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# BOOLEAN CONCEPT LATTICES AND GOOD CONTEXTS

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*Summary.* Concept lattices were introduced by R. Wille. The notion of a good context was used by M. Novotný and Z. Pawlak in the analysis of black boxes. The present paper shows that good contexts are closely related to the Boolean concept lattices, and yice versa.

*Keywords:* Concept lattice, good context, Boolean algebra.

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## 1. PRELIMINARIES

The rudiments of the formal concept analysis were built by R. Wille in [7]. Further development of this theory is due to the Research Group of Formal Concept Analysis at the Technische Hochschule Darmstadt. Let us recall that a *context* $\boldsymbol{T}$ is a triple $\boldsymbol{T} = \langle G, M, r\rangle$ where

(i) $G$ is a finite nonvoid set (of objects),

(ii) $M$ is a finite nonvoid set (of attributes),

(iii) $r$ is a correspondence from $G$ to $M$, i.e. $r \subseteq G \times M$ ($\langle g, m\rangle \in r$ means that the object $g \in G$ has the attribute $m \in M$).

Apparently the correspondence $r$ can be represented by an *incidence matrix* in a natural way.

Denote by $\mathbf{B}(M)$, $\mathbf{B}(G)$, the set of all subsets of $M$, $G$, respectively. It is well known that the mappings

$$\mathbf{B}(M) \underset{s}{\overset{t}{\rightleftarrows}} \mathbf{B}(G)$$

defined by the rules

$$t(N) = \{g \in G;\ \langle g, m\rangle \in r \text{ for any } m \in N\}\,,\quad N \subseteq M\,,$$

and

$$s(H) = \{m \in M;\ \langle g, m\rangle \in r \text{ for any } g \in H\}\,,\quad H \subseteq G\,,$$

establish a Galois connection between the posets $\langle \mathbf{B}(M), \subseteq\rangle$ and $\langle \mathbf{B}(G), \subseteq\rangle$, see e.g. [2]. Then

(i) $p = s \circ t$: $\mathbf{B}(G) \to \mathbf{B}(G)$ and $q = t \circ s$: $\mathbf{B}(M) \to \mathbf{B}(M)$ are closure operators;

(ii) $t(s(t(N))) = t(N)$ for any $N \subseteq M$ and so the set of all $p$-closed subsets in $G$, denoted by $\mathbf{C}(\boldsymbol{T})$, can be expressed in the form $\mathbf{C}(\boldsymbol{T}) = \{t(N); N \subseteq M\}$. Analogously

$\mathbf{D}(T)$, the set of all $q$-closed subsets in $M$, can be written in the form $\mathbf{D}(T) = \{s(H); H \subseteq G\}$ since $s(t(s(H))) = s(H)$ for any $H \subseteq G$.

A *concept* is a pair $\langle t(X), q(X)\rangle$, $X \subseteq M$. All concepts ordered by the rule $\langle t(X), q(X)\rangle \leqq \langle t(Y), q(Y)\rangle$ if $t(X) \subseteq t(Y)$ form the so-called *concept lattice* $\mathfrak{B}(T)$.

Further, let us recall from [3] that a set of attributes $X \subseteq M$ *depends* on a set of attributes $Y \subseteq M$ whenever the inclusion $t(X) \supseteq t(Y)$ holds. Finally, it will be useful to introduce the notion of an isomorphism of contexts.

Let $T = \langle G, M, r\rangle$ and $T' = \langle G', M', r'\rangle$ be contexts. A pair $i = \langle \alpha, \beta\rangle$ of bijections $\alpha: G \to G'$, $\beta: M \to M'$ is called an *isomorphism* of $T$ onto $T'$ whenever $\langle g, m\rangle \in r$ is equivalent to $\langle \alpha(g), \beta(m)\rangle \in r'$ for any $\langle g, m\rangle \in G \times M$. We write $i: T \underset{\to}{\cong} T'$, or briefly $T \cong T'$.

