0$;

(b) $D_\sigma h(0, 0, 1) = \langle \zeta, D_x D_\sigma M(0, 0, 1) \,.\, u_1\rangle = 2\langle \zeta, \ddot{u}_1\rangle = -2\langle \zeta, u_1\rangle = -2$.

(c) $D_\lambda h(0, 0, 1) = \langle \zeta, D_x D_\lambda M(0, 0, 1) \,.\, u_1\rangle =$

$$= \left\langle \zeta, \frac{\partial g}{\partial \lambda}(0, 0, 0)\, \dot{u}_1 + \frac{\partial f}{\partial \lambda}(0, 0, 0)\, u_1 \right\rangle =$$
$$= \frac{\partial f}{\partial \lambda}(0, 0, 0) + i\frac{\partial g}{\partial \lambda}(0, 0, 0).$$

We conclude that if the transversality condition (1.6) is satisfied, then we can split (1.28) into its real and its imaginary part, and solve for $\lambda = \lambda^*(z)$ and $\sigma = \sigma^*(z)$; both λ^* and σ^* are smooth, with $\lambda^*(0) = 0$, $\sigma^*(0) = 1$, $\lambda^*(e^{i\varphi}z) = \lambda^*(z) = \lambda^*(|z|)$ and $\sigma^*(e^{i\varphi}z) = \sigma^*(z) = \sigma^*(|z|)$. The solution sheet found in this way forms the Hopf bifurcation branch. Solutions on this sheet corresponding to z-values with the same modulus can be obtained one from the other by application of a phase shift $S_+(\varphi)$ (see (1.22)). Therefore the result can be depicted in a (λ, ϱ)-plane by the curve $\lambda = \lambda^*(\varrho)$ for $\varrho \geq 0$ bifurcating from the line $\varrho = 0$.

To conclude this section we emphasize the two conditions which allowed us to obtain the Hopf bifurcation: (i) the fact that $A(0)$ has a pair of *simple* purely imaginary eigenvalues, so that the nullspace U is two-dimensional; and (ii) the transversality condition (1.6).

2. TIME-REVERSIBILITY

A simple symmetry which may destroy the bifurcation picture given by the Hopf theorem is time-reversibility. If we assume that the functions f and g in our example equation (1.1) are such that

$$f(x, -\dot{x}, \lambda) = f(x, \dot{x}, \lambda) \tag{2.1}$$
$$g(x, -\dot{x}, \lambda) = -g(x, \dot{x}, \lambda)$$

then it follows easily that if $x(t)$ is a solution of (1.1), then so is $\tilde{x}(t) = x(-t)$. But (2.1) implies that $g(0, 0, \lambda) = 0$ for all λ, and the transversality condition (1.6) is not satisfied, since the eigenvalues given by (1.5) stay on the imaginary axis.

However, we can still carry out most of the reductions of section 1, the main difference being that the operator M has an additional symmetry. Indeed, if we define $S_-(0) \in \mathscr{L}(Z)$ by

$$(S_-(0)\, z)(t) = z(-t), \tag{2.2}$$

then it is easy to verify that (2.1) implies that

$$M(S_-(0)\, z, \lambda, \sigma) = S_-(0)\, M(z, \lambda, \sigma); \tag{2.3}$$

so M is equivariant with respect to the group action generated by the shift operators $S_+(\varphi)$ and by the time inversion operator $S_-(0)$. Since $S_+(\varphi)\, S_-(0) = S_-(0)\, S_+(-\varphi)$ this group has the form $\{S_\pm(\varphi) \mid \varphi \in \mathbf{R}\}$, with $S_-(\varphi) = S_-(0)\, S_+(\varphi)$, and is isomorphic to $O(2)$. We conclude that for time-reversible systems the bifurcation problem has an $O(2)$-equivariance.

Both v^* and G will commute with $S_-(0)$; now $S_-(0)\, \zeta = \xi$, and consequently

$$(\chi^{-1} S_-(0)\, \chi)(z) = \chi^{-1} S_-(0)\, (\mathscr{R}\mathrm{e}\, (z\zeta)) = \chi^{-1}\, \mathscr{R}\mathrm{e}\, (z\xi) = \chi^{-1}\, \mathrm{Re}\, (\bar{z}\zeta) = \bar{z}, \quad \forall z \in \mathbf{C}. \tag{2.4}$$

For the complex bifurcation function F (2.3) translates into

$$F(\bar{z}, \lambda, \sigma) = \bar{F}(z, \lambda, \sigma). \tag{2.5}$$

We can again apply the lemma of section 1, and conclude that for nontrivial solutions the bifurcation equation reduces to

$$h(z, \lambda, \sigma) = 0, \tag{2.6}$$

where now, because of (2.5), h is real-valued. By the calculations of section 1 we have $h(0, 0, 1) = 0$ and $D_\sigma h(0, 0, 1) = -2$, so that we can solve the scalar equation (2.6) for $\sigma = \sigma^*(z, \lambda)$, with σ^* smooth, $\sigma^*(0, 0) = 1$, and $\sigma^*(e^{i\varphi}z, \lambda) = \sigma^*(z, \lambda) = \sigma^*(|z|, \lambda)$.

We see that for time-reversible systems the parameter λ plays no role: for each (sufficiently small) value of λ the equation has a 2-parameter family of periodic solutions, parametrized by $z \in \boldsymbol{C}$, or, equivalently, by the „amplitude" $\varrho = |z|$ and the phase $\varphi = \arg z$. Moreover, if z is real (i.e. $z = \bar{z}$), then (2.4) and the Liapunov—Schmidt reduction imply that the corresponding solution of (1.9) satisfies $S_-(0)\,x = x$, i.e. is an even function of t. We conclude that all solutions of (1.9) near the bifurcation point become even functions of t after an appropriate phase shift.

3. HOPF BIFURCATION WITH $O(2)$-SYMMETRY

In this section we exame a situation where the first condition for Hopf bifurcation, namely that the purely imaginary eigenvalues of $A(0)$ are simple, is no longer satisfied because of some spatial symmetry of the system; our example will have an $O(2)$-symmetry.

We consider again the equation (1.1), but now we take x to be a vector in the plane; the scalar functions f and g then map $\boldsymbol{R}^2 \times \boldsymbol{R}^2 \times \boldsymbol{R}$ into $\boldsymbol{R}$. We maintain the condition (1.2), and also assume that f and g are invariant under orthogonal transformations in the plane:

$$\begin{aligned} &f(Rx, R\dot{x}, \lambda) = f(x, \dot{x}, \lambda), \\ &g(Rx, R\dot{x}, \lambda) = g(x, \dot{x}, \lambda), \qquad \forall R \in O(2); \end{aligned} \tag{3.1}$$

we have $O(2) = \{R_\pm(\Theta) \mid \Theta \in \boldsymbol{R}\}$, where $R_+(\Theta)$ and $R_-(\Theta) = R_-(0)\,R_+(\Theta)$ are represented by the matrices

$$R_+(\Theta) = \begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix} \quad \text{and} \quad R_-(\Theta) = \begin{pmatrix} \cos\Theta & \sin\Theta \\ \sin\Theta & -\cos\Theta \end{pmatrix}. \tag{3.2}$$

The conditions (3.1) are for example satisfied if f and g depend on x and $\dot{x}$ via the arguments $|x|^2$, $|\dot{x}|^2$ and $(x, \dot{x})$.

An immediate consequence of (3.1) is that each solution $x(t)$ of (1.1) generates for each $\Theta \in \boldsymbol{R}$ the other solutions $\tilde{x}(t) = R_+(\Theta)\,x(t)$ and $\tilde{\tilde{x}}(t) = R_-(\Theta)\,x(t)$. Also, linearization at $x = 0$ gives now a 4-dimensional matrix, which still has the complex conjugate eigenvalues given by (1.5); however, these eigenvalues are now double, and the results of section 1 do no longer apply, since dim $U = 4$.

As a first step towards the solutions for this case we will use the $O(2)$-symmetry of the equation to consider two particular classes of solutions, each having prescribed symmetry properties themselves; we will label these solutions respectively as *collinear* and *circular* solutions.

A. Collinear solutions are solutions such that $R_-(\Theta)\, x(t) = x(t)$, for all $t \in \boldsymbol{R}$ and for some fixed $\Theta \in \boldsymbol{R}$. By a rotation we may assume that $\Theta = 0$, i.e. we look for solutions with $x_2(t) \equiv 0$. This condition reduces us to a one-dimensional equation of the type considered in section 1. We conclude that if the transversality condition (1.6) is satisfied, then there is a Hopf bifurcation of collinear periodic solutions; each solution on the bifurcation branch generates a torus of solutions by application of rotations $R_+(\Theta)$ and by phase shifts.

