Časopis pro pěstování matematiky

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*Časopis pro pěstování matematiky*, Vol. 110 (1985), No. 2, 122--129

Persistent URL: `http://dml.cz/dmlcz/108593`

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# PERIODIC SOLUTIONS OF A CLASS OF ABSTRACT NONLINEAR EQUATIONS OF THE SECOND ORDER

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(Received January 3, 1983)

## INTRODUCTION

The aim of this paper is to prove the existence of weak periodic solutions of the abstract differential equation

$$F(u) = \varphi(u) + h\,, \tag{1}$$

where $F(u) \equiv u'' + \psi(u') + \mathscr{A}u$, $u' = \mathrm{d}u/\mathrm{d}t$, $\psi$ and $\varphi$ are nonlinear mappings of a Hilbert space $H$ into itself with linear growth and $\mathscr{A}$ is a linear elliptic operator from $V \subset H$ into $V^*$.

The results obtained here are applied to the jumping-nonlinearity problem for ordinaly and partial differential equations (many results for the linear case and further references in this field can be found in [1]).

In the case of partial differential equations $\psi$ and $\varphi$ are continuous real functions. The requirement of linear growth of $\psi$ is more restrictive than the assumptions made by Prodi, Prouse, Krylová and others (for the references see [2], see also [7]), but on the other hand here the assumptions concerning the function $\varphi$ are more general, namely the values of lim $\varphi(u)/u$ as $u \to +\infty$ and $u \to -\infty$ may be separated by two consecutive eigenvalues of the operator $\mathscr{A}$.

The present paper is divided into two parts. In the first on the equation

$$F(u) = h \tag{2}$$

is investigated and it is shown by rather elementary means that $F$ is a homeomorphism between suitable Banach spaces $X$ and $Y$ (see Assumption 2). Let us remark that a little more general result can be obtained by using the Faedo-Galerkin method, especially the assumption of the approximation of $\psi$ by Lipschitz continuous mappings can be omitted.

In the second part the existence of a solution of (1) for each right-hand side is proved by the fixed-point argument for the operator $F^{-1}$ by means of the topological degree theory.

## Assumptions

1. Let $H$, $V$ be two Hilbert spaces, $V \subset H$, $V$ dense in $H$ and let embedding $V \to H$ be compact. Let us identify $H$ with its dual in such a way that $V \subset H \subset V^*$ (for details see e.g. [3]). The scalar product and the norm in $H$ is denoted by $(\cdot, \cdot)_H$ and $\|\cdot\|_H$, respectively.

Let $\mathscr{A}: V \to V^*$ be a linear operator such that the form

$$((u, v)) \equiv (\mathscr{A}u)(v), \quad u, v \in V$$

is a scalar product on $V$.

Let $f: H \to H$ be a continuous mapping and $b > 0$ a real constant such that for every $w \in H$,

$$\psi(w) = bw + f(w).$$

Let us assume that $f$ is monotone, $(f(w), w)_H \geqq 0$ for every $w \in H$, $\lim \|f(w)\|_H / \|w\|_H = 0$ as $\|w\|_H \to +\infty$ and there exists a sequence of Lipschitz continuous mapings $f_n$ which converges uniformly to $f$ in $H$.

2. Let $T > 0$ and put $Y = L^2(0, T; H)$ with the scalar product

$$(u, v) = \int_0^T (u(t), v(t))_H \, dt \quad \text{and the norm} \quad |u| = \left(\int_0^T \|u(t)\|_H^2 \, dt\right)^{1/2},$$

$u, v \in Y$. Let us define $\Box u \equiv u'' + \mathscr{A}u$ for each $u \in L^2(0, T; V)$ such that $u' \in Y$ and $u'' \in L^2(0, T; V^*)$. Put $X = \{u \in L^2(0, T; V) \mid u' \in Y, \ \Box u \in Y, \ u(0) = u(T), u'(0) = u'(T)\}$. The norm of an element $u \in X$ is defined as $\|u\| = |u'| + |\Box u|$.

