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Časopis pro pěstování matematiky, Vol. 94 (1969), No. 3, 253--265

Persistent URL: http://dml.cz/dmlcz/108606

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BRANCHING OF SOLUTION OF ALGEBRAIC INTEGRAL EQUATION

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(Received January 27, 1967)

In this paper we shall deal with the general algebraic integral equation for the function \( y(s) \)

\[
F[\mu, y] \equiv \sum_{j=1}^{n} \sum_{a=0}^{\infty} \mu^a y^a(s) \ L_j[y^{a_1} \ldots y^{a_v}] = f(s)
\]

where

\[
L_j[y^{a_1} \ldots y^{a_v}] = \sum_{(a_1+\ldots+a_v=j)} \left( \int_a^b \int_a^b L_{a_1 \ldots a_v}(st_1 \ldots t_v) \ [y^{a_1}(t_1) \ldots y^{a_v}(t_v)] \ dt_1 \ldots dt_v \right)
\]

\( L_{a_1 \ldots a_v}(st_1 \ldots t_v) \) and \( f(s) \) are given functions and \( \mu \) is a real parameter.

The type of non-linear integral equations which are called algebraic integral equations was first introduced by W. SCHMEIDLER in [1].

For equation (1) we shall study branching of a solution which occurs for a certain value of the parameter \( \mu \).
The pair \((\mu_0, y_0(s))\) which satisfies equation (1) is called the branch point if for every \( \varepsilon > 0 \) there exists such \( \mu \) that \( |\mu - \mu_0| < \varepsilon \) and (1) has for this \( \mu \) at least two solutions which lie in the \( \varepsilon \)-neighbourhood of the solution \( y_0(s) \).

**Theorem.** Let \( L_{a_1 \ldots a_v}(st_1 \ldots t_v) \) be real functions continuous in all variables in the region \((a, b) \times \ldots \times (a, b)\) \((v + 1)\) factors for all suitable non-negative numbers \( a, a_1, \ldots, a_v \), and let \( f(s) \) be a real function continuous in \((a, b)\). Let \( y_0(s) \) be a solution of equation (1) for \( \mu = \mu_0 \) continuous in \((a, b)\) and let the discriminant of the polynomial \( F[\mu_0, y] \) be different from zero in \((a, b)\) for \( y_0(s) \). Then for equation (1) the following assertions are valid:

a) If number 1 is not an eigenvalue of the kernel

\[
L(s, t) = \frac{-1}{p(s)} \sum_{j=1}^{n} \sum_{a=0}^{\infty} \mu_0^a y_0^a(s) .
\]

\[
\sum_{(a_1+\ldots+a_v=j-a)} \left( \int_a^b \int_a^b \{ \alpha_1 L_{a_1 \ldots a_v}(st_2 \ldots t_v) \ y_0^{a_1-1}(t) \ y_0^{a_2}(t_2) \ldots y_0^{a_v}(t_v) + \right.
\]

\[ + \sum_{k=2}^{v} \alpha_k L_{a_1 \ldots a_v}(st_2 \ldots t_{k-1}t_{k+1} \ldots t_v) \ y_0^{a_1}(t) \ y_0^{a_2}(t_2) \ldots \prod_{l=2}^{v} y_0^{a_l}(t_l) \ dt_2 \ldots dt_v \]
where

\[ p(s) = \sum_{j=1}^{n} \sum_{s=0}^{j} \alpha \mu_0^{s} y_0^{s-1}(s) L_j[y_0^{s1} \ldots y_0^{s_r}], \]

then in a neighbourhood of the point \( \mu_0 \) there exists the unique solution of (1) in the form

\[ y(s) = \sum_{i=0}^{\infty} (\mu - \mu_0)^{i} y_i(s). \]

b) If number 1 is a \( p \)-multiple eigenvalue of the kernel (2) and b1) if

\[ \int_{a}^{b} \frac{\partial F[\mu, y_0]}{\partial \mu|_{\mu=\mu_0}} - \alpha_i(s) ds = 0, \quad i = 1, p, \]

is valid for the associated eigenfunctions \( \alpha_i(s) \) then in a neighbourhood of \( \mu_0 \) there exist, in general, \( 2^p \) solutions of (1) in the form (3);

b2) if any of the conditions (4) is not fulfilled then in a neighbourhood of \( \mu_0 \) there exist, in general, \( 2^p \) solutions of (1) in the form

\[ y(s) = \sum_{i=0}^{\infty} (\mu - \mu_0)^{i/2} y_i(s). \]

All the solutions are continuous in \( \langle a, b \rangle \) and tend to \( y_0(s) \) for \( \mu \to \mu_0 \).

