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ON QUADRATICALLY INTEGRABLE SOLUTIONS OF THE SECOND ORDER LINEAR EQUATION

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Abstract. Integral criteria are established for $\dim V_i(p) = 0$ and $\dim V_i(p) = 1, i \in \{0,1\}$, where $V_i(p)$ is the space of solutions u of the equation

$$u'' + p(t)u = 0$$

satisfying the condition

$$\int^{+\infty} \frac{u^2(s)}{s^i} ds < +\infty .$$

1. Main Results

Consider the equation

$$u'' + p(t)u = 0 \,, \tag{1}$$

where $p : [0,+\infty[\to] -\infty,+\infty[$ is a locally integrable function.

Under a solution of equation (1) is understood a locally absolutely continuous together with its first derivative function $u : [0,+\infty[\to] -\infty,+\infty[$ satisfying (1) almost everywhere.

Denote by $V_i(p)$ $(i = 0,1)$ the set of solutions u of equation (1) satisfying the condition

$$\int^{+\infty} \frac{u^2(s)}{s^i} ds < +\infty \,, \tag{2}$$

and denote the set of solutions u satisfying

$$\lim_{t\to+\infty} u(t) = 0$$

by $Z(p)$.

Below we give some new results on the interlocation as well as on the dimensionality of the above sets.

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Theorem 1. *Let* $i \in \{0,1\}$, $p(t) \le 0$ *in some neighbourhood of* $+\infty$, *and*

$$\int^{+\infty} \frac{ds}{s^i \left[\int_0^s \eta |p(\eta)|\, d\eta\right]^2} < +\infty\,. \tag{3}$$

Then $V_i(p) = Z(p)$ *and* $\dim V_i(p) = 1$.

Suppose that there exists a finite limit

$$\lim_{t\to+\infty} \frac{1}{t} \int_1^t \int_1^s p(\eta)\, d\eta\, ds = c_p$$

and put

$$p_* = \liminf_{t\to+\infty} t \left(c_p - \int_1^t p(s)\, ds \right), \qquad p^* = \limsup_{t\to+\infty} t \left(c_p - \int_1^t p(s)\, ds \right).$$

Theorem 2. *Let* $p_* \le -\frac{3}{4}$ *and*

$$p^* < p_* - 1 + \frac{1}{2}\sqrt{1 - 4p_*}\,. \tag{4}$$

Then $\dim V_0(p) = 1$. *If, moreover,* $p(t) \le 0$ *in some neighbourhood of* $+\infty$, *then* $V_0(p) = Z(p)$.

Theorem 3. *Let* $p_* < 0$ *and*

$$p^* < p_* - \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4p_*}\,.$$

Then $\dim V_1(p) = 1$. *If, moreover,* $p(t) \le 0$ *in some neighbourhood of* $+\infty$, *then* $V_1(p) = Z(p)$.

It is proved in [1] that if $p_* > -\frac{1}{2}$ and $p^* < \frac{1}{4}$, then $V_0(p) = \{0\}$. The following theorem makes this result more complete.

Theorem 4. *Let* $p_* < -\frac{1}{2}$ *and*

$$p^* < -\sqrt{p_*^2 - p_* - \frac{3}{4}}\,. \tag{5}$$

Then $V_0(p) = \{0\}$.

2. Proof

Proof of Theorem 1. From (3) it follows that $\int^{+\infty} s|p(s)|ds = +\infty$. However this condition is necessary and sufficient for $Z(p) \ne \{0\}$ (see [2] and [3]). On the other hand, obviously $\dim Z(p) = 1$ and $V_i(p) \subset Z(p)$. Thus it is sufficient to show that $Z(p) \subset V_i(p)$.

Suppose $u \in Z(p)$. Without loss of generality we can assume that for some $t_0 > 0$,

$$p(t) \le 0 \qquad \text{for } t > t_0\,, \tag{6}$$

$$u(t) > 0, \qquad u'(t) < 0 \qquad \text{for } t > t_0\,. \tag{7}$$

Multiplying both sides of (1) by t and integrating from t_0 to t, we obtain

$$\int_{t_0}^{t} sp(s)u(s)\,ds = tu'(t) - t_0u'(t_0) - u(t) + u(t_0) \qquad \text{for } t > t_0\,.$$

Hence on account of (6) and (7), we easily conclude that for some $r > 0$,

$$u(t)\int_0^t s|p(s)|\,ds < M \qquad \text{for } t > t_0\,.$$

Therefore, by virtue of (3) condition (2) holds. Thus the theorem is proved. □

Proof of Theorem 2. According to (4), we can find $\varepsilon > 0$ such that $p^* < -\frac{1}{2} - 2\varepsilon$ and

$$p^* < p_* - 1 + \frac{1}{2}\sqrt{1 - 4(p_* - \varepsilon)} - 3\varepsilon\,. \tag{8}$$

Suppose

$$Q(t) = t\left(c_p - \int_1^t p(s)\,ds\right) \qquad \text{for } t > 0 \tag{9}$$

and choose $t_\varepsilon > 1$ such that

$$p_* - \varepsilon < Q(t) < p^* + \varepsilon \qquad \text{for } t > t_\varepsilon\,. \tag{10}$$

