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ON TORSION CLASSES GENERATED BY RADICAL CLASSES OF LATTICE ORDERED GROUPS

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Dedicated to Academician Otakar Borůvka on the occasion of his 90th birthday

Abstract. In this paper there will be investigated a question proposed by M. Darnel [2] concerning the multiplication of torsion classes generated by radical classes of lattice ordered groups.

Key words. Lattice ordered group, torsion class, radical class.

MS Classification. 06 F 15.

1. PRELIMINARIES

For the basic terminology and notations on lattice ordered groups cf. Conrad [1] and Fuchs [3]. We recall the following notions.

A torsion class (cf. Martinez [7]) is a collection of lattice ordered groups closed with respect to convex l-subgroups, joins of convex l-subgroups, and homomorphic images. A radical class (cf. [4]) is a collection of lattice ordered groups closed with respect to convex l-subgroups, joins of convex l-subgroups, and isomorphic images.

Let $\mathscr{G}$ be the class of all lattice ordered groups and let R be a radical class. For every $G \in \mathscr{G}$ we denote by $R(G)$ the join of all convex l-subgroups of G that belong to R. Then $R(G) \in R$; moreover, $R(G)$ is an l-ideal in G (cf. [4]).

Let $\mathscr{R}$ be the collection of all radical classes. For $R, S \in \mathscr{R}$ we define $R \,.\, S$ to be the class of all lattice ordered groups H such that $G/R(G)$ belongs to S. Then $R \,.\, S$ is a radical class [4]; if both R and S are torsion classes, then $R \,.\, S$ is a torsion class as well [7].

We denote by $\mathscr{T}$ the collection of all torsion classes. Both $\mathscr{R}$ and $\mathscr{T}$ are partially ordered by inclusion. Then $\mathscr{R}$ is a lattice which is complete and Brouwerian [4]; $\mathscr{T}$ is a closed sublattice of $\mathscr{R}$ [7].

For each radical class R let R^h be the meet of all torsion classes T such that $R \subseteq T$. The torsion class R^h is said to be generated by the radical class R. The mapping $R \rightarrow R^h$ is a closure operator on the lattice $\mathscr{R}$. This closure operator was thoroughly studied in [2]. From the results of [2] we quote the following one:

1.1. Proposition. ([2], Proposition 5.7.) *For any two radical classes R and S we have $(R^h . S^h)^h = R^h . S^h$ and $(R . S)^h \subseteq R^h . S^h$.*

Next, the following open question is proposed in [2]:

It is not known if

$$R^h . S^h \subseteq (R . S)^h. \tag{1}$$

It will be shown below that the relation (1) does not hold in general. Moreover, it will be proved that the collection $\mathscr{R}_1$ of all radical classes R having the property that the relation

$$R^h . R^h \subseteq (R . R)^h \tag{2}$$

fails to hold, is nonempty.

Let X be a subcollection of $\mathscr{G}$. We denote by X_r the meet of all radical classes R_i such that $X \subseteq R_i$; then X_r is said to be the radical class generated by X.

For each $G \in \mathscr{G}$ let $c(G)$ be the system of all convex l-subgroups of G.

1.2. Proposition. (Cf. [5], Theorem 3.4.) Let $0 \neq X \subseteq \mathscr{G}$. *Assume that X is closed with respect to isomorphisms, $\{0\} \in X$ and that each lattice ordered group belonging to X is linearly ordered. Let $G \in \mathscr{G}$. Then the following conditions are equivalent:*

(i) $G \in X_r$.

(ii) *There are systems $\{A_i\}_{i \in I} \subseteq c(G)$ and $\{A_{ij}\}_{j \in J(i)} \subseteq c(A_i) \bigcap X$ for each $i \in I$, such that $A_i = \bigcup_{j \in J(i)} A_{ij}$* is valid for each $i \in I$, and $G = \sum_{i \in I} A_i$.

1.3. Proposition. ([2], Proposition 5.5.) *For any radical class R and lattice ordered group G, $R^h(G) = \{C \in c(G)$: there exists $H \in R$ and an l-ideal L of H such that $C \simeq H/L\}$.*

For each subclass X of $\mathscr{G}$ we denote by $\operatorname{Hom} X$ the class of all homomorphic images of elements of X.

1.4. Lemma. *Let X be as in Propos. 1.2. Let Y be the class of all linearly ordered groups A_i having the property that there exist linearly ordered groups A_{ij} $(j \in J(i))$ belonging to $c(A_i) \bigcap X$ such that $A_i = \bigcup_{j \in J(i)} A_i$. Then $(X_r)^h =$* $(\operatorname{Hom} Y)_r$.

Proof. This follows from Proposition 1.2, Proposition 1.3 and from [6], Lemma 3.2.

2. THE RADICAL CLASS $R(\alpha)$

The additive group of all reals (all rational numbers) with the natural linear order will be denoted by R_0 (or by R_0', respectively).

For each $i \in R_0'$ let $A_i = R_0'$ and let A^0 be the lexicographic product

$$A^0 = \Gamma A_i \qquad (i \in R_0'),$$

(cf. [3]). Let A be the subgroup of A^0 consisting of all elements of A^0 with finite support.