B. Circular solutions are solutions such that $x(t + \Theta) = R_+(\sigma\Theta)\, x(t)$, for all t and Θ, and for some fixed $\sigma \neq 0$. If they exist such solutions are necessarily periodic, with period $2\pi/|\sigma|$; they have the form $x(t) = (\varrho \cos \psi(t), -\varrho \sin \psi(t))$, with $\varrho \geq 0$ and $\psi(t) = \sigma t + c$; they rotate in the clockwise direction if $\sigma > 0$, in the anticlockwise direction if $\sigma < 0$. The conditions on ϱ, σ and λ for $x(t)$ to be a solution of (1.1) are

$$\sigma^2 = \tilde{f}(\varrho, \sigma, \lambda), \tag{3.3a}$$

$$\tilde{g}(\varrho, \sigma, \lambda) = 0, \tag{3.3b}$$

where $\tilde{f}$ and $\tilde{g}$ are defined by

$$\tilde{f}(\varrho, \sigma, \lambda) = f(\varrho(\cos \psi, -\sin \psi), -\varrho\sigma(\sin \psi, \cos \psi), \lambda) \tag{3.4}$$

and

$$\tilde{g}(\varrho, \sigma, \lambda) = g(\varrho(\cos \psi, -\sin \psi), -\varrho\sigma(\sin \psi, \cos \psi), \lambda). \tag{3.5}$$

One can easily verify that, because of (3.1), the expressions at the righthand side of (3.4) and (3.5) do not depend on ψ; also $\tilde{f}$ and $\tilde{g}$ are smooth functions, even in both ϱ and σ, and independent of σ for $\varrho = 0$; this implies in particular that

$$D_\sigma \tilde{f}(0, 0, \lambda) = D_\sigma \tilde{g}(0, 0, \lambda) = 0.$$

The equations (3.3) are satisfied for $(\varrho, \sigma, \lambda) = (0, \pm 1, 0)$; under the transversality condition (1.6) we can apply the implicit function theorem to obtain two solution branches $\sigma = \pm\sigma^*(\varrho)$, $\lambda = \lambda^*(\varrho)$, defined for ϱ sufficiently small; both σ^* and λ^* are even functions of ϱ, with $\sigma^*(0) = 1$ and $\lambda^*(0) = 0$. The branch with $\sigma = \sigma^*(\varrho) > 0$ corresponds to circular periodic solutions rotating in the clockwise direction, while solutions corresponding to the branch $\sigma = -\sigma^*(\varrho)$ rotate anticlockwise; both branches of circular solutions are mapped one into the other by the reflection $R_-(0)$. Since rotations and phase shifts have similar effects when acting on circular solutions it follows that each such cicrular solution generates a circle of circular solutions by rotation (or phase shifts).

We see that by restricting to solutions with appropriate symmetry properties we are able to obtain certain branches of small periodic solutions bifurcating from the equilibrium. The question which arises now is whether there exist any other periodic solutions near the bifurcation point, which we cannot detect by imposing a priori symmetry properties; and if such solutions exist, what are their symmetry properties? As we will see in part II the answer is that under some generic conditions there are no bifurcating periodic solutions other than the collinear and circular solutions which we have found already. In order to prove this we will use the Liapunov—Schmidt approach of section 1 to obtain the bifurcation equations which determine all bifurcating periodic solutions. After analyzing the proof of the Hopf bifurcation given in section 1, it should be no surprise to find that the new bifurcation equations have an $O(2)\times SO(2)$-symmetry; the $O(2)$-part acting by rotations in the plane, the $SO(2)$-part by phase shifts. These symmetries will then be exploited to bring the bifurcation equations in a normal form which yields easily the result mentioned above.

4. TIME-REVERSIBLE SYSTEMS WITH $O(2)$-SYMMETRY

As a second example we take the same problem with $O(2)$-symmetry as in section 3, but assume also time-reversibility; that is, we suppose that next to (3.1) also (2.1) holds. As explained in section 2 this implies that the transversality condition (1.6) fails; $A(\lambda)$ does not only have a pair of double eigenvalues on the imaginary axis for $\lambda = 0$, but these eigenvalues also stay on the imaginary axis when the parameter λ is changed.

In a similar way as in section 3 we consider collinear and circular solutions. For collinear solutions the equation reduces to a one-dimensional one which is time-reversible, and therefore of the type discussed in section 2. From the results of that section we conclude that for all λ near zero there is a one-parameter family of tori of collinear solutions: the parameter along the family is the amplitude ϱ, and each torus is generated by application of rotations $R_+(\Theta)$ and phase shifts on a particular collinear solution.

For the circular solutions the analysis of section 3, which leads to the equations (3.3), still holds. In particular the function $\tilde{g}$ is even in σ; but (2.1) is easily seen to imply that $\tilde{g}$ is also odd in σ; therefore we have $\tilde{g}(\varrho, \sigma, \lambda) = 0$ for all $(\varrho, \sigma, \lambda)$. An application of the implicit function theorem to (3.3a) then gives two symmetric solution branches $\sigma = \pm\sigma^*(\varrho, \lambda)$, with $\sigma^*(0, 0) = 1$ and $\sigma^*(-\varrho, \lambda) = \sigma^*(\varrho, \lambda)$. For each sufficiently small λ there are two one-parameter families of circles of circular solutions: the parameter is the amplitude ϱ and each circle is generated by application of rotations or phase shifts (both have similar effects) on a particular circular solution. One of the families corresponds to clockwise rotations, the other to anti-clockwise rotations; the two families are related one to the other by reflections in the plane.

As in the example of section 2 also here the parameter λ plays no role, at least when we restrict attention to circular and collinear solutions. In part II we will show that the corresponding bifurcation equations have an $O(2) \times O(2)$-symmetry, and that for generic time-reversible systems with $O(2)$-symmetry (not depending on a parameter) there are no other small periodic solutions than the families of circular and collinear solutions which we have found here. However, in one-parameter families of such equations (such as our example), a parameter change may force the system to pass through a nongeneric situation, and as we will see further bifurcations may then occur.

5. HOPF BIFURCATION WITH $SO(2)$-SYMMETRY

Let us now consider a somewhat more general second-order system, of the form

$$\ddot{x} + h(x, \dot{x}, \lambda) = 0, \tag{5.1}$$

with $x \in \boldsymbol{R}^2$, $h : \boldsymbol{R}^2 \times \boldsymbol{R}^2 \times \boldsymbol{R} \to \boldsymbol{R}^2$ smooth, and with $h(0, 0, \lambda) = 0$ for all λ. Assume also that (5.1) has an $SO(2)$-symmetry, namely:

$$h(Rx, R\dot{x}, \lambda) = Rh(x, \dot{x}, \lambda), \qquad \forall R \in SO(2), \tag{5.2}$$

where $SO(2) = \{R_+(\Theta) \mid \Theta \in \boldsymbol{R}\}$. Linearization of (5.1) at the equilibrium solution $x = 0$ gives

$$\ddot{x} + C(\lambda)\, \dot{x} + B(\lambda)\, x = 0, \tag{5.3}$$

where $B(\lambda) := D_x h(0, 0, \lambda)$ and $C(\lambda) := D_{\dot{x}} h(0, 0, \lambda)$ are 2×2-matrices. Because of (5.2) they must have the form

$$B(\lambda) = \begin{pmatrix} \beta_1(\lambda) & \beta_2(\lambda) \\ -\beta_2(\lambda) & \beta_1(\lambda) \end{pmatrix}, \qquad C(\lambda) = \begin{pmatrix} \gamma_1(\lambda) & \gamma_2(\lambda) \\ -\gamma_2(\lambda) & \gamma_1(\lambda) \end{pmatrix}. \tag{5.4}$$

for some smooth functions $\beta_i(\lambda)$ and $\gamma_i(\lambda)$ ($i = 1, 2$). The eigenvalues of the linear system (5.3) are given by the equation

$$(\mu^2 + \mu\gamma_1(\lambda) + \beta_1(\lambda))^2 + (\mu\gamma_2(\lambda) + \beta_2(\lambda))^2 = 0. \tag{5.5}$$

Suppose now that a pair of complex conjugate eigenvalues crosses the imaginary axis for $\lambda = \lambda_0$. If the corresponding purely imaginary eigenvalues have multiplicity two, then we have necessarily (i) $\gamma_1(\lambda_0) = 0$, and (ii) either $\beta_2(\lambda_0) = 0$ or $\gamma_2(\lambda_0) = 0$. For generic one-parameter problems these conditions (i) and (ii) cannot both be satisfied for one and the same parameter value λ_0. Therefore we conclude that if a pair of complex conjugate eigenvalues of the linearization (5.3) crosses the imaginary axis, then generically these eigenvalues will be simple, and the classical Hopf bifurcation theorem can be applied.

