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SOME REMARKS ON THE Ω-STABILITY FOR FAMILIES OF POLYNOMIALS

Jean Mawhin

Dedicated to the memory of Professor Otakar Borůvka

Abstract. Using Brouwer degree, we prove a more general version of the zero exclusion principle for families of polynomials and apply it to obtain very simple proofs of extensions of recent results on the Routh-Hurwitz and Schur-Cohn stability of families of polynomials.

1. Introduction

It is well know that the problem of the stability of a linear difference or differential system with constant coefficients reduces to the question of locating the roots of its characteristic polynomial in a suitable region of the complex plane. The Routh-Hurwitz and Schur-Cohn tests, which respectively correspond to the open left half-space and the open unit ball, are well known in this respect [2, 3].

In a recent work [9], Zahreddine has considered the following problem: given a path-wise connected region Ω in the complex plane and a set S of polynomials of the same degree, find conditions under which all polynomials of S have their zeros inside Ω. In the special case where S is made of all the convex combinations of two polynomials which are stable in the Routh-Hurwitz or the Schur-Cohn sense, Zahreddine has found necessary and sufficient conditions for this set S to have the same stability. His approach is algebraic and based upon some properties of resultants and standard Routh-Hurwitz or Schur-Cohn stability conditions for a complex polynomial [2, 3].

The aim of this note is to show that a very simple proof of more general version of this result can be obtained by using the elementary properties of the Brouwer degree [4]. We first use this technique to prove a more general version of the standard *zero exclusion principle* [1, 6] and then apply it to the proof of the Zahreddine's results.

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2. A zero exclusion principle

Let $\Omega \subset \mathbb{C}$ be open and $p : \mathbb{C} \to \mathbb{C}$ be a polynomial.

Definition 2.1. *We say that p is Ω-stable if $p(z) \neq 0$ for $z \notin \Omega$.*

When $\Omega = \{z \in \mathbb{C} : \Re z < 0\}$, the Ω-stability will be called the *Routh-Hurwitz stability;* when Ω is the unit open ball $B(1)$ in $\mathbb{C}$, the Ω-stability will be called the *Schur-Cohn stability.*

Let $\Lambda \subset \mathbb{R}^m$ and $\{p(\cdot,\lambda)\}_{\lambda\in}$ be a continuous family of polynomials on $\mathbb{C}$. This means that $p : \mathbb{C}\times\Lambda \to \mathbb{C}$ is continuous and, for each $\lambda \in \Lambda$, $p(\cdot,\lambda)$ is a polynomial on $\mathbb{C}$.

Definition 2.2. *We say that $\{p(\cdot,\lambda)\}_{\lambda\in}$ is Ω-stable if, for each $\lambda \in \Lambda$, $p(\cdot,\lambda)$ is Ω-stable, i.e. if $p(z,\lambda) \neq 0$ for $z \notin \Omega$ and $\lambda \in \Lambda$.*

We now state and prove a more general version of the *zero exclusion principle.*

Theorem 2.1. *Let $\Lambda \subset \mathbb{R}^m$ and $\{p(\cdot,\lambda)\}_{\lambda\in}$ be a continuous family of polynomials on $\mathbb{C}$. Assume that the following conditions hold:*

1. *Λ is connected and compact;*
2. *$p(\cdot,\lambda)$ has degree $d \geq 1$ for each $\lambda \in \Lambda$;*
3. *$p(\cdot,\lambda\ )$ is Ω-stable for some $\lambda\ \in \Lambda$;*
4. *$p(z,\lambda) \neq 0$ for each $\lambda \in \partial\Omega$ and each $\lambda \in \Lambda$.*

Then $\{p(\cdot,\lambda)\}_{\lambda\in}$ is Ω-stable.

Proof. By assumption 2, we can assume, without loss of generality, that

$$p(z,\lambda) = z^d + \sum_{k}^{d} a_k(\lambda) z^{d-k},$$

where the $a_k : \Lambda \to \mathbb{C}$ are continuous $(1 \leq k \leq d)$. Hence, if $p(z,\lambda) = 0$ for some $z \in \mathbb{C}$ and $\lambda \in \Lambda$, we have

$$|z|^d \leq \sum_{k}^{d} \alpha_k |z|^{d-k},$$

where $\alpha_k = \max_{\lambda\in}\ |a_k(\lambda)|$, $(1 \leq k \leq d)$. Consequently, $|z| \leq R$, where R is the positive root of the equation

$$R^d - \sum_{k}^{d} \alpha_k R^{d-k} = 0.$$

Let $\Omega_R = \Omega \cap B(R+1)$, with $B(R+1) \subset \mathbb{C}$ the open ball of centre 0 and radius $R+1$. By the above result and assumption 4, we have

$$p(z,\lambda) \neq 0 \text{ for each } (z,\lambda) \in \partial\Omega_R \times \Lambda.$$