Proof. If we denote

\[ \lambda = \mu - \mu_0, \quad \psi(s) = y(s) - y_0(s), \]

equation (1) can be rewritten in the form

\[ p(s) \psi(s) = p(s) \int_{a}^{b} L(s, t) \psi(t) dt - \]

\[ - \sum_{j=1}^{n} \sum_{s=0}^{j} \left\{ \alpha \mu_0^{s} y_0^{s-1}(s) L_j \left[ \sum_{i=1}^{\alpha_1} \left( \begin{array}{c} \alpha_1 \\ i_1 \\ \vdots \\ i_r \end{array} \right) \psi^{i_1} y_0^{s_1-1} \ldots \sum_{i=1}^{\alpha_r} \left( \begin{array}{c} \alpha_r \\ i_r \\ \vdots \\ i_r \end{array} \right) \psi^{i_r} y_0^{s_r-1} \right] \right\} + \]

\[ + \mu_0^{s} \sum_{m=1}^{\alpha} \left( \begin{array}{c} \alpha \\ m \end{array} \right) \psi^{m}(s) y_0^{s-m}(s) L_j \left[ \sum_{i=1}^{\alpha_1} \left( \begin{array}{c} \alpha_1 \\ i_1 \\ \vdots \\ i_r \end{array} \right) \psi^{i_1} y_0^{s_1-1} \ldots \sum_{i=1}^{\alpha_r} \left( \begin{array}{c} \alpha_r \\ i_r \\ \vdots \\ i_r \end{array} \right) \psi^{i_r} y_0^{s_r-1} \right] + \]

\[ + \sum_{k=1}^{\alpha} \left( \begin{array}{c} \alpha \\ k \end{array} \right) \lambda^k \mu_0^{s-k} \sum_{m=0}^{\alpha} \left( \begin{array}{c} \alpha \\ m \end{array} \right) \psi^{m}(s) y_0^{s-m}(s) L_j \left[ \sum_{i=1}^{\alpha_1} \left( \begin{array}{c} \alpha_1 \\ i_1 \\ \vdots \\ i_r \end{array} \right) \psi^{i_1} y_0^{s_1-1} \ldots \sum_{i=1}^{\alpha_r} \left( \begin{array}{c} \alpha_r \\ i_r \\ \vdots \\ i_r \end{array} \right) \psi^{i_r} y_0^{s_r-1} \right] + \]

\[ + \mu_0^{s} \sum_{m=2}^{\alpha} \left( \begin{array}{c} \alpha \\ m \end{array} \right) \psi^{m}(s) y_0^{s-m}(s) L_j[y_0^{s_1} \ldots y_0^{s_r}]. \]
$L_j$, $L_j'$ have the same meaning as $L_j$ under the condition that $\sum_{i=1}^{v} l_i = 0$, $\sum_{i=1}^{v} l_i = 0.1$ respectively.

Let us look for a solution of (6) in the form

$$\psi(s) = \sum_{i=1}^{\infty} \lambda^i y_i(s).$$

If we substitute (7) into (6) and compare coefficients of the same powers of $\lambda$ we obtain for $y_i(s)$ a system of nonhomogeneous linear integral equations. From the assumption on the discriminant of the polynomial $F[\mu_0, y]$ there follows that

$$p(s) = \frac{\partial F[\mu_0, y]}{\partial y(s)} \bigg|_{y=y_0}$$

is different from zero in $\langle a, b \rangle$. Then the system of equations for $y_i(s)$ can be written in the form

$$y_i(s) = \int_{a}^{b} L(s,t) y_i(t) \, dt + f_i(s), \quad l = 1, \infty$$

where

$$f_1(s) = -\frac{1}{p(s)} \frac{\partial F[\mu_0, y_0]}{\partial \mu} \bigg|_{\mu=\mu_0},$$

$$f_2(s) = -\frac{1}{p(s)} \left\langle \sum_{j=1}^{n} \sum_{a=0}^{j} \{s[y^2_1, y_1] L_j[y_0^{a+2} \ldots y_0^{a}] + \mu_0 y_0^a (Q[y^2_1] + R[y_1, y_1]) + \right.$$  

$$+ \alpha \mu_0^{a-1} y_0^{a-1} (s) (y_0(s) + \mu_0 y_1(s)) S[y_1]) + M_2[y_0] \right\rangle,$$

$$f_{l+1}(s) = -\frac{1}{p(s)} \left\langle \sum_{j=1}^{n} \sum_{a=0}^{j} \{t[y^2_1 y_1, y_1] L_j[y_0^{a+2} \ldots y_0^{a}] + \right.$$  