Of course we can still consider circular solutions, that is solutions of the form $x(t) = (\varrho \cos \psi(t), -\varrho \sin \psi(t))$, for some constant ϱ and with $\psi(t) = \sigma t + c$ for some $\sigma \neq 0$. Using (5.2) the conditions on $(\varrho, \sigma, \lambda)$ now take the form

$$(5.6) \qquad \begin{aligned} -\varrho\sigma^2 + h_1(\varrho, 0; 0, -\varrho\sigma; \lambda) &= 0, \\ h_2(\varrho, 0; 0, -\varrho\sigma; \lambda) &= 0. \end{aligned}$$

For nonzero solutions we can divide by ϱ; this gives

$$(5.7) \qquad -\sigma^2 + \tilde{h}_1(\varrho, \lambda, \sigma) = 0, \qquad \tilde{h}_2(\varrho, \lambda, \sigma) = 0,$$

with both $\tilde{h}_1(\varrho, \lambda, \sigma)$ and $\tilde{h}_2(\varrho, \lambda, \sigma)$ even in ϱ, and of the form

$$(5.8) \qquad \begin{aligned} \tilde{h}_1(\varrho, \lambda, \sigma) &= \beta_1(\lambda) - \sigma\gamma_2(\lambda) + O(\varrho^2), \\ \tilde{h}_2(\varrho, \lambda, \sigma) &= -\beta_2(\lambda) - \sigma\gamma_1(\lambda) + O(\varrho^2). \end{aligned}$$

We leave it to the reader to prove that, given $\sigma_0 \neq 0$, the following two statements are equivalent:

(i) $\pm i\sigma_0$ are eigenvalues of (5.3) for $\lambda = 0$;

(ii) either $(\varrho, \lambda, \sigma) = (0, 0, \sigma_0)$ or $(\varrho, \lambda, \sigma) = (0, 0, -\sigma_0)$ is a solution of (5.7).

It follows that we can choose the sign of σ_0 such that $(0, 0, \sigma_0)$ is a solution of (5.7). Moreover, the Jacobian of (5.7) at this solution and in the variables (λ, σ) will be different from zero if and only if the corresponding eigenvalues of (5.3) cross the imaginary axis transversally. Under such conditions we can solve (5.7) for $(\sigma, \lambda) = (\sigma^*(\varrho), \lambda^*(\varrho))$, with $\sigma^*(0) = \sigma_0$, $\lambda^*(0) = 0$, $\sigma^*(-\varrho) = \sigma^*(\varrho)$ and $\lambda^*(-\varrho) = \lambda^*(\varrho)$. This gives us a branch of circular solutions bifurcating from the origin; these circular solutions are clockwise if $\sigma_0 > 0$, anti-clockwise if $\sigma_0 < 0$. Because of the uniqueness part of the Hopf bifurcation theorem this branch of circular solutions must necessarily coincide with the branch of periodic solutions given by Hopf's theorem. So in systems with $SO(2)$-symmetry one generically has classical Hopf bifurcations, although the bifurcating periodic solutions have an additional symmetry: they are circular solutions. The same conclusion can also be obtained from a Liapunov—Schmidt reduction.

We conclude with the remark that if (5.1) has an $O(2)$-symmetry, i.e. if (5.2) holds for all $R \in O(2)$, then $\beta_2(\lambda) \equiv 0$ and $\gamma_2(\lambda) \equiv 0$ in (5.4). Then all eigenvalues of (5.3) have multiplicity two. This shows that the example of section 3 forms in some sense a „generic example" of a system with $O(2)$-symmetry. For such systems one needs to make an analysis of the full four-dimensional bifurcation equations in order to get the complete bifurcation picture; we will make such analysis in part II of this paper.

PART II

1. HYPOTHESES AND NOTATIONS

In this second part of the paper we will make a full bifurcation analysis for the problem of the bifurcation of periodic solutions from the trivial solution for the equation

$$\ddot{x} + g(x, \dot{x}, \lambda)\,\dot{x} + f(x, \dot{x}, \lambda)\,x = 0, \tag{1.1}$$

where $x \in \mathbf{R}^2$, $\lambda \in \mathbf{R}$ is a small parameter, and the scalar functions $f : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}$ are smooth and such that $f(0, 0, 0) > 0$ and $g(0, 0, 0) = 0$; by a time rescale we may suppose that $f(0, 0, 0) = 1$. Our main hypothesis will be that f and g are rotationally invariant, i.e. they satisfy

$$f(Rx, R\dot{x}, \lambda) = f(x, \dot{x}, \lambda), \qquad g(Rx, R\dot{x}, \lambda) = g(x, \dot{x}, \lambda), \qquad \forall R \in O(2). \tag{1.2}$$

We consider two different cases, corresponding to the hypotheses (H1) and (H2), respectively:

(H1) f and g satisfy (1.2) together with the transversality condition (see part I):

$$\frac{\partial g}{\partial \lambda}(0, 0, 0) \neq 0. \tag{1.3}$$

(H2) f and g satisfy (1.2) together with the following condition for time reversibility (see part I):

$$f(x, -\dot{x}, \lambda) = f(x, \dot{x}, \lambda), \qquad g(x, -\dot{x}, \lambda) = -g(x, \dot{x}, \lambda). \tag{1.4}$$

Remark that for $O(2)$-symmetric systems a transversality condition such as (1.3) will be generically satisfied; however, if we also impose the symmetry condition (1.4) then (1.3) fails, since (1.4) implies that $g(0, 0, \lambda) = 0$ for all λ. We will refer to the case (H2) as the time-reversible case.

We will obtain the bifurcation equations for our problem via an equivariant Liapunov—Schmidt reduction (see [6]). We will then exploite the symmetry properties of these bifurcation equations to discuss their solution set. This will give us the solution branches found in a heuristic way in part I, together with conditions which imply that there are no further bifurcating periodic solutions. As already remarked in part I the main ideas of our approach lead to similar results for general systems with $O(2)$-symmetry; for details on such extensions and generalizations we refer to the papers [2, 3, 5, 7, 8]. The distinctive points of our approach are: (i) the use of appropriate complex coordinates; (ii) the way in which we obtain the normal form of the bifurcation equations; (iii) the introduction of time-reversibility in the problem.

We are interested in small periodic solutions of (1.1), with period near 2π. By a time rescale (see part I) we can reformulate this problem as that of finding all small 2π-periodic solutions of the equation

$$\sigma^2\ddot{x} + \sigma g(x, \sigma\dot{x}, \lambda)\,\dot{x} + f(x, \sigma\dot{x}, \lambda)\,x = 0, \tag{1.5}$$

for values of (λ, σ) near $(0, 1)$. We write C^0 as a nonlinear operator equation in the following way. Let $Z = C^0_{2\pi}(\boldsymbol{R}^2)$ and $X = C^2_{2\pi}(\boldsymbol{R}^2)$ be the Banach spaces of all continuous, respectively C^2-differentiable 2π-periodic mappings $z : \boldsymbol{R} \to \boldsymbol{R}^2$, equipped with the C^0-, respectively C^2-supremum norm. Then we have to solve the equation

$$M(x, \lambda, \sigma) = 0 \tag{1.6}$$

for $(x, \lambda, \sigma) \in X \times \boldsymbol{R}^2$ near $(0, 0, 1)$, where $M : X \times \boldsymbol{R}^2 \to Z$ is defined by

(1.7)

$$M(x, \lambda, \sigma)\,(t) := \sigma^2\ddot{x}(t) + \sigma g(x(t), \sigma\dot{x}(t), \lambda)\,\dot{x}(t) + f(x(t), \sigma\dot{x}(t), \lambda)x(t), \qquad \forall t \in \boldsymbol{R}.$$

For each $\varphi \in \boldsymbol{R}$ we define a phase shift operator $S_+(\varphi) \in \mathscr{L}(Z)$ by $(S_+(\varphi)\,z)\,(t) = z(t + \varphi)$; we also use the time inversion operator $S_-(0) \in \mathscr{L}(Z)$ defined by $(S_-(0)\,z)\,(t) = z(-t)$, and we put $S_-(\varphi) = S_-(0)\,S_+(\varphi)$. The group $\{S_\pm(\varphi) \mid \varphi \in \boldsymbol{R}\}$ is isomorphic to $O(2)$. As already remarked in part I we have $SO(2) = \{R_+(\Theta) \mid \Theta \in \boldsymbol{R}\}$ and $O(2) = \{R_\pm(\Theta) / \Theta \in \boldsymbol{R}\}$, where $R_+(\Theta)$ and $R_-(\Theta) = R_-(0)\,R_+(\Theta)$ are linear operators on $\boldsymbol{R}^2$ represented by the matrices given in (I.3.2). We will also consider $R_+(\Theta)$ and $R_-(\Theta)$ as linear operators on Z, acting in the obvious way: $(R_\pm(\Theta)\,z)\,(t) = R_\pm(\Theta)\,z(t)$.