Now let us turn our attention to the good contexts. The notion of a good contex was introduced by M. Novotný and Z. Pawlak in [6] as a context corresponding to the so-called good black box, see [5] for good black boxes. To make this paper selfcontained we take a simple characterization of good contexts from [6; Thm 4.6] as a fundamental definition for our further investigations. Denote $\Theta = \{\langle x, y\rangle \in M \times M; t(\{x\}) = t(\{y\})\}$. Evidently, $\Theta$ is an equivalence on $M$, put $[X]\,\Theta = \bigcup_{x \in X} [x]\,\Theta$ for any $X \subseteq M$. Now we are ready to formulate

**Definition 1.** Let $T = \langle G, M, r\rangle$ be a context. $T$ is called a *good context* if $t(X) = t(Y)$ implies $[X]\,\Theta = [Y]\,\Theta$ for any $X, Y \subseteq M$.

If $M_\Theta$ denotes an arbitrary set of representatives of the equivalence $\Theta$ on $M$ then clearly $\mathfrak{B}(\langle G, M, r\rangle) \cong \mathfrak{B}(\langle G, M_\Theta, r \cap G \times M_\Theta\rangle)$ holds. Roughly speaking, the concept lattice $\mathfrak{B}(\langle G, M, r\rangle)$ does not depend on the repeating columns in the incidence matrix representing $r$. For these reasons the contexts satisfying the equality $\Theta = \omega_M$ are prefered. They are named *reduced* contexts in the present paper. The following simplification is at hand.

**Proposition 1.** *Let $T = \langle G, M, r\rangle$ be a context. The following conditions are equivalent:*

(1) *$T$ is a reduced good context;*

(2) *the mapping $t$: $\mathbf{B}(M) \to \mathbf{B}(G)$ is injective.*

Proof. Immediate.

## 2. GOOD CONTEXTS AND THE DEPENDENCE OF ATTRIBUTES

The following simple observation will be frequently used in the sequel:

**Lemma 1.** *Let $T = \langle G, M, r\rangle$ be an arbitrary context. Then for any subsets $X_1, \ldots, X_k \subseteq M$ we have $t(\bigcup_{i=1}^{k} X_i) = \bigcap_{i=1}^{k} t(X_i)$. In particular, $t(X) = \bigcap_{x \in X} t(\{x\})$ holds for any $X \subseteq M$.*

Proof. Trivial.

**Theorem 1.** *Let $T = \langle G, M, r\rangle$ be a context. The following conditions are equivalent:*

(1) *$T$ is a reduced good context;*

(2) *$t(\{m\}) \not\supseteq t(M \setminus \{m\})$ for every $m \in M$;*

(3) *a set of attributes $X \subseteq M$ depends on a set of attributes $Y \subseteq M$ iff $X \subseteq Y$.*

Proof. (1) $\Rightarrow$ (2): Suppose on the contrary that $t(\{m\}) \supseteq t(M \setminus \{m\})$ for an element $m \in M$. Then $t(M) = t((M \setminus \{m\}) \cup \{m\}) = t(M \setminus \{m\}) \cap t(\{m\}) = t(M \setminus \{m\})$. Since $M \neq M \setminus \{m\}$ the mapping $t$: $\mathbf{B}(M) \to \mathbf{B}(G)$ is not injective, a contradiction.

(2) $\Rightarrow$ (3): Suppose that $t(X) \supseteq t(Y)$ and $X \setminus Y \neq \emptyset$ for some subsets $X, Y \subseteq M$. Then for any element $x \in X \setminus Y$ we have $t(\{x\}) \supseteq t(X) \supseteq t(Y) \supseteq t(M \setminus \{x\})$, a contradiction.

(3) $\Rightarrow$ (1): The required implication "$X = Y$ whenever $t(X) = t(Y)$" is a direct consequence of the hypothesis "$X \subseteq Y$ whenever $t(X) \supseteq t(Y)$". The proof is complete.

