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A REMARK ON COMPACT SYMPLECTIC MANIFOLDS NOT ADMITTING COMPLEX STRUCTURES

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Abstract. We study the behaviour of the *-Ricci tensor ϱ^* and the Ricci tensor ϱ of some compact symplectic manifolds and prove that, in general, ϱ^* is neither symmetric nor skewsymmetric.

Key words. Symplectic manifolds, complex manifolds.

MS Classification. 53 C 15, 53 C 55.

1. INTRODUCTION

Many examples of compact symplectic manifolds with no Kähler structure are now known (see [12], [13], [3], [4], [5], [8], [16]). In the non-compact case, it is well known that the tangent bundle of a non-flat Riemannian manifold admits a non-Kähler almost Kähler structure (hence, a symplectic structure) (see [7], [14]).

Recently, M. Fernández, M. Gotay and A. Gray ([8]) gave the first examples of compact 4-dimensional manifolds that have symplectic structures but no complex structures (see [15], [18], [2] for another examples of almost complex manifolds with no complex structures). These manifolds E^4 are circle bundles over circle bundles over a 2-dimensional torus.

As it is well known, the *-Ricci tensor ϱ^* and the Ricci tensor ϱ of a Kähler manifold coincide. Then ϱ^* is symmetric for a Kähler manifold. The same is true for the Kodaira and Thurston manifolds (see [1]).

However, ϱ^* is neither symmetric nor skewsymmetric for the tangent bundle of a Riemannian manifold (for a proof, see [1]; this fact can also be deduced from [9]). In this paper, we study the behaviour of ϱ^* on the compact symplectic manifolds E^4 and prove that ϱ^* is neither symmetric nor skewsymmetric.

2. THE MANIFOLDS E^4 ([8])

Let us recall the following theorem due to Kobayashi:

Theorem ([10], [11]). *Let M be a manifold. Then there is a one to one correspondence between equivalence classes of circle bundles over M and the integral cohomology group $H^2(M, Z)$. Furthermore, given an integral 2-form Ω on M there is a circle bundle $\pi\colon E \to M$ with connection form ω such that Ω is the curvature of ω (that is $\pi^*\Omega = \mathrm{d}\omega$).*

Now, let α and β be parallel (hence harmonic) 1-forms on T^2 such that $[\alpha]$ and $[\beta]$ are generators of $H^1(T^2, Z) = Z \oplus Z$. Then for any integer n there is a circle bundle $\pi\colon E_n^3 \to T^2$ with connection form γ such that $\mathrm{d}\gamma = n\alpha \wedge \beta$. (Let us agree to use the same notation for differential forms on T^2 and their pullbacks to E_n^3. In fact we shall presently consider another bundle $E^4 \to E_n^3$ then we consider forms on T^2 and E_n^3 to be forms on E^4 as well). When $n = 0$ the space E_n^3 is the 3-torus; when $n \neq 0$, E_n^3 is a compact quotient $\Gamma_n \backslash H_n$, where H_n is the Lie group of matrices of the form

$$\begin{pmatrix} 1 & a & -c/n \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

and Γ_n is the subgroup of H_n consisting of those elements for which a, b and c are integers (see[8]). In the following we only consider the case $n \neq 0$.

Now, Kobayashi's theorem says that the circle bundles over E_n^3 are classified by $H^2(E_n^3, Z)$. But the Gysin sequence can be used to compute the integral cohomology groups $H^i(E_n^3, Z)$ of $E_n^3 (n \neq 0)$:

$$H^0(E_n^3, Z) = Z, \qquad H^1(E_n^3, Z) = Z \oplus Z,$$

$$H^2(E_n^3, Z) = Z \oplus Z \oplus Z_{|n|}, \qquad H^3(E_n^3, Z) = Z.$$

Hence we can use Kobayashi's theorem and conclude that for every pair of in tegers p and q there is a circle bundle $E^4 \to E_n^3$ with connection form η such that $\mathrm{d}\eta = p\alpha \wedge \gamma + q\beta \wedge \gamma$. (We note that $p\alpha \wedge \gamma + q\beta \wedge \gamma$ is not exact on E_n^3 but onE^4 we have $\mathrm{d}\eta = p\alpha \wedge \eta + q\beta \wedge \gamma$).

As consequence, the minimal model of E^4 is $M(E^4) = \{\alpha, \beta, \gamma, \eta / \mathrm{d}\alpha = \mathrm{d}\beta = 0, \mathrm{d}\gamma = n\alpha \wedge \beta, \mathrm{d}\eta = p\alpha \wedge \gamma + q\beta \wedge \gamma\}$ for $n \neq 0$ (see [8]).

Since $M(E^4)$ is not formal if p or q is different from zero, we have, from the Main theorem of [6] that E^4 can have no Kähler structure. Furthermore, if $p \neq 0$ or $q \neq 0$, the first Betti number of E^4 is even, say $b_1(E^4) = 2$. Hence, froma result of Kodaira (see [12], theorem 25), we deduce that E^4 can have no complex structure.

Nevertheless E^4 has many symplectic forms. For example,

$$F = (a\alpha + b\beta) \wedge \gamma + (e\alpha + f\beta) \wedge \eta,$$

where a, b, e, f are constants such that $fp - eq = 0$ and $af - be \neq 0$, is a symplectic form on E^4.