$$+ \mu_0^{a} y_0^{a-1}(s) (2Q[y_1, y_1] + T[y_1, y_1]) + \alpha \mu_0^{a-1} y_0^{a-1}(s) (\mu_0 y_1(s) + y_0(s)) S[y_1] +$$  

$$+ \alpha \mu_0^{a} y_0^{a-1}(s) y_1(s) S[y_1]) + M_{l+1}[y_0, y_1, \ldots, y_{l-1}] \right\rangle =$$

$$= K[y_1] - \frac{1}{p(s)} M_{l+1}[y_0, y_1, \ldots, y_{l-1}], \quad l = 2, \infty$$

where the notation

$$Q[y_1, y_1] = L_j \left[ \sum_{k=1}^{y \chi_k} \left( \frac{\chi_k}{2} \right) y_0^{\chi_k-2}(t_k) y_1(t_k) y_1(t_k) \prod_{i=1}^{y} y_0^{i}(t_i) \right],$$

$$R[y_1, y_1] = L_j \left[ \sum_{k=1}^{y-1} \sum_{m=k+1}^{y} \chi_k \chi_m y_0^{\chi_m-1}(t_k) y_1(t_k) y_0^{\chi_m-1}(t_m) y_1(t_m) \prod_{i=1}^{y} y_0^{i}(t_i) \right],$$

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\[ S[y_i] = L_j \left[ \sum_{k=1}^{\nu} \alpha_k y_0^{\alpha_k-1}(t_k) y_i(t_k) \prod_{i=1}^{\nu} y_0^{\alpha_i}(t_i) \right], \]
\[ T[y_1, y_{l+1}] = L_j \left[ \sum_{k=1}^{\nu} \sum_{m=1}^{\nu} \alpha_k \alpha_m y_0^{\alpha_k+\alpha_m-1}(t_k) y_1(t_k) y_0^{\alpha_m-1}(t_m) y_{l+1}(t_m) \prod_{i=1}^{\nu} y_0^{\alpha_i}(t_i) \right], \]
\[ q[y_1 y_1] = (\nu_0^\mu) y_0^{\nu_0^\mu-2}(s)y_1(s)y_1(s), \quad r[y_1] = \alpha^2 y_0^{\mu-1} y_0^{\nu_0^\mu-1}(s)y_1(s), \]
\[ s[y_1^2, y_1] = q[y_1^2] + r[y_1], \quad t[y_1, y_{l+1}, y_{l+1}] = 2q[y_1 y_{l+1}] + r[y_{l+1}], \quad l = 1, \infty, \]

has been introduced.

Solving system (8) it is necessary to distinguish whether number 1 is or is not an eigenvalue of the kernel \( L(s, t) \).

a) If number 1 is not the eigenvalue of \( L(s, t) \) it is possible to solve all equations of system (8) uniquely and to write the solutions continuous in \( \langle a, b \rangle \) in the form

\[ y_l(s) = \int_a^b \Gamma(s, t) f_1(t) \, dt + f_1(s), \quad l = 1, \infty \]

where \( \Gamma(s, t) \) is the continuous resolving kernel of \( L(s, t) \). So it is possible to construct the series (7) formally.

Now we shall prove that the constructed series converges absolutely and uniformly according to \( s \) and \( \lambda \) in \( \langle a, b \rangle \) for \( \lambda \) sufficiently small. Let us choose such numbers \( A, B, C \) and \( D \) that for \( s \in \langle a, b \rangle \)

\[ |p(s)|^{-1} < A, \quad \int_a^b |\Gamma(s, t)| \, dt < B, \quad |y_0(s)| < C, \]
\[ \int_a^b \int_a^b |L_{s_1 \ldots s_\nu} (s_1 \ldots s_\nu)| \, dt_1 \ldots dt_\nu \leq D_{s_1 \ldots s_\nu}, \quad \max_{s_1 \ldots s_\nu} D_{s_1 \ldots s_\nu} = D \]

is valid and denote

\[ \max_s |\psi(s)| = \Psi, \quad \max (1, B) = L. \]