3. Let $C \subset H$ be a closed cone, i.e. a closed set with the properties $C + C \subset C$, $aC \subset C$ for each $a \geqq 0$, $C \cap (-C) = \{0\}$. This cone induces a semiordering $\leqq$: $v \leqq w$ iff $w - v \in C$. Assume that it has the following properties:

a) For every $w \in H$ there exist $w^+ = \sup\{w, 0\}$ and $w^- = \sup\{-w, 0\}$ such that $(w^+, w^-) = 0$, and the mapping $w \mapsto w^+$ is continuous from $H$ into $H$.

b) Denote $\mathscr{C} = \{w \in Y \mid w(t) \in C \text{ a.e.}\}$. Put $w^+(t) = (w(t))^+$ for $t \in [0, T]$. We assume that $(v^+, v') = 0$ for every $v \in X$.

4. Let $\sigma(\mathscr{A}) = \{\lambda_k\}_{k=1}^{\infty}$, $\lambda_k < \lambda_{k+1}$, $\lambda_k \to +\infty$ as $k \to +\infty$, be the spectrum of $\mathscr{A}$ and let $m_k$ be the multiplicity of the eigenvalue $\lambda_k$. Let us denote by $w_k^i$, $k = 1, 2, \ldots$, $i = 1, \ldots, m_k$, an eigenfunction of $\mathscr{A}$ corresponding to the eigenvalue $\lambda_k$, $\mathscr{A}w_k^i = \lambda_k w_k^i$. Assume that $w_k^i \in C$ or $w_k^i \in -C$ only if $k = 1$.

5. Let $g: H \to H$ be a continuous mapping, $\lim \|g(w)\|_H / \|w\|_H = 0$ as $\|w\|_H \to +\infty$, such that there exist real numbers $\mu, \nu$, $\varphi(u) = \mu u^+ - \nu u^- + g(u)$.

**Lemma 1.** *Let the assumptions* 1 *and* 2 *be fulfilled. Then for* $u \in X$ *we have*

$$(\Box u, u) = -|u'|^2 + \int_0^T ((u, u)) \, dt \textit{ and } (\Box u, u') = 0.$$

Proof. Let $\varrho$: $]-T/2, T/2[ \to [0, +\infty[$ be a $C^\infty$-function with compact support in $]-T/2, T/2[$. Let us define the sequence of "$T$-periodic mollifiers"

$$\varrho_n(t) = n \sum_{k=-\infty}^{+\infty} \varrho(n(t - kT)),$$

$$u_n(t) = \int_0^T \varrho_n(t - s)\, u(s)\, ds .$$

The lemma is valid for $u_n$. The passage to the limit $n \to +\infty$ completes the proof.

### Remarks

1. Let the assumptions 1 and 2 be fulfilled. Then $X$ is a Banach space and the embedding $X \to Y$ is compact. The last assertion follows from the fact that $X$ is continuously embedded into $X_1 = \{u \in L^2(0, T; V) \mid u' \in Y\}$ and from the "compactness lemma" of [3].

2. For $u \in Y$ the mappings $u \mapsto \tilde{f}(u)$, $\tilde{f}(u)(t) = f(u(t))$ and $u \mapsto \tilde{g}(u)$, $\tilde{g}(u)(t) = g(u(t))$ are continuous operators from $Y$ into $Y$ (see [4]).

3. In the examples below, the methods of verification of the assumption 4 are explained in [5].

### Examples

1. Let $G \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary, $H = L^2(G)$, $\psi$, $\varphi$ continuous real functions, $C = \{u \in H \mid u(x) \geqq 0 \text{ a.e.}\}$, $V = H_0^1(G)$, $\mathscr{A} = -\Delta$. One proves the existence of a $T$-periodic solution to the boundary-value problem $u_{tt} + \psi(u_t) - \Delta u = \varphi(u) + h$, $u = 0$ on $\partial G$, for arbitrary $h \in Y = L^2((0, T) \times G)$, where $\lim \psi(u)/u = b$ as $u \to \pm\infty$ and $\lim \varphi(u)/u$ is equal to $\mu$ as $u \to +\infty$ and to $\nu$ as $u \to -\infty$.