It follows directly from these definitions and hypotheses that

$$M(\gamma x, \lambda, \sigma) = \gamma M(x, \lambda, \sigma), \qquad \forall \gamma \in \Gamma, \tag{1.8}$$

where $\Gamma \subset \mathscr{L}(Z)$ is the group generated by $R_\pm(\Theta)$ and $S_+(\varphi)$ in case (H1), and the group generated by $R_\pm(\Theta)$ and $S_\pm(\varphi)$ in case (H2). In the first case Γ is isomorphic to $O(2) \times SO(2)$, in the time-reversible case it is isomorphic to $O(2) \times O(2)$.

Finally we will denote by $Z^c = C^0_{2\pi}(\boldsymbol{C}^2)$ the complexification of Z, and use in Z^c the inner product given by

$$\langle u, v\rangle := \frac{1}{2\pi}\int_0^{2\pi} (u(t), v(t))\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \Big(\sum_{i=1}^{2} \bar{u}_i(t)\,v_i(t)\Big)\,\mathrm{d}t, \qquad \forall u, v \in Z^c. \tag{1.9}$$

2. THE LIAPUNOV–SCHMIDT REDUCTION

We now perform on (1.6) a Liapunov—Schmidt reduction (see [6]) similar to the one worked out in section I.1. We start by observing that $L := D_x M(0, 0, 1) \in \mathscr{L}(Y, Z)$ is explicitly given by $(Lx)\,(t) = \ddot{x}(t) + x(t)$ and has a 4-dimensional nulspace U, with basis $\{u_1, \tilde{u}_1, u_2, \tilde{u}_2\}$ given by

$$\begin{aligned} u_1(t) &= \mathrm{col}(\cos t, -\sin t), \qquad & \tilde{u}_1(t) &= \mathrm{col}(\sin t, \cos t),\\ u_2(t) &= \mathrm{col}(\cos t, \sin t), & \tilde{u}_2(t) &= \mathrm{col}(\sin t, -\cos t). \end{aligned} \tag{2.1}$$

We define $\zeta_1 \in Z^c$, $\zeta_2 \in Z^c$ and $P \in \mathscr{L}(Z)$ by

$$\zeta_j(t) = u_j(t) + i\tilde{u}_j(t) = e^{it}\,\mathrm{col}(1, (-1)^{j+1}i), \qquad j = 1, 2 \tag{2.2}$$

and

$$Pz = Re(\langle \zeta_1, z\rangle \zeta_1 + \langle \zeta_2, z\rangle \zeta_2), \qquad \forall z \in Z. \tag{2.3}$$

It is easily seen that P is a continuous projection onto $U = N(L)$, while Floquet theory implies that $N(P) = R(L)$. So we have

$$Z = N(P) \oplus R(P), \qquad X = U \oplus V, \tag{2.4}$$

with $V = N(P) \cap X$; also, L maps V isomorphically onto $R(L) = N(P)$. Remark that

$$PR_{\pm}(\Theta) = R_{\pm}(\Theta)\,P, \qquad PS_{\pm}(\varphi) = S_{\pm}(\varphi)\,P, \qquad \forall \Theta, \varphi \in \boldsymbol{R}. \tag{2.5}$$

This implies that all symmetry operators leave U and V invariant.

Using the splittings (2.4) we can now perform an equivariant Liapunov—Schmidt reduction in precisely the same way as we did in section I.1. The result is the following: there exist a neighborhood $\tilde{\Omega}$ of $(0, 0, 1)$ in $X \times \boldsymbol{R} \times \boldsymbol{R}$, a neighborhood Ω of $(0, 0, 1)$ in $U \times \boldsymbol{R} \times \boldsymbol{R}$, and smooth mappings $v^* : \Omega \to V$ and $G : \Omega \to U$ such that

$$\begin{aligned}&\{(x, \lambda, \sigma) \in \tilde{\Omega} \mid M(x, \lambda, \sigma) = 0\} = \\ &= \{(u + v^*(u, \lambda, \sigma), \lambda, \sigma) \mid (u, \lambda, \sigma) \in \Omega \text{ and } G(u, \lambda, \sigma) = 0\}.\end{aligned} \tag{2.6}$$

The mappings M, v^* and G are related to each other by

$$(I - P)\,M(u + v^*(u, \lambda, \sigma), \lambda, \sigma) = 0, \qquad \forall (u, \lambda, \sigma) \tag{2.7}$$

and

$$G(u, \lambda, \sigma) = PM(u + v^*(u, \lambda, \sigma), \lambda, \sigma), \qquad \forall (u, \lambda, \sigma). \tag{2.8}$$

The bifurcation mapping $G : \Omega \subset U \times \boldsymbol{R} \times \boldsymbol{R} \to U$ has the following properties:

(i) $G(0, \lambda, \sigma) = 0, \forall \lambda, \sigma$;
(ii) $D_u G(0, 0, 1) = 0$;
(iii) $G(\gamma u, \lambda, \sigma) = \gamma G(u, \lambda, \sigma), \forall \gamma \in \Gamma$;

similar properties hold for v^*.

By (2.6) the problem is reduced to that of solving the Γ-equivariant bifurcation equation

$$G(u, \lambda, \sigma) = 0 \tag{2.9}$$

for $(u, \lambda, \sigma) \in U \times \boldsymbol{R} \times \boldsymbol{R}$ near $(0, 0, 1)$. As in part I we will write down (2.8) in complex coordinates. To do so we identify U with $\boldsymbol{C}^2$, considered as a *real* vector-space, via the linear isomorphism

$$\chi : \boldsymbol{C}^2 \to U, \qquad (z_1, z_2) \mapsto \chi(z_1, z_2) := Re(z_1\zeta_1 + z_2\zeta_2). \tag{2.10}$$

Then (2.9) takes the form

$$F(z_1, z_2, \lambda, \sigma) := \chi^{-1}G(\chi(z_1, z_2), \lambda, \sigma) = 0. \tag{2.11}$$

It follows from (2.8) and the definition of P that $F = (F_1, F_2) : \boldsymbol{C}^2 \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{C}^2$ is explicitly given by

$$F_j(z_1, z_2, \lambda, \sigma) = \langle \zeta_j, M(\chi(z_1, z_2) + v^*(\chi(z_1, z_2), \lambda, \sigma), \lambda, \sigma) \rangle. \qquad j = 1, 2 \tag{2.12}$$

As for the symmetry operators, we will identify each $\gamma \in \Gamma$ with $\chi^{-1}\gamma\chi \in \mathscr{L}(\boldsymbol{C}^2)$. We have

$$\begin{array}{llll} S_+(\varphi)\,\zeta_1 = e^{i\varphi}\zeta_1, & S_+(\varphi)\,\zeta_2 = e^{i\varphi}\zeta_2, & S_-(0)\,\zeta_1 = \bar{\zeta}_2, & S_-(0)\,\zeta_2 = \bar{\zeta}_1, \\ R_+(\Theta)\,\zeta_1 = e^{i\Theta}\zeta_1, & R_+(\Theta)\,\zeta_2 = e^{-i\Theta}\zeta_2, & R_-(0)\,\zeta_1 = \zeta_2, & R_-(0)\,\zeta_2 = \zeta_1, \end{array} \tag{2.13}$$

and therefore

$$\begin{array}{ll} S_+(\varphi)\,(z_1, z_2) = (e^{i\varphi}z_1, e^{i\varphi}z_2), & S_-(0)\,(z_1, z_2) = (\bar{z}_2, \bar{z}_1), \\ R_+(\Theta)\,(z_1, z_2) = (e^{i\Theta}z_1, e^{-i\Theta}z_2), & R_-(0)\,(z_1, z_2) = (z_2, z_1). \end{array} \tag{2.14}$$

The equivariance of G induces the equivariance of F:

$$F(\gamma(z_1, z_2), \lambda, \sigma) = \gamma F(z_1, z_2, \lambda, \sigma), \qquad \forall \gamma \in \Gamma. \tag{2.15}$$