Furthermore F is the Kähler form of the almost Hermitian structure $(\langle , \rangle, J)$ over E^4 where $\langle , \rangle$ is the Riemannian metric given by

$$\langle , \rangle = \alpha^2 + \beta^2 + \gamma^2 + \eta^2$$

and J is the almost complex structure on E^4 given by

$$JX = aZ + eT, \qquad JY = bZ + fT,$$
$$JZ = -aX - bY, \qquad JT = -eX - fY,$$

$\{X, Y, Z, T\}$ being the orthonormal basis of vector fields on E^4 dual to $\{\alpha, \beta, \gamma, \eta\}$ and the constants a, b, e, f satisfying the additional relations

$$a^2 + b^2 = b^2 + f^2 = e^2 + f^2 = a^2 + e^2 = 1, \qquad ab + ef = ae + bf = 0.$$

Since F is symplectic then $(E^4, \langle , \rangle, J)$ is an almost Kähler manifold.

3. THE *-RICCI TENSOR OF $(E^4, \langle , \rangle J)$

In the sequel, we denote by ∇ the Levi-Civita connection on $(E^4, \langle , \rangle, J)$. A simple computation shows that ∇ is determined by the following relations:

$$\nabla_X Y = -\nabla_Y X = -\frac{n}{2} Z,$$

$$\nabla_X Z = \frac{n}{2} Y - \frac{p}{2} T, \qquad \nabla_Z X = \frac{n}{2} Y + \frac{p}{2} T,$$

$$\nabla_X T = \nabla_T X = \frac{p}{2} Z,$$

$$\nabla_Y Z = -\frac{n}{2} X - \frac{q}{2} T, \qquad \nabla_Z Y = -\frac{n}{2} X + \frac{q}{2} T,$$

$$\nabla_Y T = \nabla_T Y = \frac{q}{2} Z,$$

$$\nabla_Z T = \nabla_T Z = -\frac{p}{2} X - \frac{q}{2} Y,$$

being zero the other covariant derivatives.

Hence the curvature tensor R of ∇ is given by

$$R(X, Y, X, Y) = \frac{3}{4} n^2,$$

$$R(X, Y, X, T) = \quad R(Y, Z, Z, T) = \frac{1}{4} np,$$

$$R(X, Y, Y, T) = -R(X, Z, Z, T) = \frac{1}{4} nq,$$

$$R(X, Z, X, Z) = -\frac{1}{4} n^2 + \frac{3}{4} p^2,$$

$$R(X, Z, Y, Z) = -3R(X, T, Y, T) = \frac{3}{4} pq,$$

$$R(X, T, X, T) = -\frac{1}{4} p^2,$$

$$R(Y, T, Y, T) = -\frac{1}{4} q^2,$$

$$R(Y, Z, Y, Z) = -\frac{1}{4} n^2 + \frac{3}{4} q^2,$$

$$R(Z, T, Z, T) = -\frac{1}{4} (p^2 + q^2).$$

Next, we compute the *-Ricci tensor of $(E^4, \langle , \rangle, J)$. Let us recall that the *-Ricci tensor ϱ^* of the almost Hermitian manifold $(E^4, \langle , \rangle, J)$ is given by

$$\varrho^*(U, V) = R(U, X, JV, JX) + R(U, Y, JV, JY) + R(U, Z, JV, JZ) + \\ + R(U, T, JV, JT).$$

A long but straightforward computation shows that ϱ^* is given by

$$\varrho^*(X, X) = -\frac{1}{4} a^2 n^2 + \frac{1}{4} (3a^2 - e^2) p^2 - efpq,$$

$$\varrho^*(Y, Y) = -\frac{1}{4} b^2 n^2 + \frac{1}{4} (3b^2 - f^2) q^2 - efpq,$$

$$\varrho^*(Z, Z) = -\frac{1}{4} n^2 + \frac{3}{4} a^2 p^2 + \frac{3}{4} b^2 q^2 + \frac{3}{2} abpq,$$

$$\varrho^*(T, T) = -\frac{1}{4} (ep + fq)^2,$$

$$\varrho^*(X, Y) = -\frac{1}{4} abn^2 - efp^2 + \frac{1}{4} (3b^2 - f^2) pq,$$

$$\varrho^*(X, Z) = -\varrho^*(Z, X) = -\frac{1}{4} benp - \frac{1}{4} bfnq,$$

$$\varrho^*(X, T) = -\varrho^*(T, X) = -\frac{1}{4} efnp - \frac{1}{4} f^2nq,$$

$$\varrho^*(Y, X) = -\frac{1}{4} abn^2 - efq^2 + \frac{1}{4}(3a^2 - e^2)\, pq,$$

$$\varrho^*(Y, Z) = -\varrho^*(Z, Y) = \frac{1}{4} aenp + \frac{1}{4} afnq,$$

$$\varrho^*(Y, T) = -\varrho^*(T, Y) = \frac{1}{4} e^2np + \frac{1}{4} efnq,$$

$$\varrho^*(Z, T) = -3\varrho^*(T, Z) = \frac{3}{4} aep^2 + \frac{3}{4} bfq^2 + \frac{3}{4}(af + be)\, pq.$$

These identities show that, in general, ϱ^* is neither symmetric nor skewsymmetric. In fact, if we put $a = f = q = 0$, $b^2 = e^2 = 1$, $p \neq 0$, $n \neq 0$, then we have

$$\varrho^*(X, X) = -\frac{1}{4} p^2 \neq 0$$

and

$$\varrho^*(Y, T) = -\varrho^*(T, Y) = \frac{1}{4} np \neq 0.$$

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