2. Analogous problem arises with $V = H^2(G) \cap H_0^1(G)$, $\mathscr{A} = \Delta^2$.

3. Let $\psi$, $\varphi$, $C$ be as above, $H = L^2(0, l)$, $l > 0$, $\tilde{\varphi}(u) = \varphi(u) - u$, $\mathscr{A}u = -d^2u/dx^2 + u$, $V = \{u \in H^1(0, l) \mid u(0) = u(l)\}$. Then the equation (1) represents the generalized periodic problem for the nonlinear telegraph equation

$$u_{tt} + \psi(u_t) - u_{xx} = \tilde{\varphi}(u) + h .$$

4. Let $H, \psi, \varphi, \tilde{\varphi}, C$ be as in Example 3, $V = \{u \in H^2(0, l) \mid u(0) = u(l), u'(0) = u'(l)\}$,

$$\mathscr{A}u = \frac{d^4u}{dx^4} + u .$$

Then the equation (1) becomes the generalized periodic problem for the nonlinear beam equation

$$u_{tt} + \psi(u_t) + u_{xxxx} = \tilde{\varphi}(u) + h .$$

5. Let $\psi, \varphi$ be continuous mappings from $\mathbb{R}^N$ into $\mathbb{R}^N$, $H = V = \mathbb{R}^N$, $C = \{u \in \mathbb{R}^N \mid u = (u^1, \ldots, u^N),\ u^i \geqq 0$ for each $i = 1, \ldots, N\}$. Let $A$ be a symmetric positive definite $(N \times N)$-matrix $A = \{a_{ij}\}$, $a_{ij} > 0$ for each $i, j = 1, \ldots, N$. One proves the existence of periodic solutions of the system of ordinary differential equations

$$u'' + \psi(u') + A^{-1}u = \varphi(u) + h .$$

## I. INVERSION THEOREM

In this section we require the assumptions 1 and 2 to be satisfied. Our aim is to prove

**Theorem 1.** *$F$ is a homeomorphism from $X$ onto $Y$.*

Obviously, $F$ is a continuous mapping and for every $u, v \in X$ we have

$$(3) \qquad \begin{cases} |u' - v'| \leqq 1/b|F(u) - F(v)| , \\ |\Box(u - v)| \leqq |F(u) - F(v)| + |f(u') - f(v')| . \end{cases}$$

The assertion of Theorem 1 is a consequence of (3) and of the following two lemmas.

**Lemma 2.** *Let $f: H \to H$ be Lipschitz continuous. Then $F: X \to Y$ is a homeomorphism.*

Proof. Let $\|f(w) - f(z)\|_H \leqq L\|w - z\|_H$ for every $w, z \in H$. For $s \in [0, 1]$, $u \in X$ put

$$(4) \qquad F_s(u) \equiv \Box u + bu' + s \,.\, f(u') ,$$

and $c_1 = 1 + (L + 1)/b$, $c_2 = \max\{1, b + L\}$.

The inequality

$$(5) \qquad 1/c_1\|u - v\| \leqq |F_s(u) - F_s(v)| \leqq c_2\|u - v\|$$

holds for each $u, v \in X$.