Then for \( \Psi \) the following equation can be obtained from (6) and (12):

\[ F[\lambda, \Psi] = \Psi - 2LAD \sum_{j=1}^n \sum_{s=0}^n |\mu_0|^s \left\{ C^j H_1[\Psi] + G_2[\Psi] + G_1[\Psi] H_1[\Psi] + (\Psi \lambda + C)^j \sum_{k=1}^{\nu} \left( \frac{\lambda}{|\mu_0|} \right)^k \right\} = 0 \]
where

\[ H_m[\Psi] = \sum_{(a_1 + \ldots + a_v = j - \alpha)} \sum_{l_1 = 0}^{a_1} \ldots \sum_{l_v = 0}^{a_v} (\Psi C^{-1})^{l_1} \ldots (\Psi C^{-1})^{l_v}, \]

\[ m = 1 \text{ if } \sum_{i=1}^v l_i \neq 0, \quad m = 2 \text{ if } \sum_{i=1}^v l_i \neq 0, 1, \]

\[ G_m[\Psi] = \sum_{k=m}^{\infty} \frac{\Psi}{k} C^{j-k}, \quad m = 1, 2. \]

Let us look for a solution of (13) in the form of the power series expansion

\[ \Psi = \sum_{l=1}^{\infty} \lambda^l k_l. \]

Substituting (14) into (13) and comparing coefficients of the same powers of \( \lambda \) we have the following system of equations for \( k_l \)

\[ k_1 = 2LAD \sum_{j=1}^{n} \sum_{a=0}^{j} \left( a j \mu_0 |a|^{a-1} C^j \right), \]

\[ k_2 = 2LA \left( D \sum_{j=1}^{n} \sum_{a=0}^{j} \left( |j| k_1 + \sum_{k=1}^{v} \left( \sum_{m=1}^{v} \alpha_k \alpha_m \right) \right) + \alpha j |j| |j|^{a-1} C^{j-1} k_1 \right) + M_2[C], \]

\[ k_1 = 2LA \left( D \sum_{j=1}^{n} \sum_{a=0}^{j} \left( |j| k_1 + \sum_{k=1}^{v} \left( \sum_{m=1}^{v} \alpha_k \alpha_m \right) \right) + \alpha j |j|^{a-1} C^{j-1} k_1 \right) + M_1[C, k_1, k_2, \ldots, k_{l-2}], \]

where \( M_l[C, k_1, k_2, \ldots, k_{l-2}] \) are upper bounds for \( M_l[y_0, y_1, \ldots, y_{l-2}] \). From these relations and from (9), (10), (11) and (12) it is obvious that

\[ |y_l(s)| < k_l \quad \text{for} \quad s \in \langle a, b \rangle, \quad l = 1, \infty. \]

This implies that the region of convergence of (14) is the region of convergence of (7). From the implicit function theorem according to

\[ \frac{\partial F[\lambda, \Psi]}{\partial \Psi} = 1 \quad \text{for} \quad \lambda = \Psi = 0 \]

there follows that from (13) it is possible to determine \( \Psi \) as an unambiguous and continuous function of \( \lambda \) so that the series (14) has a finite radius of convergence. As
the series (14) is a majorant for (7), the series (7) converges absolutely and uniformly according to $s$ and $\lambda$ in $\langle a, b \rangle$ and in a neighbourhood of the point $\lambda = 0$. Hence, the series (3) represents the unique solution of (1) in the neighbourhood of $\mu = \mu_0$ continuous in $\langle a, b \rangle$ which tends to $y_0(s)$ for $\mu \to \mu_0$.

b) Let number 1 be a $p$-multiple eigenvalue of the kernel $L(s, t)$ with continuous eigenfunctions $\varphi_i(s)$ $(i = 1, p)$ and with continuous associated eigenfunctions $\alpha_i(s)$ $(i = 1, p)$. If equations (8) are to have solutions it is necessary and sufficient that

$$\int_a^b f_i(s) \alpha_i(s) \, ds = 0, \quad l = 1, \infty, \quad i = 1, p,$$

be valid.

b1) Let us assume that (16) is valid for $l = 1$. Then the solution of the first equation from (8) can be written according to the third Fredholm’s theorem

$$y_1(s) = g_1(s) + \sum_{i=1}^p C_i^1 \varphi_i(s)$$

where

$$g_1(s) = \int_a^b \Phi(s, t) f_1(t) \, dt + f_1(s), \quad (l = 1, \infty, \text{for } l = 2, \infty \text{ see further});$$