Hence the Brouwer degree $\deg[p(\cdot,\lambda),\Omega_R,0]$ [4] is well defined for each $\lambda \in \Lambda$, and its value is independent of λ. On the other hand, it is well known (see [4]) that $\deg[p(\cdot,\lambda),\Omega_R,0]$ is equal to the number of zeros of $p(\cdot,\lambda)$ in Ω_R, counted with their multiplicities. By assumption 3 and the definition of Ω-stability, every possible zero of $p(\cdot,\lambda\ )$ lies in Ω, and hence in Ω_R. Consequently, $\deg[p(\cdot,\lambda\ ),\Omega_R,0]=d$, and, by the homotopy invariance of Brouwer degree, $\deg[p(\cdot,\lambda),\Omega_R,0]=d$ for each $\lambda\in\Lambda$. This implies that all the zeros of $p(\cdot,\lambda)$ are in Ω_R, hence in Ω, and each $p(\cdot,\lambda)$ is Ω-stable.

Remark 2.1. It is easy to get rid of the assumption of compactness for Λ in Theorem 2.1.

3. The first Schur transform of a polynomial and its properties

To a polynomial $p(z)$ on $\mathbb{C}$, one can associate, with Schur [8], the polynomial on $\mathbb{C}$

$$p^*(z):=\overline{p(-\overline{z})},$$

that Zahreddine [9] calls the *paraconjugate* of p and that we will call the *first Schur transform* of p. Notice that

$$p^{**}(z):=(p^*)^*(z)=\overline{p^*(-\overline{z})}=\overline{\overline{p(z)}}=p(z),$$

and that, for $c\in\mathbb{C}$ and another polynomial q over $\mathbb{C}$, one has

$$(cp)^*(z)=\overline{c}p^*(z),\quad (p+q)^*(z)=p^*(z)+q^*(z).$$

Define, with Zahreddine [9],

$$N(z)=\frac{1}{2}\left[p(z)+(-1)^dp^*(z)\right],\quad D(z)=\frac{1}{2}\left[p(z)-(-1)^dp^*(z)\right], \tag{1}$$

so that

$$p(z)=N(z)+D(z), \tag{2}$$

$$p^*(z)=(-1)^d[N(z)-D(z)], \tag{3}$$

$$N^*(z)=\frac{1}{2}\left[p^*(z)+(-1)^dp(z)\right]=(-1)^dN(z),$$

$$D^*(z)=\frac{1}{2}\left[p^*(z)-(-1)^dp(z)\right]=(-1)^d\quad D(z).$$

Lemma 3.1. *z is a common zero to N and D if and only if z and $-\overline{z}$ are zeros of p.*

Proof. If z is a common zero to N and D, then, by (2) and (3), we have $p(z\ )=0=p(-\overline{z})$. If z and $-\overline{z}$ are zeros of p, then $p(z\ )=0=p^*(z\ )$, and, by (1), we have $N(z\ )=D(z\ )=0$.

Corollary 3.1. *Any zero of p which lies on the imaginary axis is a common zero of N and D.*

Proof. For such a zero z, one has $z=-\overline{z}$.

The resultant [5] of two polynomials p and q over $\mathbb{C}$ will be denoted by $R[p,q]$.

4. The Routh-Hurwitz stability of a family of polynomials

Let $\Lambda \subset \mathbb{R}^m$ be compact and connected and let $\{p(\cdot,\lambda)\}_{\lambda\in}$ be a continuous family of polynomials on $\mathbb{C}$ such that $p(\cdot,\lambda)$ has degree d for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, let

$$p(z,\lambda) = N(z,\lambda) + D(z,\lambda)$$

be the decomposition defined by (2). The following result generalizes, with a simpler proof, Theorem 3.1 of [9].

Theorem 4.1. *The family* $\{p(\cdot,\lambda)\}_{\lambda\in}$ *is Routh-Hurwitz-stable if and only if* $p(\cdot,\lambda\ )$ *is Routh-Hurwitz-stable for some* $\lambda\ \in \Lambda$ *and* $R[N(\cdot,\lambda), D(\cdot,\lambda)] \neq 0$ *for each* $\lambda \in \Lambda$.