We know that $F_0$ is a linear isomorphism between $X$ and $Y$. Let us suppose that for some $s \in [0, 1]$ the mapping $F_s$ is a homeomorphism from $X$ onto $Y$. Then for arbitrary $\varepsilon > 0$ the equation

$$(6) \qquad F_{s+\varepsilon}(u) = h$$

is equivalent to

$$u = F_s^{-1}(h - \varepsilon f(u')) .$$

The existence of a solution of (6) for arbitrary $h \in Y$ is therefore ensured by the Banach contraction principle, whenever we choose $\varepsilon < (Lc_1)^{-1}$. Since $\varepsilon$ is independent on $s$, the mapping $F_s\colon X \to Y$ is onto for each $s \in [0, 1]$, and by (5) the proof is complete. ■

**Lemma 3.** *Let $\{f_n\}$ be a sequence of continuous mappings from $H$ into $H$ which converges uniformly to $f$ in $H$. Let $h$ be an arbitrary element of $Y$ and let $u_n$ be the solution of*

$$\Box u_n + bu_n' + f_n(u_n') = h\,.$$

*Then $u_n$ converge in $X$ to the solution $u$ of the equation*

$$\Box u + bu' + f(u') = h\,.$$

Proof. For $n \neq m$ we have $\Box(u_n - u_m) + b(u_n' - u_m') + f(u_n') - f(u_m') = f(u_n') - f_n(u_n') - f(u_m') + f_m(u_m')$. Hence, $\{u_n\}$ is a fundamental sequence in $X$. Let us denote by $u$ the limit of $u_n$. We have $\Box u + bu' + f(u') = h + \Box(u - u_n) + b(u' - u_n') + f(u') - f_n(u_n')$ for arbitrary $n$, and the proof follows immediately. ■

## II. JUMPING NONLINEARITY

Throughout this section we make use of the assumptions 1–5.

Denote by $A_0$ the set of all $(\mu, \nu) \in R^2$ such that the equation

$$\Box u + bu' = \mu u^+ - \nu u^- \tag{7}$$

has only the trivial $T$-periodic solution (i.e. $u \equiv 0$), and

$$A_1 = (]-\infty, \lambda_1[^2 \cup ]\lambda_1, \lambda_2]^2 \cup \bigcup_{k=2}^{\infty} [\lambda_k, \lambda_{k+1}]^2) \setminus \bigcup_{k=2}^{\infty} \{(\lambda_k, \lambda_k)\}\,.$$

The following lemma is an easy consequence of Lemma 1.

**Lemma 4.** *Let $u \in X$ be a solution of* (7). *Then $u =$ const., $u \in V$, and*

$$\mathscr{A}u = \mu u^+ - \nu u^-\,. \tag{8}$$

**Lemma 5.** *Let $\lambda \notin \sigma(\mathscr{A})$. Put* $R_\lambda = \|(\mathscr{A} - \lambda\,\mathrm{Id})^{-1}\|_{(H\to H)} \equiv \sup\limits_{\substack{u\in H\\ |u|=1}} |(\mathscr{A} - \lambda\,\mathrm{Id})^{-1} u|$.

*Then*

(a) $R_\lambda = [\operatorname{dist}(\lambda, \sigma(\mathscr{A}))]^{-1}$;

(b) *if* $|(\mathscr{A} - \lambda\,\mathrm{Id})^{-1} u| = R_\lambda \,.\, |u|$, *then* $u = \sum\limits_{k\in\mathscr{K}_\lambda} \sum\limits_{i=1}^{m_k} u_k^i w_k^i$, *where* $u_k^i \in \mathbb{R}^1$, $\mathscr{K}_\lambda = \{k \mid |\lambda - \lambda_k| = \operatorname{dist}(\lambda, \sigma(\mathscr{A}))\}$.

The proof of Lemma 5 is immediate if we represent $u$ in the form of the series $\sum_{k=1}^{\infty} \sum_{i=1}^{m_k} u_k^i w_k^i$. The set $\mathscr{K}_\lambda$ contains two points in the case $\lambda = \frac{1}{2}(\lambda_k + \lambda_{k+1})$ and one point in the other cases.