$\Phi(s, t)$ is the continuous resolving kernel of the kernel

$$L(s, t) - \sum_{i=1}^p \varphi_i(s) \alpha_i(t).$$

For the determination of constants $C_i^1$ $(i = 1, p)$ we obtain from conditions (16) by solving the second equation from (8) after substituting (17) into $f_2(s)$ the following system of $p$ nonlinear equations

$$\int_a^b \frac{\alpha_i(s)}{p(s)} \left< \sum_{j=1}^n \sum_{s=0}^l \left\{ s \left[ (g_1 + \sum_{m=1}^p C_m \varphi_m) s \right] + i \right\} \right> \left[ \left( g_1 + \sum_{m=1}^p C_m \varphi_m \right) \lambda s + \sum_{m=1}^p C_m \varphi_m \right] \lambda s + \sum_{m=1}^p C_m \varphi_m \right] +$$

$$+ \mu_0 y_0^s(s) \left( \mathcal{Q} \left[ \left( g_1 + \sum_{m=1}^p C_m \varphi_m \right)^2 \right] + R \left[ g_1 + \sum_{m=1}^p C_m \varphi_m, g_1 + \sum_{m=1}^p C_m \varphi_m \right] \right) +$$

$$+ x \mu_0 y_0^s(s) \left( \mathcal{Q} \left[ \left( g_1 + \sum_{m=1}^p C_m \varphi_m \right)^2 \right] + R \left[ g_1 + \sum_{m=1}^p C_m \varphi_m, g_1 + \sum_{m=1}^p C_m \varphi_m \right] \right) +$$

$$+ M_2[y_0] \right> \, ds = 0, \quad i = 1, p.$$
The solution of the \( l \)-th \((l \geq 2)\) equation of system (8) can be written in the form

\[
y_l(s) = g_l(s) + \sum_{i=1}^{p} C_l^i \phi_i(s) .
\]

From conditions (16) by solving the \((l + 1)\)-st equation from (8) after substituting (19) into \( f_{l+1}(s) \) we obtain the nonhomogeneous system of \( p \) linear equations for \( C_l^i \)

\[
\int_a^b \alpha_i(s) K \left[ \sum_{m=1}^{p} C_m^l \phi_m \right] ds = m_l^i , \quad i = 1, p
\]

where

\[
m_l^i = \int_a^b \alpha_i(s) \left( \frac{1}{p(s)} M_{l+1}(y_0, y_1, \ldots, y_{l-1}) - K[g_l] \right) ds .
\]

Under the assumption that the determinant of system (20) is different from zero it is possible to determine \( C_l^i \) \((i = 1, p)\) uniquely in the form

\[
C_l^i = \sum_{k=1}^{p} F_{ik} m_k^l , \quad i = 1, p ,
\]

and so to determine the functions \( y_{lj}(s) \) \((j = 1, 2^p)\) uniquely.

Therefore it is possible to construct \( 2^p \) series of the type (7). The convergence of these series may be proved in the following way. Let us consider two sequences \( \{u_i\}^\infty_{i=0} , \{v_i\}^\infty_{i=0} \) of such numbers \( u_i , v_i \) that

\[
|y_1(s)| \leq |g_1(s)| + \sum_{i=1}^{p} |C_l^i| |\phi_i(s)| < u_0 + v_0 \quad \text{for} \quad s \in \langle a, b \rangle , \quad l = 1, \infty
\]

and

\[
|y_{l+1}(s)| \leq |g_{l+1}(s)| + \sum_{i=1}^{p} |C_{l+1}^i| |\phi_i(s)| < u_l + v_l \]

holds. Let us choose constants \( u_0 , v_0 \) so that the inequality mentioned above is valid and for determination of constants \( u_l , v_l \) \((l = 1, \infty)\) consider the function

\[
E(z) = AD \sum_{j=1}^{n} \sum_{a=0}^{j} |\mu|^a \left\{ C^j H_2[z] + G_2[z] + \sum_{k=1}^{\alpha} \left( \frac{\lambda}{|\mu_0|} \right)^k (z + C)^l + \right.
\]

\[
\left. + G_1[z] H_1[z] - \frac{\alpha \lambda}{|\mu_0|} C^l \right\}
\]

where the notations \( G_1[z] , H_1[z] , G_2[z] \) and \( H_2[z] \) have the same meaning as in (13). If we put instead of \( z \)

\[
z = \sum_{l=0}^{\infty} \lambda^{l+1} (u_l + v_l)
\]
into (22) and if we expand the expression obtained by powers of $\lambda$ then

$$E(z) = \sum_{l=2}^{\infty} \lambda^l E_l$$

where

$$E_2 = AD \left\{ D \sum_{j=1}^{n} \sum_{a=0}^{j} \left\{ |\mu_0|^a C^{j-2}(u_0 + v_0)^2 \left[ \sum_{(a_1 + \ldots + a_{v-1})=j-2} \left( \sum_{k=1}^{v} \frac{\alpha_k}{2} \right)^2 \right. \right. \right.$$

$$+ \sum_{k=1}^{v} \sum_{m=k+1}^{v-1} \alpha_k \alpha_m \left( \frac{\alpha}{2} \right) + \alpha (j - \alpha) \right. + \alpha j |\mu_0|^{a-1} C^{j-1}(u_0 + v_0) \right\} + \bar{M}[C] \right\},$$