Proof. *Necessity.* If $R[N(\cdot,\lambda\ ), D(\cdot,\lambda\ )] = 0$ for some $\lambda\ \in \Lambda$, then, by a classical result [5], $N(\cdot,\lambda\ )$ and $D(\cdot,\lambda\ )$ have a common zero $z\ $, so that, by Lemma 3.1,

$$p(z\ ,\lambda\ ) = 0 = p(-\overline{z\ },\lambda\ ).$$

Now $\Re z\ \geq 0$ if and only if $\Re(-\overline{z\ }) \leq 0$, and hence $p(\cdot,\lambda\ )$ is not Routh-Hurwitz-stable.

Sufficiency. Assume now that $R[N(\cdot,\lambda), D(\cdot,\lambda)] \neq 0$ for each $\lambda \in \Lambda$. Then, for each $\lambda \in \Lambda$, $N(\cdot,\lambda)$ and $D(\cdot,\lambda)$ have no common zeros, which, by Corollary 3.1, implies that $p(\cdot,\lambda)$ has no zero on the imaginary axis, which is the boundary of $\{z \in \mathbb{C} : \Re z < 0\}$. The conclusion follows then from Theorem 2.1.

We obtain immediately as a special case the Theorem 3.1 of Zahreddine. Let

$$p\ (z) = z^d + \sum_{k}^{d} a_k z^{d-k}, \quad p\ (z) = z^d + \sum_{k}^{d} a_k z^{d-k}$$

be two monic Routh-Hurwitz-stable polynomials. Let

$$p_j(z) = N_j(z) + D_j(z), \quad (j = 0, 1)$$

be their respective decompositions (2) and let

$$p(z,\lambda) = (1-\lambda)p\ + \lambda p\ , \quad (0 \leq \lambda \leq 1),$$

be their convex combinations. It is immediate to check that if, for each $\lambda \in [0,1]$,

$$p(z,\lambda) = N(z,\lambda) + D(z,\lambda)$$

is the decomposition (2) of $p(\cdot,\lambda)$, then

$$N(z,\lambda) = (1-\lambda)N\ (z) + \lambda N\ (z), \quad D(z,\lambda) = (1-\lambda)D\ (z) + \lambda D\ (z), \quad (0 \leq \lambda \leq 1).$$

Corollary 4.1. *Assume that* p *and* q *are Routh-Hurwitz-stable. Then their convex combinations* $(1-\lambda)p + \lambda q$, $(\lambda \in [0,1])$, *are Routh-Hurwitz-stable if and only if* $R[(1-\lambda)N\ + \lambda N\ , (1-\lambda)D\ + \lambda D\ ] \neq 0$ *for each* $\lambda \in]0,1[$.

5. The second Schur transform of a polynomial and its properties

To a monic polynomial

$$p(z) = z^d + \sum_{k}^{d} a_k z^{d-k}$$

on $\mathbb{C}$, one can associate, with Schur [7], the polynomial on $\mathbb{C}$

$$p\ \ (z) := z^d \overline{p\left(\frac{1}{\overline{z}}\right)},$$

that we may call the *second Schur transform* of p. Notice that

$$p\quad (z) := (p\ \ )\ \ (z) = z^d \overline{p\ \ \left(\frac{1}{\overline{z}}\right)} = z^d \frac{1}{\overline{z}^d}\overline{\overline{p(z)}} = p(z),$$

that $p\ \ (0) = 1$ and that, for $c \in \mathbb{C}$ and another monic polynomial q over $\mathbb{C}$, one has

$$(cp)\ \ (z) = \overline{c}p\ \ (z), \quad (p+q)\ \ (z) = p\ \ (z) + q\ \ (z).$$

Define, with Zahreddine [9],

$$H(z) = \frac{1}{2}\left[p(z) + p\ \ (z)\right], \quad K(z) = \frac{1}{2}\left[p(z) - p\ \ (z)\right], \tag{4}$$

so that

$$p(z) = H(z) + K(z), \tag{5}$$

$$p\ \ (z) = H(z) - K(z), \tag{6}$$

$$H\ \ (z) = \frac{1}{2}\left[p\ \ (z) + p(z)\right] = H(z),$$

$$K\ \ (z) = \frac{1}{2}\left[p\ \ (z) - p(z)\right] = -K(z).$$

Lemma 5.1. *If $z\ $ is a common zero to H and K then $z\ \neq 0$ and $z\ $ and $\frac{\;}{\overline{z_0}}$ are zeros of p. If $z\ \neq 0$ and $\frac{\;}{\overline{z_0}}$ are zeros of p, then $z\ $ is a common zero to H and K.*