**Lemma 6.** $A_1 \subset A_0$.

Proof. Let us consider two cases.

a) $(\mu, \nu)$ lies in the interior of $A_1$. Put $\lambda = \frac{1}{2}(\mu + \nu)$, $\varkappa = \frac{1}{2}(\mu - \nu)$. Let $u \in V$ be a solution of (8). Obviously $u = u^+ - u^-$, hence

$$(\mathscr{A} - \lambda \operatorname{Id}) u = \varkappa(u^+ + u^-) \tag{9}$$

and Lemma 5 (a) yields

$$|u| \leqq |\varkappa| / \operatorname{dist}(\lambda, \sigma(\mathscr{A}))\, |u| .$$

Since $|\varkappa| < \operatorname{dist}(\lambda, \sigma(\mathscr{A}))$, necessarily $u = 0$.

b) $(\mu, \nu) \in \partial A_1$. Set $|\varkappa| = \operatorname{dist}(\lambda, \sigma(\mathscr{A}))$. Assume $\mu = \lambda_{k+1}$, $\nu = \lambda_k$, $k > 1$ (the other cases are analogous). Lemma 5 (b) implies

$$u^+ + u^- = \sum_{i=1}^{m_k} u_k^i w_k^i + \sum_{i=1}^{m_{k+1}} u_{k+1}^i w_{k+1}^i .$$

The fact that $u$ is a solution of (9) implies

$$u = u^+ - u^- = -\sum_{i=1}^{m_k} u_k^i w_k^i + \sum_{i=1}^{m_{k+1}} u_{k+1}^i w_{k+1}^i .$$

Finally, we obtain $u^+ = \frac{1}{2} \sum_{i=1}^{m_{k+1}} u_{k+1}^i w_{k+1}^i$, $u^- = \frac{1}{2} \sum_{i=1}^{m_k} u_k^i w_k^i$, hence $u^+$ and $u^-$ are eigenfunctions of the operator $\mathscr{A}$. Using the assumption 4 we obtain $u = 0$. ∎

Let us define the system of operators $F_s \colon X \to Y$ as in (4). For $h \in Y$ put $u_s = F_s^{-1}(h)$, $s \in [0, 1]$. Then

$$F_0^{-1} h - F_s^{-1} h = s F_0^{-1}(f(u_s')) .$$

Making use of the a priori estimate $|u_s'| \leqq (1/b)\, |h|$, from the assumption $\|f(w)\|_H / \|w\|_H \to 0$ as $\|w\|_H \to +\infty$ we deduce that for each $\varepsilon > 0$ there exists $K_\varepsilon$ such that

$$\|F_0^{-1} h - F_s^{-1} h\| \leqq \varepsilon |h| + K_\varepsilon . \tag{10}$$

**Lemma 7.** *Let $(\mu, \nu) \in A_0$. Then there exists $m > 0$ such that for every $u \in X$ the inequality*

(11) $$|u - F_0^{-1}(\mu u^+ - \nu u^-)| \geqq 3\, m|u|$$

*holds.*

Proof. Let us suppose that (11) does not hold. Then there exists a sequence $\{u_j\}$, $|u_j| = 1$, $\lim_{j\to\infty} |u_j - F_0^{-1}(\mu u_j^+ - \nu u_j^-)| = 0$. Let us choose the sequence $\{u_j\}$ in such a way that $F_0^{-1}(\mu u_j^+ - \nu u_j^-) \to u_0$ in $X$ and $F_0^{-1}(\mu u_j^+ - \nu u_j^-) \to u_0$ in $Y$. Consequently $u_j \to u_0$ in $Y$, $|u_0| = 1$ and $u_0 = F_0^{-1}(\mu u_0^+ - \nu u_0^-)$ is a nontrivial solution of (7), which contradicts the assumption $(\mu, \nu) \in A_0$. ∎

Let us define $A_2$ as the set of all $(\mu, \nu) \in A_0$ such that there exists a continuous curve $(a(z), b(z)) \subset A_0$, $z \in [0, 1]$, $a, b \in C([0, 1])$, $a(0) = \mu$, $b(0) = \nu$, $a(1) = b(1) = \lambda \notin \sigma(\mathscr{A})$. Obviously $A_1 \subset A_2$ and from Lemma 7 it follows that $A_2$ and $A_0$ are open sets in $\mathbb{R}^2$.