$$E_{l+2} = K(u_l + v_l) + A\bar{M}_{l+2}[C, u_0 + v_0, \ldots, u_{l-1} + v_{l-1}], \quad l = 1, \infty$$

where

$$K = AD \sum_{j=1}^{n} \sum_{a=0}^{j} \left\{ |\mu_0|^a C^{j-2}(u_0 + v_0)^2 \left[ \sum_{(a_1 + \ldots + a_{v-1})=j-2} \left( \sum_{k=1}^{v} \frac{\alpha_k}{2} \right)^2 \right. \right. \right.$$

$$+ \sum_{k=1}^{v} \sum_{m=k+1}^{v-1} \alpha_k \alpha_m \left( \frac{\alpha}{2} \right) + \alpha (j - \alpha) \right. + \alpha j |\mu_0|^{a-1} C^{j-1} \right\}$$

and $\bar{M}_{l+2}$ are upper bounds for $M_{l+2}$. From relations (9) and (12) there follows that $E_l$ are upper bounds for the functions $f_l(s)$.

Further, let us choose such numbers $N, \Phi, \beta$ and $\beta_1$ that

$$\max_{i,k} |F_{ik}| = N,$$

$$\int_a^b |\Phi(s, t)| \, dt < \Phi, \quad s \in \langle a, b \rangle,$$

$$\max_i |\varphi_i(s)| < \beta, \quad \max_i \int_a^b |\alpha_i(s)| \, ds = \beta_1$$

is valid and designate $\max (1, \Phi) = K$. Then we determine $u_l (l = 1, \infty)$ from the equation

$$u_l = 2KE_{l+1}.$$

If we take into account that

$$|g_{l+1}(s)| < \max_s |f_{l+1}(s)| \left( 1 + \int_a^b |\Phi(s, t)| \, dt \right) < 2KE_{l+1}$$

then $|g_{l+1}(s)| < u_l$. 260
The constants $v_t (l = 1, \infty)$ can be determined from the equation

$$ (1 + dK) v_t = dE_{t+2} $$

where

$$ d = p^2 \beta_1 N. $$

As there is

$$ \left| C_i^{l+1} \right| \leq N \sum_{k=1}^{p} \left| m_k^{l+1} \right| < N \beta_1 P \mathcal{Q}, \quad \sum_{i=1}^{p} \left| C_i^{l+1} \right| \left| \varphi_i(s) \right| < d \mathcal{Q} $$

where

$$ \mathcal{Q} = K u_1 + A M_{l+2} [C, u_0 + v_0, \ldots, u_{l-1} + v_{l-1}] $$

and $v_t = d \mathcal{Q}$ according to equation (27), we have $\sum_{i=1}^{p} \left| C_i^{l+1} \right| \left| \varphi_i(s) \right| < v_t$ and so $\left| v_{l+1}(s) \right| < u_l + v_t$.

If we introduce the notation

$$ u = \sum_{i=1}^{\infty} \lambda^{l+1} u_i, \quad v = \sum_{i=1}^{\infty} \lambda^{l+1} v_i, $$

then $z = \lambda (u_0 + v_0) + u + v$ and the determination of $u_t, v_t$ from (26) and (27) is equivalent to solving the following system for $u, v$:

$$ u = 2KE(\lambda (u_0 + v_0) + u + v), $$
$$ (1 + dK) \lambda v = d(E(\lambda (u_0 + v_0) + u + v) - \lambda^2 E_2) $$

in the form (28). If we perform the substitution

$$ u = \lambda U, \quad v = \lambda V $$

in (29) and devide the first equation by $\lambda$ and the second one by $\lambda^2$ we obtain for $U, V$ the system

$$ \Phi_1 \equiv U - \frac{2K}{\lambda} E(\lambda (u_0 + v_0 + U + V)) = 0, $$
$$ \Phi_2 \equiv (1 + dK) V - d \left( \frac{1}{\lambda^2} E(\lambda (u_0 + v_0 + U + V)) - E_2 \right) = 0. $$