Taking γ equal to $R_+(\varphi)\,S_+(\varphi)$, $R_+(-\varphi)\,S_+(\varphi)$ and $R_-(0)$ gives the following symmetry properties, which will be particularly useful for our further analysis:

$$F_1(e^{i\varphi}z_1, z_2, \lambda, \sigma) = e^{i\varphi}F_1(z_1, z_2, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R}, \tag{2.16}$$

$$F_1(z_1, e^{i\varphi}z_2, \lambda, \sigma) = F_1(z_1, z_2, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R}, \tag{2.17}$$

and

$$F_2(z_1, z_2, \lambda, \sigma) = F_1(z_2, z_1, \lambda, \sigma). \tag{2.18}$$

In the time-reversible case we can also take $\gamma = R_-(0)\,S_-(0)$ which gives

$$F_1(\bar{z}_1, \bar{z}_2, \lambda, \sigma) = \bar{F}_1(z_1, z_2, \lambda, \sigma). \tag{2.19}$$

3. COLLINEAR AND CIRCULAR SOLUTIONS

Before analyzing the full bifurcation equations (2.11) we briefly indicate how one obtains from them the branches of collinear and circular solutions found in part I. Collinear solutions are solutions such that $R_-(\Theta)\,x = x$ for some $\Theta \in \boldsymbol{R}$; they correspond to solutions of (2.11) for which $z_1 = e^{-i\Theta}z_2$ for some Θ, i.e. for which $|z_1| = |z_2|$. By an appropriate rotation we may restrict to collinear solutions for which $R_-(0)\,x = x$; that means that we take $z_1 = z_2 = z$ in the bifurcation equation. By (2.18) this bifurcation equation reduces then to a single complex equation

$$H(z, \lambda, \sigma) := F_1(z, z, \lambda, \sigma) = 0. \tag{3.1}$$

It follows from (2.16) and (2.17) that H is $SO(2)$-equivariant:

$$H(e^{i\varphi}z, \lambda, \sigma) = e^{i\varphi}H(z, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R}; \tag{3.2}$$

in the time-reversible case we also have

$$H(\bar{z}, \lambda, \sigma) = \bar{H}(z, \lambda, \sigma), \tag{3.3}$$

i.e. H is $O(2)$-equivariant. The equation (3.1) can now be solved by exactly the same approach as in sections I.1 and I.2.

Circular solutions of (1.6) are solutions such that either

$$R_+(\varphi)\, S_+(-\varphi)\, x = x, \qquad \forall \varphi \in \boldsymbol{R}, \tag{3.4}$$

or

$$R_+(\varphi)\, S_+(\varphi)\, x = x, \qquad \forall \varphi \in \boldsymbol{R}. \tag{3.5}$$

Solutions satisfying (3.4) correspond to solutions of (2.11) for which $(z_1, z_2) = R_+(\varphi)\, S_+(-\varphi)\, (z_1, z_2) = (z_1, e^{-2i\varphi}z_2)$, for all $\varphi \in \boldsymbol{R}$, i.e. for which $z_2 = 0$. But then the equivariance of F implies that $F(z_1, 0, \lambda, \sigma) = R_+(\varphi)\, S_+(-\varphi) \times F(z_1, 0, \lambda, \sigma)$ for all $\varphi \in \boldsymbol{R}$; that is, we have $F_2(z_1, 0, \lambda, \sigma) = 0$ for all (z_1, λ, σ). The same result also follows from (2.16) and (2.18). So, for solutions satisfying (3.4) the bifurcation equation reduces to

$$H_1(z_1, \lambda, \sigma) := F_1(z_1, 0, \lambda, \sigma) = 0. \tag{3.6}$$

Again, (2.16) implies that H_1 is $SO(2)$-equivariant, and even $O(2)$-equivariant in the time-reversible case (see (2.19)). The existence of a branch of circular solutions satisfying (3.4) then follows once more by the approach of sections I.1 and I.2.

In exactly the same way we must, for circular solutions satisfying (3.5), take $z_1 = 0$ in (2.11), which then reduces to

$$H_2(z_2, \lambda, \sigma) := F_2(0, z_2, \lambda, \sigma) = 0. \tag{3.7}$$

But (2.18) shows that $H_2(z_2, \lambda, \sigma) = H_1(z_2, \lambda, \sigma)$, so that (3.7) is in fact the same equation as (3.6), and therefore has the same solution branch. This reflects the fact that circular solutions satisfying (3.4) are by reflections $R_-(\Theta)$ taken into circular solutions satisfying (3.5), and vice versa.

It is easily seen that both the collinear and the circular solutions are such that

$$x(t + \pi) = -x(t), \qquad \forall t \in \boldsymbol{R}. \tag{3.8}$$

In fact, (3.8) holds for all solutions (x, λ, σ) of (1.6) in a sufficiently small neighborhood of the bifurcation point. Indeed, we have $S_+(\pi)\,(z_1, z_2) = R_+(\pi)\,(z_1, z_2)$, for all $z_1, z_2 \in \boldsymbol{C}$; this implies via the equivariant Liapunov—Schmidt reduction that each solution of (1.6) will be such that $R_+(\pi)\, x = S_+(\pi)\, x$, i.e. will satisfy (3.8).

As a final remark we observe that for each $(z_1, z_2) \in \boldsymbol{C}^2$ we can find Θ, φ, ϱ_1, $\varrho_2 \in$

$\in \boldsymbol{R}$ such that

$$(z_1, z_2) = (e^{i(\Theta+\varphi)}\varrho_1, e^{i(-\Theta+\varphi)}\varrho_2) = R_+(\Theta)\, S_+(\varphi)\,(\varrho_1, \varrho_2);$$

also $R_-(0)\, S_-(0)\,(\varrho_1, \varrho_2) = (\varrho_1, \varrho_2)$. It follows that in the time-reversible case, when $R_-(0)\, S_-(0)$ belongs to the symmetry group Γ, we can find for each solution x of (1.6) some appropriate Θ, $\varphi \in \boldsymbol{R}$ such that $x_0 := R_+(\Theta)\, S_+(\varphi)\, x$ is a solution belonging to the subspace

$$X_0 := \{x \in X \mid R_-(0)\, S_-(0)\, x = x\} =$$
$$= \{x \in X \mid x_1(-t) = x_1(t),\, x_2(-t) = -x_2(t)\}.$$

Then it is sufficient to solve (2.9) for $u \in U_0 := U \cap X_0 = \operatorname{span}\{u_1, u_2\}$, or, equivalently, solve (2.11) for $(z_1, z_2) \in \boldsymbol{R}^2$. This approach has been worked out in [7]: it leads to a 2-dimensional bifurcation problem ($\dim U_0 = 2$) with a D_4-symmetry; that is, with the symmetry of a square.

4. NORMAL FORM OF THE BIFURCATION MAPPING

In this section we use the equivariance properties (2.16)—(2.19) to bring the bifurcation mapping F in an appropriate normal form. We start with some general lemma's on real- and complex-valued functions of a real or complex variable, and dependent on a parameter λ belonging to a bounded open subset Λ of a finite-dimensional Banach space. Let us remind also that we consider $\boldsymbol{C}$ as a real vector-space, and that statements on smoothness always refer to smoothness for mappings between real Banach spaces.

Our first lemma describes a classical result of Whitney [11] on even functions. The proof uses Borel's theorem (see e.g. [1]) and the result has been extended to functions which are invariant under a general compact group by G. Schwartz [4].

Lemma 1. *Let $h : \boldsymbol{R} \times \Lambda \to \boldsymbol{R}$ be smooth and even: $h(\varrho, \lambda)$. Then there exists a smooth function $\tilde{h} : \boldsymbol{R} \times \Lambda \to \boldsymbol{R}$ such that*

$$h(\varrho, \lambda) = \tilde{h}(\varrho, \lambda), \qquad \forall (\varrho, \lambda) \in \boldsymbol{R} \times \Lambda. \tag{4.1}$$

Proof. First we consider the case where h is flat at $\varrho = 0$: $D_\varrho^k h(0, \lambda) = 0$ for all $k \in \boldsymbol{N}$ and all $\lambda \in \Lambda$. We define $\tilde{h} : \boldsymbol{R} \times \Lambda \to \boldsymbol{R}$ by

$$\tilde{h}(\xi, \lambda) := h(|\,\xi\,|^{1/2}, \lambda), \qquad \forall (\xi, \lambda) \in \boldsymbol{R} \times \Lambda. \tag{4.2}$$

It is clear that (4.1) holds and that $\tilde{h}$ is smooth for $\xi \neq 0$. To prove that h is also smooth at $\xi = 0$ it is sufficient to show that

$$\lim_{\substack{\to \\ \xi \neq 0}} D_\xi^k \tilde{h}(\xi, \lambda) = 0, \qquad \forall \lambda \in \Lambda, \forall k \in \boldsymbol{N}. \tag{4.3}$$

Now (4.3) is obvious for $k = 0$. By induction it is easy to prove that, for $k \geq 1$ and $\xi \neq 0$, $D_\xi^k \tilde{h}(\xi, \lambda)$ is given by a linear combination (with different coefficients in $\xi > 0$ and in $\xi < 0$) of terms of the form

$$D_\varrho^j h(|\xi|^{1/2}, \lambda) \, . \, |\xi|^{j/2-k},$$

with $1 \leqq j \leqq k$. Since h is flat at $\varrho = 0$ and $\bar{\Lambda}$ compact we can find for each (j, k) with $1 \leqq j \leqq k$ some constant $C_{j,k} \geq 0$ and some $\varepsilon_{j,k} > 0$ such that

$$|D_\varrho^j h(\varrho, \lambda)| \leqq C_{j,k} |\varrho|^{2k}, \qquad \forall \varrho : |\varrho| < \varepsilon_{j,k}.$$

This implies (4.3).

