Josef Niederle
A note on tolerance lattices of finite chains

Časopis pro pěstování matematiky, Vol. 106 (1981), No. 1, 75--78

A NOTE ON TOLERANCE LATTICES OF FINITE CHAINS

JOSEF NIEDERLE, Brno

(Received November 13, 1978)

In this paper, tolerance lattices of finite chains are characterized as free distributive lattices over a family of finite partial lattices.

Notation. $L(n + 1)$ is the distributive lattice of all $(n + 1)$-tuples of natural numbers $[x^0, x^1, \ldots, x^n]$ satisfying $x^0 = 0$, $x^i \leqq x^{i-1} + 1$ for $i = 1, \ldots, n$.

Remarks. $i < j$ implies $x^j \leqq x^i + j - i$.

$L(n + 1)$ is isomorphic to the tolerance lattice of an $(n + 1)$-element chain (regarded as a lattice) ([2]).

Notation. U_n is a partial lattice defined as follows:

1° The underlying set of U_n is the $2n$-element set $\{O, I, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\}$,

2° $O \wedge u = O$, $u \wedge I = u$ for all $u \in U_n$,

3° $a_i \wedge a_j = a_{\min(i,j)}$, $b_i \wedge b_j = b_{\min(i,j)}$ for all $i, j \in \{1, \ldots, n - 1\}$,

4° $a_i \wedge b_j = O$ for all $i, j \in \{1, \ldots, n - 1\}$ satisfying $i + j \leqq n$.

In the following, the notation $a_0 = b_0 = O$, $a_n = b_n = I$ will be used. It is clear that conditions 3° and 4° remain valid for all $i, j \in \{0, 1, \ldots, n\}$.

Proposition. *$L(n + 1)$ is generated by a partial sublattice isomorphic to U_n.*

Proof. Denote $a_i = [0, 1, \ldots, i, 0, \ldots, 0]$, i.e. $a_i^k = k$ for $k \leqq i$ and $a_i^k = 0$ for $k > i$, and $b_i = [0, \ldots, 0, 1, \ldots, i]$, i.e. $b_i^k = 0$ for $k < n - i$ and $b_i^k = k + i - n$ for $k \geqq n - i$. Then $O = [0, \ldots, 0]$, i.e. $O^k = 0$, and $I = [0, 1, \ldots, n]$. i.e. $I^k = k$. The set $\{O, I, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\}$ satisfies 1°–4°, so it can be regarded as a partial sublattice of $L(n + 1)$ isomorphic to U_n. Let $x = [x^0, x^1, \ldots, x^n] \in L(n + 1)$. Put $y = \bigvee_{k=1}^{n} (a_k \wedge b_{x^k - k + n})$. Then $y^i = \bigvee_{k=1}^{n} (a_k^i \wedge b_{x^k - k + n}^i) \geqq a_i^i \wedge b_{x^i - i + n}^i = i \wedge \wedge x^i = x^i$. As $a_k^i \wedge b_{x^k - k + n}^i \leqq x^i$, $y^i \leqq x^i$ must hold and therefore $y^i = x^i$. Hence $y = x$. Q.E.D.

Definition. A pair of indices $[i, j]$ is *n-significant* if $i + j > n$. A pair of indices is *maximal n-significant* in a set M of pairs of indices if it is a maximal element in the subset of all n-significant elements of M.

Remark. Clearly, a homomorphic image of a partial lattice satisfying $2°-4°$ satisfies $2°-4°$ as well.

Lemma 1. *Let a distributive lattice D contain a subset $U = \{a_0 = b_0 = O, a_n = b_n = I, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\}$, not necessarily a $2n$-element set, the elements of which satisfy $2°-4°$. Let $d \in D$, $d = \bigvee(a_{i_k} \wedge b_{j_k})$. Then $d = \bigvee(a_{e_m} \wedge b_{f_m})$, where $\{[e_m, f_m]\}_m$ is the set of all maximal n-significant elements of $\{[i_k, j_k]\}_k$.*

Proof. If there is no n-significant element in $\{[i_k, j_k]\}_k$, then $\{[e_m, f_m]\}_m = \emptyset$ and $d = O$. $\bigvee_{m \in \emptyset}(a_{e_m} \wedge b_{f_m})$ can be put equal to O in this case. If there is at least one n-significant element in $\{[i_k, j_k]\}_k$, then evidently

$$\bigvee_m(a_{e_m} \wedge b_{f_m}) \leqq \bigvee_k(a_{i_k} \wedge b_{j_k}) = \bigvee_p(a_{i_p} \wedge b_{j_p}) \vee \bigvee_q(a_{i_q} \wedge b_{j_q}) =$$
$$= O \vee \bigvee_q(a_{i_q} \wedge b_{j_q}) \leqq \bigvee_m(a_{e_m} \wedge b_{f_m})$$

where $[i_p, j_p]$ are all n-nonsingificant pairs and $[i_q, j_q]$ are all n-significant pairs in $\{[i_k, j_k]\}_k$. Q.E.D.

Lemma 2. *Let a distributive lattice D be generated by its subset $U = \{a_0 = b_0 = O, a_n = b_n = I, a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}\}$, not necessarily a $2n$-element set, the elements of which satisfy $2°-4°$. Then every element $d \in D$ can be represented in the form $d = \bigvee_k(a_{i_k} \wedge b_{j_k})$, where the indices i_k form an increasing finite sequence and the indices j_k form a decreasing finite sequence, both of the same length, and $i_k + j_k > n$ for all k.*

Proof. By Lemma 1, every element $d \in D$ can be represented in the form $d = \bigvee_m(a_{e_m} \wedge b_{f_m})$, where all pairs of indices $[e_m, f_m]$ are maximal n-significant in $\{[e_m, f_m]\}_m$. By ordering all binomials in this representation by indices e, the needed representation can be obtained. Q.E.D.