Let $\alpha = -\frac{1}{2} - p^* - 2\varepsilon$. It is evident that $\alpha > 0$ and

$$\alpha - \sqrt{\alpha} + p^* + \varepsilon < 0\,. \tag{11}$$

Due to (8), it is easy to see that $\alpha + \sqrt{\alpha} + p_* - \varepsilon < 0$. If along with this we take into account (11), then from (10) we get

$$-\alpha - \sqrt{\alpha} < Q(t) < -\alpha + \sqrt{\alpha} \qquad \text{for } t > t_\varepsilon\,,$$

and therefore, $Q^2(t) + 2\alpha Q(t) + \alpha(\alpha - 1) < 0$ for $t > t_\varepsilon$. In view of this, it is clear that the function $w(t) = t^\alpha$ satisfies the inequality

$$w''(t) \le -\frac{Q^2(t)}{t^2}w(t) - \frac{2Q(t)}{t}w'(t) \qquad \text{for } t > t_\varepsilon\,.$$

Consequently, the equation

$$v'' = -\frac{Q^2(t)}{t^2}v - \frac{2Q(t)}{t}v' \tag{12}$$

has a solution v satisfying the inequalities

$$0 < v(t) < t^\alpha \qquad \text{for } t > t_\varepsilon \tag{13}$$

(see, e.g., [4]).

It can be directly checked that the function

$$u(t)=v(t)\exp\left(\int_1^t \frac{Q(s)}{s}\,ds\right) \qquad \text{for } t>t_\varepsilon \tag{14}$$

is a solution of equation (1). By (10) and (13), there are $M>0$ and $t_1>t_\varepsilon$ such that

$$0<u(t)<Mt^{\alpha+p^*+\varepsilon} \qquad \text{for } t>t_1\,.$$

Hence, taking into consideration how α is, we conclude that $u\in V_0(p)$. Therefore we have proved that $V_0(p)\neq\{0\}$.

Since $u(t)>0$ for $t>t_1$, we have $\dim V_0(p)\le 1$ (see, e.g., [1]). However $\dim V_0(p)=1$, since $V_0(p)\neq\{0\}$.

Let us now suppose that $p(t)\le 0$ in some neighbourhood of $+\infty$. Then it is obvious that $\dim Z(p)\le 1$ and $V_0(p)\subset Z(p)$. Hence in view of the fact that $\dim V_0(p)=1$, we obtain $V_0(p)=Z(p)$. This completes the proof of the theorem. □

The proof of Theorem 3 is omitted, since it is analogous to that of Theorem 2 with the only difference $\alpha=-p^*-\varepsilon$.

Proof of Theorem 4. Assume the contrary. Let u be a nontrivial solution of equation (1) and $u\in V_0(p)$. According to (5) and applying Theorem 1.6 from [5], equation (1) is nonoscillatory. Thus without loss of generality we can assume that $u(t)>0$ for $t>t_0$. Choose $\varepsilon\in\,]0,\frac12[$ and $t_\varepsilon>t_0$ such that (10) holds and

$$p_*-\varepsilon<-\frac12\,, \tag{15}$$

$$p^*+\varepsilon<-\sqrt{(p_*-\varepsilon)^2-(p_*-\varepsilon)-\frac34}\,. \tag{16}$$

It is evident that the function v defined by (14) and (9) is a solution of equation (12). According to our assumption,

$$v(t)>0 \qquad \text{for } t>t_0\,, \tag{17}$$

$$\int^{+\infty} v^2(s)\exp\left(2\int_1^s \frac{Q(\eta)}{\eta}\,d\eta\right)ds<+\infty\,. \tag{18}$$

Let us show that for some $t_1>t_0$,

$$v'(t)>0 \qquad \text{for } t>t_1\,. \tag{19}$$

Indeed, if there exists $t_*>t_1$ such that $v'(t_*)\le 0$, then by virtue of the equality

$$\left(v'(t)\exp\left(2\int_1^t \frac{Q(\eta)}{\eta}\,d\eta\right)\right)'=-\frac{Q^2(t)}{t^2}\exp\left(2\int_1^t \frac{Q(\eta)}{\eta}\,d\eta\right)v(t) \qquad \text{for } t>0\,,$$

we have

$$v'(t) < 0 \qquad \text{for } t > t_* . \tag{20}$$

Then in view of (15), from (12) we find $v''(t) < 0$ for $t > t_*$. But this together with (20) contradicts inequality (17).

By (10), (17), and (19), from (12) we get

$$v''(t) \le -\frac{(p^* + \varepsilon)^2}{t^2} v(t) - \frac{2(p_* - \varepsilon)}{t} v'(t) \qquad \text{for } t > t_1 .$$

Consequently, the equation

$$w'' = -\frac{(p^* + \varepsilon)^2}{t^2} w - \frac{2(p_* - \varepsilon)}{t} w' \tag{21}$$

has a solution w satisfying the inequalities

$$0 < w(t) < v(t) \qquad \text{for } t > t_1 .$$

Hence due to (10) and (18), we have

$$\int^{+\infty} w^2(s) s^{2(p_* - \varepsilon)}\, ds < +\infty . \tag{22}$$

On the other hand, we can easily check that the functions $w_k(t) = t^{\lambda_k}$, $k = 1, 2$, where

$$\lambda_k = \frac{1}{2} \left[1 - 2(p_* - \varepsilon) - (-1)^k \sqrt{(1 - 2(p_* - \varepsilon))^2 - 4(p^* + \varepsilon)^2} \right] , \qquad k = 1, 2,$$

are linearly independent solutions of equation (21). By (16), $2\lambda_1 + 2(p_* - \varepsilon) > -1$ and therefore,

$$\int^{+\infty} w_k^2(s) s^{2(p_* - \varepsilon)}\, ds = +\infty , \qquad k = 1, 2 .$$

Thus neither of nontrivial solutions of equation (21) satisfies condition (22). This concludes the proof of the theorem. □

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