Let α be a cardinal, $\alpha \geqq \aleph_0$. Let I_α be the first ordinal with card $I_\alpha = \alpha$ and let J_α be a linearly ordered set dual to I_α. For each $j \in J_\alpha$ let $B_j = R_0$. Put

$$B^0 = \Gamma B_j \qquad (j \in J_\alpha)$$

and let B be the subgroup of B^0 consisting of all elements of B^0 with finite support Put

$$G = B \circ A,$$

where $\circ$ denotes the operation of lexicographic product. Let X be the class of all linearly ordered groups G' such that either $G' = \{0\}$ or G' is isomorphic to G. Put $R = X_r$.

From the construction of the linearly ordered group G we obtain immediately:

2.1. Lemma. *Let Y be as in Lemma 1.4. Then $Y = X$.*

Lemma 2.1 and Lemma 1.4 yield:

2.2. Lemma. $R^h = (\operatorname{Hom} X)_r$.

2.3. Lemma. *Let $\{0\} \neq G' \in \mathscr{G}$. Then G' belongs to* Hom X *if and only if some of the following conditions is fulfilled:*

(i) $G' \simeq G$.

(ii) *There exists a dual ideal J_1 of the linearly ordered set R_0' such that $G' \simeq B \circ A'$, where $A' = \Gamma A_i$ $(i \in J_1)$.*

(iii) *There exists a subset J_2 of J_α such that either $J_2 = \emptyset$ or J_2 is an ideal of the linearly ordered set J_α such that $G' \simeq \Gamma B_j$ $(j \in J_\alpha \setminus J_2)$.*

Proof. This is an obvious consequence of the structure of the linearly ordered group G.

Let Y' be defined analogously as Y (in Lemma 1.4) with the distinction that instead of X we take now the class Hom X into account. Then from 2.3 we infer:

2.4. Lemma. $Y' = \operatorname{Hom} X$.

Since each element of Hom X is a linearly ordered group, from 2.2, 2.4 and 1.2 we obtain:

2.5. Lemma. *Let $H \in \mathscr{G}$. Then the following conditions are equivalent:*
(i) *H belongs to R^h.*
(ii) *H is a direct sum of linearly ordered groups belonging to* Hom X.

2.6. Proposition. *Let H_1 be a linearly ordered group having a strong unit. Then the following conditions are equivalent:*
(i) *H_1 is an l-subgroup of some element of R^h.*
(ii) *H_1 belongs to* Hom X.

Proof. This is a consequence of 2.5 and [6], Lemma 3.6.

In view of 2.3 we have $B \in R^h$, whence

$$B \circ B \in R^h \,.\, R^h. \tag{3}$$

The linearly ordered group G and the radical class R depend from the cardinal α; when we want to emphasize this fact then we write $G = G(\alpha)$ and $R = R(\alpha)$.

3. THE RADICAL CLASS $(R \,.\, R)^h$

We apply the same denotations as above. Put $H_2 = B \circ B$. We want to verify that H_2 does not belong to $(R \,.\, R)^h$.

By way of contradiction, suppose that $H_2 \in (R \,.\, R)^h$. Let A_i and A_{ij} be as in 1.2 with the distinction that we have now H_2 instead of G and $R \,.\, R$ instead of X. Without loss of generality we may suppose that $A_i \neq \{0\}$ for each $i \in I$. Since H_2 is linearly ordered, the set I must be a one-element set, $I = \{i\}$ and $H_2 = A_i$. Next, H_2 cannot be represented as a join of proper convex l-subgroups of H_2; thus $H_2 = A_{ij}$ for some $j \in J(i)$. Hence $H_2 \in R \,.\, R$. Therefore

$$H_2/R(H_2) \in R. \tag{4}$$

Let H_3 be a convex l-subgroup of H_2, $H_3 \neq \{0\}$. Then H_3 cannot be represented as a join of its proper convex l-subgroups, and clearly there is no convex subgroup of G isomorphic to H_3. Therefore $R(H_2) = \{0\}$ and hence in view of (4), H_2 belongs to R. By analogous argument as above (using 1.2) we would obtain that H_2 is isomorphic to a convex l-subgroup of G, which is a contradiction. Thus we have

3.1. Lemma. *H_2 does not belong to $(R \,.\, R)^h$.*

In view of (3) and 3.1 we obtain:

3.2. Corollary. *$R^h \,.\, R^h$ fails to be a subclass of $(R \,.\, R)^h$.*

3.3. Lemma. *Let α and β be cardinals, $\aleph_0 \leqq \alpha < \beta$. Then $G(\beta)$ does not belong to $R(\alpha)$.*

Proof. This is an easy consequence of [2], Lemma 5.4.

Let $\mathscr{R}_1$ be as in Section 1. From 3.2 and 3.3 we infer:

3.4. Theorem. *The mapping $\alpha \to R(\alpha)$ is an injective mapping of the class of all infinite cardinals into the collection $\mathscr{R}_1$.*

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