Notation. The representation from Lemma 2 will be denoted by a tilde over $\bigvee$, i.e. $\tilde{\bigvee}$.

Proposition. *In the case of $L(n + 1)$, the representation mentioned in Lemma 2 is unique.*

Proof. Let $d = \tilde{\bigvee\limits_k}(a_{i_k} \wedge b_{j_k})$ be such a representation. Then it holds

$$(*) \qquad d^{i_k} \neq 0\,, \quad d^{i_k+1} \leqq d^{i_k}\,,$$

because

$$d^{i_k} = b_{j_k}^{i_k} > 0\,, \quad d^{i_k+1} = \bigvee_{l>k}(a_{i_l} \wedge b_{j_l}) = \bigvee_{l>k} b_{j_l}^{i_k+1} < b_{j_k}^{i_k+1} = d^{i_k} + 1\,.$$

Conversely, let d^i satisfy $(*)$, i.e. $d^i \neq 0$, $d^{i+1} \leqq d^i$. Then there exists a k such that $i = i_k$ and $b_{j_k}^i = d^i$, because a k must exist such that $i \leqq i_k$ and $b_{j_k}^i = d^i$, and if $i < i_k$, then $d^{i+1} = d^i + 1$. Hence i_k and j_k are uniquely determined by the coordinates of the element d. Q.E.D.

Theorem. *$L(n + 1)$ is a free distributive lattice over U_n.*

Proof. Let D be a distributive lattice, φ a homomorphism of the partial lattice U_n into D. Clearly, the only possible homomorphic extension of φ on the whole $L(n + 1)$ is the mapping

$$\bar{\varphi} = (\tilde{\bigvee_k}(a_{i_k} \wedge b_{j_k}) \mapsto \tilde{\bigvee_k}(\varphi a_{i_k} \wedge \varphi b_{j_k}))\,.$$

$\bar{\varphi}$ is a lattice homomorphism:
Join:

$$\bar{\varphi}(\tilde{\bigvee_k}(a_{i_k} \wedge b_{j_k}) \vee \tilde{\bigvee_l}(a_{g_l} \wedge b_{h_l})) = \bar{\varphi}(\tilde{\bigvee_m}(a_{e_m} \wedge b_{f_m})) = \tilde{\bigvee_m}(\varphi a_{e_m} \wedge \varphi b_{f_m})\,,$$

where $[e_m, f_m]$ are exactly all maximal n-significant elements in $\{[i_k, j_k]\}_k \cup \{[g_l, h_l]\}_l$.

$$\bar{\varphi}(\tilde{\bigvee_k}(a_{i_k} \wedge b_{j_k})) \vee \bar{\varphi}(\tilde{\bigvee_l}(a_{g_l} \wedge b_{h_l})) =$$

$$= \tilde{\bigvee_k}(\varphi a_{i_k} \wedge \varphi b_{j_k}) \vee \tilde{\bigvee_l}(\varphi a_{g_l} \wedge \varphi b_{h_l}) = \tilde{\bigvee_m}(\varphi a_{e_m} \wedge \varphi b_{f_m})\,,$$

where $[e_m, f_m]$ are exactly all maximal n-significant elements in $\{[i_k, j_k]\}_k \cup \{[g_l, h_l]\}_l$.
Hence $\bar{\varphi}$ is join-preserving.
Meet:

$$\bar{\varphi}(\tilde{\bigvee_k}(a_{i_k} \wedge b_{j_k}) \wedge \tilde{\bigvee_l}(a_{g_l} \wedge b_{h_l})) = \bar{\varphi}(\tilde{\bigvee_m}(a_{e_m} \wedge b_{f_m})) = \tilde{\bigvee_m}(\varphi a_{e_m} \wedge \varphi b_{f_m})\,,$$

where $[e_m, f_m]$ are exactly all maximal n-significant elements in $\{[\min(i_k, g_l), \min(j_k, h_l)]\}_{k,l}$.

$$\bar{\varphi}(\tilde{\bigvee_k}(a_{i_k} \wedge b_{j_k})) \wedge \bar{\varphi}(\tilde{\bigvee_l}(a_{g_l} \wedge b_{h_l})) =$$

$$= \tilde{\bigvee_k}(\varphi a_{i_k} \wedge \varphi b_{j_k}) \wedge \tilde{\bigvee_k}(\varphi a_{g_l} \wedge \varphi b_{h_l}) = \tilde{\bigvee_m}(\varphi a_{e_m} \wedge \varphi b_{f_m})\,,$$

where $[e_m, f_m]$ are exactly all maximal n-significant elements in $\{[\min(i_k, g_l), \min(j_k, h_l)]\}_{k,l}$. Hence $\bar{\varphi}$ is meet-preserving. Q.E.D.

References

[1] *G. Szász:* Introduction to lattice theory. Akadémiai Kiadó Budapest, 1963.
[2] *I. Chajda - J. Dalík - J. Niederle - V. Veselý - B. Zelinka:* How to draw tolerance lattices of finite chains. Arch. Math. (Brno), *16* (1980), 161—165.

Author's address: 615 00 Brno 15, Viniční 60.