Proof. If $z\ $ is a common zero to H and K, then, by (5) and (6), we have $0 = p(z\ ) = p\ \ (z\ )$, so that $z\ \neq 0$ and $p\left(\frac{\;}{\overline{z_0}}\right) = 0$. If $z\ \neq 0$ and $\frac{\;}{\overline{z_0}}$ are zeros of p, then $p(z\ ) = 0 = p\ \ (z\ )$, and, by (4), we have $H(z\ ) = K(z\ ) = 0$.

Corollary 5.1. *Any zero $z\ $ of p which lies on the unit circle is a common zero of H and K.*

Proof. For such a zero $z\ $, one has $0 \neq z\ = \frac{\;}{\overline{z_0}}$.

6. The Schur-Cohn-stability of a family of polynomials

Let $\Lambda \subset \mathbb{R}^m$ be compact and connected and let $\{p(\cdot,\lambda)\}_{\lambda\in}$ be a continuous family of polynomials on $\mathbb{C}$ such that $p(\cdot,\lambda)$ has degree d for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, let

$$p(z,\lambda) = H(z,\lambda) + K(z,\lambda)$$

be the decomposition defined by (5). The following result generalizes, with a simpler proof, Theorem 4.1 of [9].

Theorem 6.1. *The family $\{p(\cdot,\lambda)\}_{\lambda\in}$ is Schur-Cohn-stable if and only if $p(\cdot,\lambda\ )$ is Schur-Cohn-stable for some $\lambda\ \in \Lambda$ and $R[H(\cdot,\lambda), K(\cdot,\lambda)] \neq 0$ for each $\lambda \in \Lambda$.*

Proof. *Necessity.* If $R[H(\cdot,\lambda\ ), K(\cdot,\lambda\ )] = 0$ for some $\lambda\ \in \Lambda$, then, by a classical result [5], $H(\cdot,\lambda\ )$ and $K(\cdot,\lambda\ )$ have a common zero $z\ $, so that by Lemma 5.1, $z\ \neq 0$ and

$$p(z\ ,\lambda\ ) = 0 = p\left(\frac{1}{\overline{z}},\lambda\ \right).$$

Now $|z\ | \geq 1$ if and only if $\left|\frac{}{\overline{z_0}}\right| \leq 1$, and hence $p(\cdot,\lambda\ )$ is not Schur-Cohn-stable.

Sufficiency. Assume now that $R[H(\cdot,\lambda), K(\cdot,\lambda)] \neq 0$ for each $\lambda \in \Lambda$. Then, for each $\lambda \in \Lambda$, $K(\cdot,\lambda)$ and $H(\cdot,\lambda)$ have no common zeros, which, by Corollary 5.1, implies that $p(\cdot,\lambda)$ has no zero on the unit circle, which is the boundary of $B(1)$. The conclusion follows then from Theorem 2.1.

We obtain immediately as a special case the Theorem 4.1 of Zahreddine. Let

$$p\ (z) = z^d + \sum_{k}^{d} a_k z^{d-k}, \quad p\ (z) = z^d + \sum_{k}^{d} a_k z^{d-k}$$

be two monic Schur-Cohn-stable polynomials. Let

$$p_j(z) = H_j(z) + K_j(z), \quad (j = 0, 1)$$

be their respective decompositions (5) and let

$$p(z,\lambda) = (1-\lambda)p\ + \lambda p\ , \quad (0 \leq \lambda \leq 1),$$

be their convex combinations. It is immediate to check that if, for each $\lambda \in [0,1]$,

$$p(z,\lambda) = H(z,\lambda) + K(z,\lambda)$$

is the decomposition (5) of $p(\cdot,\lambda)$, then

$$H(z,\lambda) = (1-\lambda)H\ (z) + \lambda H\ (z), \quad K(z,\lambda) = (1-\lambda)K\ (z) + \lambda K\ (z), \quad (0 \leq \lambda \leq 1).$$

Corollary 6.1. *Assume that p and q are Schur-Cohn-stable. Then their convex combinations $(1-\lambda)p + \lambda q$, $(\lambda \in [0,1])$, are Schur-Cohn-stable if and only if $R[(1-\lambda)H\ + \lambda H\ , (1-\lambda)K\ + \lambda K\ ] \neq 0$ for each $\lambda \in]0,1[$.*

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