For the general case we denote by $\Sigma a_k(\lambda)\, \varrho^{2k}$ the Taylor expansion of h at $\varrho = 0$. By Borel's theorem there exists a smooth $\tilde{h}_1 = \tilde{h}_1(\xi, \lambda) : \boldsymbol{R} \times \Lambda \to \boldsymbol{R}$ having $\Sigma a_k(\lambda)\, \xi^k$ as Taylor expansion at $\xi = 0$. Let $h_2(\varrho, \lambda) = h(\varrho, \lambda) - \tilde{h}_1(\varrho^2, \lambda)$; then h_2 is even and flat at $\varrho = 0$. By the first part of the proof there exists a smooth $\tilde{h}_2 : \boldsymbol{R} \times \Lambda \to \boldsymbol{R}$ such that $h_2(\varrho, \lambda) = \tilde{h}_2(\varrho^2, \lambda)$. Then $\tilde{h}(\xi, \lambda) = \tilde{h}_1(\xi, \lambda) + \tilde{h}_2(\xi, \lambda)$ satisfies the requirements of the lemma. Remark that the condition (4.1) determines $\tilde{h}(\xi, \lambda)$ uniquely only for $(\xi, \lambda) \in [0, \infty) \times \Lambda$.

Lemma 2. *Let $H : \boldsymbol{C} \times \Lambda \to \boldsymbol{C}$ be smooth and such that*

$$H(e^{i\varphi} z, \lambda) = H(z, \lambda), \qquad \forall \varphi \in \boldsymbol{R}. \tag{4.4}$$

Then there exists a smooth function $h : \boldsymbol{R} \times \boldsymbol{C} \to \boldsymbol{C}$ such that

$$H(z, \lambda) = h(|z|^2, \lambda), \qquad \forall (z, \lambda) \in \boldsymbol{C} \times \Lambda. \tag{4.5}$$

Proof. Define $\tilde{H} : \boldsymbol{R} \times \Lambda \to \boldsymbol{C}$ by $\tilde{H}(\varrho, \lambda) := H(\varrho, \lambda)$; it follows from (4.4) with $\varphi = \pi$ that $\tilde{H}$ is even in ϱ; an application of lemma 1 to the real and imaginary part of $\tilde{H}$ gives us the existence of a smooth function $h : \boldsymbol{R} \times \Lambda \to \boldsymbol{C}$ such that $\tilde{H}(\varrho, \lambda) = h(\varrho^2, \lambda)$. By (4.4) we have then for each $(z, \lambda) \in \boldsymbol{C} \times \Lambda$ that $H(z, \lambda) = H(|z|, \lambda) = h(|z|^2, \lambda)$.

Lemma 3. *Let $H : \boldsymbol{C} \times \Lambda \to \boldsymbol{C}$ be smooth and such that*

$$H(e^{i\varphi} z, \lambda) = e^{i\varphi} H(z, \lambda), \qquad \forall \varphi \in \boldsymbol{R}. \tag{4.6}$$

Then there exists a smooth function $h : \boldsymbol{R} \times \Lambda \to \boldsymbol{C}$ such that

$$H(z, \lambda) = h(|z|^2, \lambda)\, z, \qquad \forall (z, \lambda) \in \boldsymbol{C} \times \Lambda. \tag{4.7}$$

If moreover $H(\bar{z}, \lambda) = \bar{H}(z, \lambda)$ for all (z, λ) then h can be chosen to be real-valued.

Proof. Let $\tilde{H}(z, \lambda) = H(z, \lambda)\, \bar{z}$; then $\tilde{H}$ satisfies the requirement of lemma 2, and we have $\tilde{H}(z, \lambda) = \tilde{h}(|z|^2, \lambda)$ for some smooth $\tilde{h} : \boldsymbol{R} \times \Lambda \to \boldsymbol{C}$. Since $\tilde{H}(0, \lambda) = 0$ it follows that also $\tilde{h}(0, \lambda) = 0$, and therefore we can write $\tilde{h}(\xi, \lambda) = \xi h(\xi, \lambda)$ for some smooth $h : \boldsymbol{R} \times \Lambda \to \boldsymbol{C}$. Then (4.7) follows from $H(z, \lambda)\, \bar{z} = \tilde{H}(z, \lambda) = |z|^2 h(|z|^2, \lambda)$. If $H(\bar{z}, \lambda) = \bar{H}(z, \lambda)$ for all (z, λ), then (4.7) still holds if we replace $h(\xi, \lambda)$ by $\frac{1}{2}\,[h(\xi, \lambda) + \bar{h}(\xi, \lambda)]$; this shows that we can find a real-valued function $h(\xi, \lambda)$ satisfying (4.7).

Lemma 4. *Let $h : \boldsymbol{R}^2 \times \Lambda \to \boldsymbol{C}$ be smooth. Then there exist unique smooth mappings $h_1 : \boldsymbol{R}^2 \times \Lambda \to \boldsymbol{C}$ and $h_2 : \boldsymbol{R}^2 \times \Lambda \to \boldsymbol{C}$ such that*

(i) $h(\xi_1, \xi_2, \lambda) = h_1(\xi_1, \xi_2, \lambda) + (\xi_1 - \xi_2) h_2(\xi_1, \xi_2, \lambda)$;
(ii) $h_i(\xi_2, \xi_1, \lambda) = h_i(\xi_1, \xi_2, \lambda)$, $i = 1, 2$.

Proof. Let $h_1(\xi_1, \xi_2, \lambda) := \frac{1}{2} [h(\xi_1, \xi_2, \lambda) + h(\xi_2, \xi_1, \lambda)]$ and $\tilde{h}_2(\xi_1, \xi_2, \lambda) := \frac{1}{2} [h(\xi_1, \xi_2, \lambda) - h(\xi_2, \xi_1, \lambda)]$. Since $\tilde{h}_2(\xi_1, \xi_2, \lambda) = 0$ if $\xi_1 = \xi_2$ we have $\tilde{h}_2(\xi_1, \xi_2, \lambda) = (\xi_1 - \xi_2) h_2(\xi_1, \xi_2, \lambda)$, for some smooth h_2. The functions h_1 and h_2 defined in this way satisfy the requirements (i) and (ii); the same conditions also easily imply the uniqueness of h_1 and h_2.

Now we apply the foregoing lemma's to obtain a normal form for the bifurcation mapping $F(z_1, z_2, \lambda, \sigma)$; because of (2.18) it is sufficient to consider the function $F_1(z_1, z_2, \lambda, \sigma)$. Restricting all variables to a bounded neighborhood of (0, 0, 0, 1) it follows from (2.17) and lemma 2 that

$$F_1(z_1, z_2, \lambda, \sigma) = f_1(z_1, |z_2|^2, \lambda, \sigma) \tag{4.8}$$

for some smooth $f_1 : \boldsymbol{C} \times \boldsymbol{R} \times \boldsymbol{R}^2 \to \boldsymbol{C}$. Using (2.16) we may replace $f_1(z_1, \xi_2, \lambda, \sigma)$ in (4.8) by

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-i\varphi} f_1(e^{i\varphi} z_1, \xi_2, \lambda, \sigma)\, d\varphi; \tag{4.9}$$

this symmetrization allows us to assume that

$$f_1(e^{i\varphi} z_1, \xi_2, \lambda, \sigma) = e^{i\varphi} f_1(z_1, \xi_2, \lambda, \sigma), \qquad \forall \varphi \in \boldsymbol{R}. \tag{4.10}$$

An application of lemma 3 then shows that

$$f_1(z_1, \xi_2, \lambda, \sigma) = g_1(|z_1|^2, \xi_2, \lambda, \sigma) z_1 \tag{4.11}$$

for some smooth $g_1 : \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}^2 \to \boldsymbol{C}$. This gives

$$F_1(z_1, z_2, \lambda, \sigma) = g_1(|z_1|^2, |z_2|^2, \lambda, \sigma) z_1. \tag{4.12}$$