For system (30) we use the implicit function theorem. If system (30) is to determine unambiguous continuous functions $U(\lambda), V(\lambda)$ in a neighbourhood of the point $\lambda = 0$ it is necessary and sufficient that

$$ \Delta = \frac{D(\Phi_1, \Phi_2)}{D(U, V)} \neq 0 \quad \text{for} \quad \lambda = U = V = 0. $$
As for $\lambda = U = V = 0$,
\[
\frac{\partial \Phi_1}{\partial U} = 1, \quad \frac{\partial \Phi_2}{\partial V} = 0, \quad \frac{\partial \Phi_2}{\partial U} = -dK, \quad \frac{\partial \Phi_2}{\partial V} = 1
\]
holds and so $A = 1$, the assumptions of the theorem mentioned above are fulfilled and system (30) has only one continuous solution $U, V$ in the form of the series
\[
U = \sum_{i=1}^{\infty} \lambda^i u_i, \quad V = \sum_{i=1}^{\infty} \lambda^i v_i
\]
which have a finite radius of convergence in a neighbourhood of the point $\lambda = 0$. The same is valid for series (23). As this series is a majorant for (7), series (7) converges absolutely and uniformly according to $s$ and $\lambda$ in $[a, b]$ and in a neighbourhood of the point $\mu_0$ and because of the continuity of its terms the limit functions $\psi_j(s)$ are continuous in $[a, b]$. Hence, in the neighbourhood of the point $\mu = \mu_0$ there exist, in general, $2^p$ solutions of equation (1) in the form (3) which tend to $y_0(s)$ for $\mu \to \mu_0$.

b2) If any of conditions (16) for $l = 1$ is not fulfilled it is not possible to solve equations (8) and the problem of determination of the number of solutions of equation (6) for $\mu$ from a neighbourhood of $\mu_0$ becomes more complicated. Such solutions can be sought in the form
\[
\psi(s) = \sum_{i=1}^{\infty} (\mu - \mu_0)^{1/k} y_i(s)
\]
where $k$ is a positive integer. The functions $y_i(s)$ can be determined from a system of linear integral equations obtained with the aid of substitution (32) in (6) and by comparison of coefficients of the same powers of $(\mu - \mu_0)^{1/k}$. For example, for $k = 2$, i.e.
\[
\psi(s) = \sum_{i=1}^{\infty} v^i y_i(s), \quad v = (\mu - \mu_0)^{1/2}
\]
we obtain the system
\[
y_1(s) - \int_a^b L(s, t) y_1(t) \, dt = h_1(s), \quad l = 1, \infty
\]
where
\[
h_1(s) = 0,
\]
\[
h_2(s) = \frac{-1}{p(s)} \sum_{j=1}^{n} \sum_{\alpha=0}^{j} (\mu_0^{\alpha} y_0^{\alpha}(s)) \left( Q[y_1^2] + R[y_1, y_1] + q[y_1^3] L[y_0^{\alpha}, \ldots, y_0^{\alpha}] + r[y_1] S[y_1] + M_2[y_0] \right),
\]
The solution of the first equation from (34) can be written in the form

\[ y_1(s) = \sum_{i=1}^{p} D_i^1 \phi_i(s). \]

If other equations from (34) are to have solutions it is necessary and sufficient to fulfil the conditions

\[ \int_a^b h_i(s) \alpha_i(s) \, ds = 0, \quad i = 1, p, \quad l = 2, \infty. \]

If we substitute (36) into (37) when \( l = 2 \) we obtain the following system of \( p \) non-linear equations for \( D_i^1 \) (\( i = 1, p \))

\[ \int_a^b \alpha_i(s) \left< \sum_{j=1}^{p} \sum_{a=0}^{1} \{ \mu_{a,y}^y(s) \{ Q[\sum_{m=1}^{p} D_{m=1}^1 \phi_{m=1}^m] + R[\sum_{m=1}^{p} D_{m=1}^1 \phi_{m=1}^m, \sum_{m=1}^{p} D_{m=1}^1 \phi_{m=1}^m] \right) + \\
+ q[\sum_{m=1}^{p} D_{m=1}^1 \phi_{m=1}^m] L_j[y_0, \ldots, y_0] + r'[\sum_{m=1}^{p} D_{m=1}^1 \phi_{m=1}^m] S[\sum_{m=1}^{p} D_{m=1}^1 \phi_{m=1}^m] \right) + \\
+ M_2[y_0] \right) \, ds = 0. \]

From (38) we obtain, in general, \( 2^p \) systems \( D_i^1 \) (\( i = 1, p \)) and so we have \( 2^p \) functions \( y_i(s) \)

\[ y_{1j}(s) = \sum_{i=1}^{p} D_{i=1}^1 \phi_{i=1}(s), \quad j = 1, 2^p. \]