**Theorem 2.** *Let $(\mu, \nu) \in A_2$. Then the equation (1) has at least one solution $u \in X$ for every right-hand side $h \in Y$.*

Proof. Let $(\mu, \nu) \in A_2$ and $h \in Y$ be given. For any $r, s \in [0, 1]$ and $u \in Y$ we have

$$\begin{aligned} &|F_s^{-1}(\mu u^+ - \nu u^- + r(g(u) + h)) - F_0^{-1}(\mu u^+ - \nu u^-)| \leqq \\ &\leqq r|F_0^{-1}(g(u) + h)| + |F_s^{-1}(\mu u^+ - \nu u^- + r(g(u) + h)) - \\ &\qquad - F_0^{-1}(\mu u^+ - \nu u^- + r(g(u) + h))| . \end{aligned}$$

Using the assumption 5 and (10) we conclude that there exists a constant $K_m > 0$ such that

(12) $$|F_s^{-1}(\mu u^+ - \nu u^- + r(g(u) + h)) - F_0^{-1}(\mu u^+ - \nu u^-)| \leqq m|u| + K_m .$$

Put $R = K_m/m$. The inequalities (11) and (12) imply that

(13) $$|u - F_s^{-1}(\mu u^+ - \nu u^- + r(g(u) + h))| \geqq m|u|$$

for every $u \in Y$, $|u| \geqq R$. The operators $F_s^{-1}$ may be considered as compact mappings from $Y$ into $Y$. The property (13) enables us to define the topological degree of the mapping $u \mapsto u - F_s^{-1}(\mu u^+ - \nu u^- + r(g(u) + h))$ in $Y$ with respect to the ball $B_R(0) = \{u \in Y \mid |u| \leqq R\}$ and to the point 0 for every $r, s \in [0, 1]$.

Let $(a(z), b(z)) \subset A_2$, $z \in [0, 1]$, be a curve such that $a(0) = \mu$, $b(0) = \nu$, $a(1) = b(1) = \lambda \notin \sigma(\mathscr{A})$. Then the homotopy property of the topological degree yields

$$\begin{aligned} &d(u - F_1^{-1}(\varphi(u) + h), B_R(0), 0) = d(u - F_1^{-1}(\mu u^+ - \nu u^-), B_R(0), 0) = \\ &= d(u - F_0^{-1}(\mu u^+ - \nu u^-), B_R(0), 0) = d(u - \lambda F_0^{-1}(u), B_R(0), 0) . \end{aligned}$$

The mapping $Id - \lambda F_0^{-1}$ is linear, consequently its degree is odd. This ensures the existence of $u \in Y$ such that $u = F_1^{-1}(\varphi(u) + h)$. Hence, $u \in X$ and $u$ is a solution of (1). The theorem is proved. ∎

**Corollary.** *Let $k > 1$. Then there exists $\varepsilon > 0$ such that for arbitrary $(\mu, \nu) \in R^2$, $\lambda_k - \varepsilon < \nu < \lambda_k$, $\lambda_k + \varepsilon < \mu < \lambda_{k+1} + \varepsilon$ and for each $h \in Y$ there exists at least one solution of* (1).

## Remarks

4. In special cases there is possible to describe the set $A_2$ precisely. In the situation of Example 1 with $N = 1$ and Example 3 this problem was solved by Fučík (see e.g. [1]). He found a countable system $\{S_k\}$, $k \geqq 2$ of continuous curves in $]\lambda_1, +\infty[^2$, $(\lambda_k, \lambda_k) \in S_k$, such that $A_2 = (]-\infty, \lambda_1[^2 \cup ]\lambda_1, +\infty[^2) \setminus \bigcup_{k=2}^{\infty} S_k$.

5. In [6] it is proved that in the cases of Examples 2 ($N = 1$) and 4 there exists a system of curves with the same property as above, but it is not found explicitly.


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