Moreover, we can apply lemma 4 to g_1 and rewrite (4.12) in the form

$$F_1(z_1, z_2, \lambda, \sigma) = [h_1(\varrho_1^2, \varrho_2^2, \lambda, \sigma) + (\varrho_1^2 - \varrho_2^2) h_2(\varrho_1^2, \varrho_2^2, \lambda, \sigma)] z_1, \tag{4.13}$$

where $\varrho_i = |z_i|$ $(i = 1, 2)$ and the smooth functions $h_i : \boldsymbol{R} \times \boldsymbol{R} \times \boldsymbol{R}^2 \to \boldsymbol{C}$ are such that

$$h_i(\xi_2, \xi_1, \lambda, \sigma) = h_i(\xi_1, \xi_2, \lambda, \sigma), \qquad i = 1, 2. \tag{4.14}$$

This gives then finally, via (2.18):

$$F_2(z_1, z_2, \lambda, \sigma) = [h_1(\varrho_1^2, \varrho_2^2, \lambda, \sigma) - (\varrho_1^2 - \varrho_2^2) h_2(\varrho_1^2, \varrho_2^2, \lambda, \sigma)] z_2. \tag{4.15}$$

In the time-reversible case we have also (2.19); this allows us to use the symmetrization

$$\frac{1}{4\pi}\int_0^{2\pi} e^{-i\varphi}[f_1(e^{i\varphi}z_1, \xi_2, \lambda, \sigma) + \bar{f}_1(e^{-i\varphi}\bar{z}_1, \xi_2, \lambda, \sigma)]\, d\varphi \tag{4.16}$$

instead of (4.9). Then f_1 not only satisfies (4.10) but also

$$f_1(\bar{z}_1, \xi_2, \lambda, \sigma) = \bar{f}_1(z_1, \xi_2, \lambda, \sigma). \tag{4.17}$$

Then lemma 3 implies that the function g_1 in (4.11) and (4.12) is real-valued; consequently also h_1 and h_2 are real-valued in that case.

5. CALCULATION OF $h_1(0, 0, \lambda, \sigma)$ AND $h_2(0, 0, \lambda, \sigma)$

For our further analysis we will need $h_1(0, 0, \lambda, \sigma)$ and $h_2(0, 0, \lambda, \sigma)$; it follows from (4.13) that in order to determine those expressions we must calculate $F_1(z_1, z_2, \lambda, \sigma)$ up to third order terms in z_1 and z_2. The following lemma gives a first step in this calculation.

Lemma 5. *The bifurcation mapping $G(u, \lambda, \sigma)$ defined by* (2.8) *is such that*

$$G(u, \lambda, \sigma) = PM(u, \lambda, \sigma) + O(\| u \|^5). \tag{5.1}$$

Proof. Define $\tilde{G} : U\times \mathbf{R}\times \mathbf{R} \to U$ by $\tilde{G}(u, \lambda, \sigma) = PM(u, \lambda, \sigma)$. Then we have to prove that $D_u^k G(0, \lambda, \sigma) = D_u^k\tilde{G}(0, \lambda, \sigma)$ for $0 \leq k \leq 4$ and for all λ, σ. First we remark that both G and $\tilde{G}$ commute with $\gamma = R_+(\pi) = -I$; this implies $D_u^{2k}G(0, \lambda, \sigma) = D_u^{2k}\tilde{G}(0, \lambda, \sigma) = 0$ for all $k \in N$ and all λ, σ. Also v^* and M have this symmetry property, and therefore $D_x^{2k}M(0, \lambda, \sigma) = 0$ and $D_u^{2k}v^*(0, \lambda, \sigma) = 0$ for all $k \in N$.

Next the definition of $M(x, \lambda, \sigma)$ implies that

$$D_x M(0, \lambda, \sigma) \,.\, w = \sigma^2\ddot{w} + \sigma g(0, 0, \lambda)\,\dot{w} + f(0, 0, \lambda)\, w, \qquad \forall w \in X. \tag{5.2}$$

It is easily seen from this that $D_x M(0, \lambda, \sigma)$ leaves U invariant, and maps V into $N(P)$. Differentiating (2.7) at $u = 0$ then shows that $D_u v^*(0, \lambda, 0) = 0$ for all λ, σ. Now we differentiate (2.8) three times in u; putting $u = 0$ in the formulae and using the foregoing results then shows that $D_u G(0, \lambda, \sigma) = D_u\tilde{G}(0, \lambda, \sigma)$ and $D_u^3 G(0, \lambda, \sigma) = D_u^3\tilde{G}(0, \lambda, \sigma)$. This completes the proof.

Now we take $z_1 = \varrho_1 \in \mathbf{R}$ and $z_2 = \varrho_2 \in \mathbf{R}$ in (4.13), and combine with (2.11), (2.12) and lemma 5. This gives

$$\begin{aligned}&[h_1(\varrho_1^2, \varrho_2^2, \lambda, \sigma) + (\varrho_1^2 - \varrho_2^2)\, h_2(\varrho_1^2, \varrho_2^2, \lambda, \sigma)]\,\varrho_1 = \\ &= \langle \zeta_1, M(\varrho_1 u_1 + \varrho_2 u_2, \lambda, \sigma)\rangle + 0((| \varrho_1 | + | \varrho_2 |)^5).\end{aligned} \tag{5.3}$$

In order to calculate the expression at the right hand side of (5.3) up to third order terms in ϱ_1 and ϱ_2 we need to consider the Taylor expansion of the functions f and g appearing in (1), The condition (1.1) on those functions implies that

$$f(x, \dot{x}, \lambda) = \alpha(\lambda) + \beta_1(\lambda) \mid x \mid^2 + \beta_2(\lambda) (x, \dot{x}) + \beta_3(\lambda) \mid \dot{x} \mid^2 + \text{h.o.t.} \tag{5.4}$$

and

$$g(x, \dot{x}, \lambda) = \gamma(\lambda) + \delta_1(\lambda) \mid x \mid^2 + \delta_2(\lambda) (x, \dot{x}) + \delta_3(\lambda) \mid \dot{x} \mid^2 + \text{h.o.t.} \tag{5.5}$$

for some smooth functions $\alpha(\lambda) = f(0, 0, \lambda)$, $\beta_i(\lambda)$, $\gamma(\lambda) = g(0, 0, \lambda)$ and $\delta_i(\lambda)$ $(i = 1, 2, 3)$. Using these expansions a lengthy but straightforward calculation gives:

$$\begin{aligned} \langle \zeta_1, M(\varrho_1 u_1 + \varrho_2 u_2, \lambda, \sigma)\rangle &= \varrho_1(-\sigma^2 + i\sigma\gamma(\lambda) + \alpha(\lambda)) + \\ &+ \varrho_1(\varrho_1^2 + \varrho_2^2)(\beta_1(\lambda) + \sigma^2\beta_3(\lambda) + i\sigma\delta_1(\lambda) + i\sigma^3\delta_3(\lambda)) + \\ &+ \varrho_1\varrho_2^2(\beta_1(\lambda) + i\sigma\beta_2(\lambda) - \sigma^2\beta_3(\lambda) - i\sigma\delta_1(\lambda) + \sigma^2\delta_2(\lambda) + i\sigma^3\delta_3(\lambda)) + \text{h.o.t.} \end{aligned} \tag{5.6}$$

Together with (5.3) and the symmetry property (4.14) this implies

$$h_1(0, 0, \lambda, \sigma) = -\sigma^2 + i\sigma\gamma(\lambda) + \alpha(\lambda) \tag{5.7}$$

and

$$h_2(0, 0, \lambda, \sigma) = -\frac{1}{2}(\beta_1(\lambda) + i\sigma\beta_2(\lambda) - \sigma^2\beta_3(\lambda) - i\sigma\delta_1(\lambda) + \sigma^2\delta_2(\lambda) + i\sigma^3\delta_3(\lambda)). \tag{5.8}$$

It follows in particular from (5.7) that

$$h_1(0, 0, 0, 1) = 0, \quad D_\sigma h_1(0, 0, 0, 1) = -2, \quad D_\lambda h_1(0, 0, 0, 1) = \alpha'(0) + i\gamma'(0). \tag{5.9}$$

In the time-reversible case the additional conditions (1.3) on f and g imply that

$$\beta_2(\lambda) = 0, \ \gamma(\lambda) = 0, \ \delta_1(\lambda) = 0, \ \delta_3(\lambda) = 0, \ \forall \lambda. \tag{5.10}$$

Then the right hand sides of (5.6), (5.7) and (5.8) become real-valued, in accordance with the fact that h_1 and h_2 are real-valued in that case.

It remains to discuss the solution set of the bifurcation equation (2.11), with $F = (F_1, F_2)$ given by (4.13) and (4.15). We will do this in the last two sections of this paper.