The solution of the \( l \)-th (\( l \geq 3 \)) equation from (34) can be written

\[ y_i(s) = g_i(s) + \sum_{i=1}^{p} D_i^1 \phi_i(s) \]

where

\[ g_i(s) = \int_a^b \Phi(s, t) h_i(t) \, dt + h_i(s); \]
\( \Phi(s,t) \) has the same meaning as in the section bl). For \( D_i^l (i = 1, p) \) we obtain from conditions (37) after substituting (39) into \( h_{i+1}(s) \) the system of \( p \) linear equations

\[
\sum_{m=1}^{p} D_i^l \int_a^b \alpha_i(s) G[\varphi_m] \, ds = d_i^l, \quad i = 1, p
\]

where

\[
d_i^l = \int_a^b \alpha_i(s) \left( \frac{1}{p(s)} M_{i+1}[y_0, y_1, \ldots, y_{l-1}] - G[g_i] \right) \, ds.
\]

Under the assumption that the determinant of the system (40) is different from zero it is possible to determine \( D_i^l \) uniquely in the form

\[
D_i^l = \sum_{k=1}^{p} H_{ik} d_k^l
\]

and so to determine the functions \( y_{ij}(s) (j = 1, 2^p) \) uniquely.

Therefore it is possible to construct, in general, \( 2^p \) series of the type (33). The proof of convergence of these series in a neighbourhood of the point \( v = 0 \) will be carried out analogically as that in the section bl). Let us choose such a constant \( v_0 \) that

\[
|y_1(s)| \leq \sum_{i=1}^{p} |D_i^l| |\varphi_i(s)| < v_0 \quad \text{for} \quad s \in (a, b)
\]

is valid. Further, let us consider the function

\[
S(z) = AD \sum_{j=1}^{n} \sum_{a=0}^{j} |\mu_a|^a \left\{ C^j H_2[z] + G_2[z] + \sum_{k=1}^{a} \binom{a}{k} \left( \frac{v^2}{|\mu_0|} \right)^k (z + C)^j + G_1[z] H_1[z]\right\}
\]

where the symbols \( G_1[z], G_2[z], H_1[z], H_2[z] \) have the same meaning as in (13). If we put instead of \( z \)

\[
z = vv_0 + \sum_{i=1}^{\infty} v^{l+1}(u_i + v_i)
\]

in \( S(z) \) and expand the expression obtained in powers of \( v \), we obtain

\[
S(z) = \sum_{i=2}^{\infty} v^i S_i
\]

where \( S_i \) are upper bounds for the functions \( h_i(s) \) if \( (u_{i-1} + v_{l-1}) \) are upper bounds for \( y_i(s) (l = 2, \infty) \).
Let us choose such a number $H$ that
\[
\max_{i, k} |H_{i k}| = H
\]
holds. Then we determine $u_l$ and $v_l$ ($l = 1, \infty$) from the equations
\[
(45) \quad u_l = 2KS_{l+1}, \quad (1 + e\bar{G}) v_l = eS_{l+2}
\]
where $e = p^2\beta_1 H$ and
\[
\bar{G} = AD \sum_{j=1}^{n} \sum_{\alpha=0}^{j} |\alpha_0|^2 C^{j-2} v_0 \left\{ \left( \sum_{(\alpha_1 + \ldots + \alpha_p) = j-\alpha} \left( \frac{\alpha_k}{2} \right) + \sum_{k=1}^{\alpha} \sum_{m=1}^{\alpha_k} \alpha_k \alpha_m \right) + 2 \left( \frac{\alpha}{2} \right) + 2\alpha(j - \alpha) \right\}.
\]
We can easily see that
\[
|y_{l+1}(s)| < u_l + v_l.
\]
If we introduce the notation
\[
(46) \quad U = \sum_{l=1}^{\infty} v^l u_l, \quad V = \sum_{l=1}^{\infty} v^l v_l,
\]
then $z = v(v_0 + U + V)$ and the determination of $u_l, v_l$ from equations (45) is equivalent to the solution of the system for $U, V$
\[
(47) \quad \Phi_1 \equiv U - \frac{2K}{v} S(v(v_0 + U + V)) = 0
\]
\[
\Phi_2 \equiv (1 + e\bar{G}) V - e \left( \frac{1}{v^2} S(v(v_0 + U + V)) - S_2 \right) = 0
\]
in the form (46). As for $v = U = V = 0$, there is $A = 1$, the assumption (31) is fulfilled and from system (47) it is possible to determine $U$ and $V$ as unambiguous and continuous functions of $v$. From analogical considerations as in the section b1) there follows that the series (33) converges absolutely and uniformly according to $s$ and $\nu$ in $\langle a, b \rangle$ and in a neighbourhood of the point $v = 0$ to functions $(y_j(s) - y_0(s))$ ($j = 1, 2^p$) which are continuous in $\langle a, b \rangle$.

Hence, the assertions of the theorem are proved.

Reference


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