6. HOPF BIFURCATION WITH $O(2)$-SYMMETRY

First we consider the case of hypotheses (H1); remark that the transversality condition (1.3) means that $\gamma'(0) \neq 0$ in (5.9). Because of the factorizations (4.13) and (4.15) of F_1 and F_2 we can, next to the trivial solution $z_1 = z_2 = 0$, consider three types of solutions.

(a) Solutions with $\mid z_1 \mid = \varrho_1 \neq 0$ and $z_2 = 0$. For such solutions (2.11) reduces to a single complex equation

$$h_1(\varrho_1^2, 0, \lambda, \sigma) + \varrho_1^2 h_2(\varrho_1^2, 0, \lambda, \sigma) = 0. \tag{6.1}$$

Because of (5.9) we can split (6.1) in its real and imaginary parts and apply the implicit function theorem to solve for $\lambda = \lambda_1^*(\varrho_1^2)$ and $\sigma = \sigma_1^*(\varrho_1^2)$, with $\lambda_1^*(0) = 0$ and $\sigma_1^*(0) = 1$. This gives us the branch of clockwise circular solutions discussed in part I and in section 3; remark that multiplication of (6.1) by z_1 gives (3.6).

(b) Solutions with $z_1 = 0$ and $|z_2| = \varrho_2 \neq 0$. Now the bifurcation equation reduces to

$$h_1(0, \varrho_2^2, \lambda, \sigma) + \varrho_2^2 h_2(0, \varrho_2^2, \lambda, \sigma) = 0. \tag{6.2}$$

Because of (4.14) this is the same equation as (6.1); therefore it has the solutions $\lambda = \lambda_1^*(\varrho_2^2)$, $\sigma = \sigma_1^*(\varrho_2^2)$. This gives a branch of anti-clockwise circular solutions, which can also be obtained by reflecting the solution branch found under (a).

(c) Solutions with $|z_1| = \varrho_1 \neq 0$ and $|z_2| = \varrho_2 \neq 0$. For such solutions (2.11) reduces to the system

$$\begin{aligned} h_1(\varrho_1^2, \varrho_2^2, \lambda, \sigma) &= 0 \\ (\varrho_1^2 - \varrho_2^2)\, h_2(\varrho_1^2, \varrho_2^2, \lambda, \sigma) &= 0. \end{aligned} \tag{6.3}$$

Now suppose that $h_2(0, 0, 0, 1) \neq 0$, or equivalently

$$(\beta_1(0) + \delta_2(0) - \beta_3(0)) + i(-\delta_1(0) + \beta_2(0) + \delta_3(0)) \neq 0. \tag{6.4}$$

Then (6.3) has, in a neighborhood of (0, 0, 0, 1), only solutions with $\varrho_1^2 = \varrho_2^2 =: \varrho^2$, and (6.3) reduces to

$$h_1(\varrho^2, \varrho^2, \lambda, \sigma) = 0. \tag{6.5}$$

Again by (5.9) we can solve (6.5) for $\lambda = \lambda_2^*(\varrho^2)$ and $\sigma = \sigma_2^*(\varrho^2)$, with $\lambda_2^*(0) = 0$ and $\sigma_2^*(0) = 1$. Since for these solutions we have $|z_1| = |z_2|$, we have found here the branch of collinear solutions discussed in part I and in section 3.

We conclude with a remark about the condition (6.4); since we have already imposed the condition $\gamma(0) = g(0, 0, 0) = 0$, it follows that generically the condition (6.4) will be satisfied. We conclude that for generic $O(2)$-symmetric systems of the form (1) one has only bifurcation of collinear and circular solutions. This property can be extended to general $O(2)$-symmetric systems (see [2]).

7. TIME-REVERSIBLE SYSTEMS WITH $O(2)$-SYMMETRY

In this last section we consider the time-reversible case, i.e. we assume ($H2$). Then the functions h_1 and h_2 in (4.13) and (4.15) are real-valued. For solutions with $z_1 = 0$ or $z_2 = 0$ we obtain again the equations (6.1) and (6.2), which are now real equations. Since $D_\sigma h_1(0, 0, 0, 1) = -2 \neq 0$ we can solve for $\sigma = \bar{\sigma}(\varrho^2, \lambda)$. This gives us two branches of circular solutions, connected by reflections.

For solutions with $|z_1| = \varrho_1 \neq 0$ and $|z_2| = \varrho_2 \neq 0$ we have to consider the system (6.3) which consists now of two real equations. Let

$$\psi(\lambda) = \beta_1(\lambda) + \delta_2(\lambda) - \beta_3(\lambda). \tag{7.1}$$

If $\psi(0) \neq 0$ then $h_2(\varrho_1^2, \varrho_2^2, \lambda, \sigma) \neq 0$ for all $(z_1, z_1, \lambda, \sigma)$ near $(0, 0, 0, 1)$, and (6.3) has only solutions with $\varrho_1^2 = \varrho_2^2 =: \varrho^2$; for such solutions we have to solve the real equation (6.5). By the implicit function theorem it has a solution of the form $\sigma = \bar{\sigma}(\varrho^2, \lambda)$, with $\bar{\sigma}(0, 0) = 1$, and corresponding to a branch of collinear solutions.

We conclude that if $\psi(0) \neq 0$ then we have for all sufficiently small λ two one-parameter families of circles of circular solutions and a one-parameter family of tori of collinear solutions (see part I, the parameter is in each case the "amplitude" ϱ). There is no qualitative change in the picture when λ passes through zero.

However, in time-reversible systems with $O(2)$-symmetry of the form (1) the condition $\gamma(0) = 0$ is automatically satisfied, since $\gamma(\lambda) = 0$ for all λ; therefore in generic such systems there may be parameter values λ_0 at which $\psi(\lambda_0) = 0$. Assuming that $\lambda = 0$ is such a critical value (i.e. $\psi(0) = 0$) we see that then (6.3) may have solutions for which $\varrho_1^2 \neq \varrho_2^2$. Indeed, for such solutions (6.3) reduces to the system

$$h_1(\varrho_1^2, \varrho_2^2, \lambda, \sigma) = 0, \qquad h_2(\varrho_1^2, \varrho_2^2, \lambda, \sigma) = 0. \tag{7.2}$$

This system has the solution $(\varrho_1, \varrho_2, \lambda, \sigma) = (0, 0, 0, 1)$, and if we assume that

$$\frac{\partial(h_1, h_2)}{\partial(\lambda, \sigma)}(0, 0, 0, 1) = \alpha'(0)(\beta_3(0) - \delta_2(0)) - \psi'(0) \neq 0 \tag{7.3}$$

then it may be solved for $\sigma = \hat{\sigma}(\varrho_1^2, \varrho_2^2)$ and $\lambda = \hat{\lambda}(\varrho_1^2, \varrho_2^2)$, with $\hat{\sigma}(0, 0) = 1$, $\hat{\lambda}(0, 0) = 0$, $\hat{\sigma}(\xi_2, \xi_1) = \hat{\sigma}(\xi_1, \xi_2)$ and $\hat{\lambda}(\xi_2, \xi_1) = \hat{\lambda}(\xi_1, \xi_2)$. That means that for each $(z_1, z_2) \in C^2$ we can find $(\lambda, \sigma) = (\hat{\lambda}, \hat{\sigma})$ such that (1.6) has a solution of the form $x = u + O(\| u \|^3)$, with $u = \chi(z_1, z_2)$. Such solutions have no other symmetries than the ones mentioned in section 3 : $S^+(\pi)x = -x$ and $R_-(\Theta)S_-(\varphi)x = x$ for some $\Theta, \varphi \in \boldsymbol{R}$. In a (ϱ_1, ϱ_2)-plane we can depict the situation as follows. To fix the ideas we will assume that $D_{\xi_1}\hat{\lambda}(0, 0) > 0$.

For each λ there are four solution lines in the (ϱ_1, ϱ_2)-plane: the coordinate axes $\varrho_1 = 0$ and $\varrho_2 = 0$, corresponding to circular solutions, and the diagonals $\varrho_1 = \pm\varrho_2$, corresponding to collinear solutions. These are the only solutions near the origin if $\lambda \leqq 0$. For fixed $\lambda > 0$ there is also a small closed curve of solutions, given by the equation $\hat{\lambda}(\varrho_1^2, \varrho_2^2) = \lambda$. This closed curve encircles the origin and connects the branches of circular and collinear solutions. Each point on the curve represents a torus of periodic solutions, obtained by application of rotations and phase shifts; the curve shrinks down to the origin as λ decreases to zero. Observe finally that this picture has a D_4-symmetry, in accordance with the remark made at the end of section 3.

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