Remark 1. For $|M| > 1$ condition (2) from Theorem 1 can be reformulated as follows:

(2′) *the subsets $t(\{m\})$, $t(M \setminus \{m\}) \subseteq G$ form an antichain for every $m \in M$.*

**Corollary 1.** *Let $T = \langle G, M, r\rangle$ be a reduced good context. Then for any sets $H \supseteq G$ and $N \subseteq M$, $N \neq \emptyset$ the following assertions hold:*

(1) *the subcontext $\langle G, N, r \cap G \times N\rangle$ of $T$ is a reduced good context;*

(2) *any supercontext $\langle H, M, u\rangle$, $u \cap G \times M = r$, of $T$ is a reduced good context.*

Remark 2. It follows directly from Theorem 1 that the subsets $t(\{m\})$, $m \in M$, form an antichain whenever $\langle G, M, r\rangle$ is a reduced good context. The converse is false, see e.g. the context of equality $\langle G, G, =\rangle$, $|G| \neq 2$. Something more will be proved for the above mentioned subsets $t(\{m\})$, $m \in M$. To do this we need the context of inequality $\langle G, G, \neq\rangle$.

Example 1. The *context of inequality* $\langle G, G, \neq\rangle$, i.e. the context represented by the following incidence matrix is evidently a reduced good context, see e.g.

$$\begin{array}{c|ccccc}
\neq & & & G & & \\
\hline
 & 0 & 1 & 1 & \dots & 1 \\
 & 1 & 0 & 1 & \dots & 1 \\
G & 1 & 1 & 0 & \dots & 1 \\
 & \dots & & & & \cdot \\
 & 1 & 1 & 1 & \dots & 0
\end{array}$$

Theorem 1 (2). To emphasize the crucial role of the context $\langle G, G, \neq \rangle$ we will denote by $t_0, s_0$ the mappings $t, s$, respectively, introduced in Section 1. Notice that $t_0(A) = s_0(A) = G \setminus A$ and thus $s_0(t_0(A)) = t_0(s_0(A)) = A$ for any subset $A \subseteq G$. Further, one can easily verify the well-known fact (see also [4]) that $\mathfrak{B}(\langle G, G, \neq \rangle) \cong \cong \mathbf{2}^{|G|}$.

## 3. GOOD CONTEXTS AND IRREDUNDANT SUBSET SYSTEMS

**Definition 2.** Let $S$ be a nonvoid set, $k$ a positive integer. We say that the nonvoid subsets $X_i \subseteq S$, $i = 1, \ldots, k$, form an *irredundant subset system* in $S$ whenever $X_i \nsubseteq \bigcup_{\substack{j=1 \\ j \neq i}}^{k} X_j$ for every $i = 1, \ldots, k$.

**Lemma 2.** *Let $\{X_i;\ i \leqq k\}$ be an irredundant subset system in a set $S$. Then there is a subset $\{x_i;\ i \leqq k\}$ of $S$ such that $x_i \in X_j$ iff $i = j$ for $i, j \in \{1, \ldots, k\}$.*

Proof. Let $\{X_i;\ i \leqq k\}$ be an irredundant subset system in $S$. Fix $i \leqq k$ and suppose that every element $x \in X_i$ belongs to some $X_{j(x)} \neq X_i$. Then $X_i \subseteq \bigcup_{\substack{j=1 \\ j \neq i}}^{k} X_j$, a contradiction.

**Definition 3.** Let $\{X_i;\ i \leqq k\}$ be an irredundant subset system in a set $S$. A subset $\{x_i;\ i \leqq k\}$ of $S$ with the property exhibited in Lemma 2 is called a *representative set* of $\{X_i;\ i \leqq k\}$; $x_i$ is named a *representative* of $X_i$ for $i = 1, \ldots, k$.

**Lemma 3.** *Let $\{X_i;\ i \leqq k\}$ be an irredundant subset system in a finite set $S$. Then*
(a) $k \leqq |S|$;
(b) *if $k = |S|$ then $X_i$ is a singleton for $i = 1, \ldots, k$.*

Otline of the proof: Apply the concept of a representative set.

If there is a danger of misunderstanding we will write $t_T$ instead of $t$ to denote the mapping $t\colon \mathbf{B}(M) \to \mathbf{B}(G)$ determined by the context $T = \langle G, M, r \rangle$.

**Theorem 2.** *Let $T = \langle G, M, r \rangle$ be a reduced good context, $\{X_i;\ i \leqq n\}$ an irredundant subset system in $M$, and $\{y_i;\ i \leqq n\}$ an arbitrary n-element set. Then the context $U = \langle G, \{y_i;\ i \leqq n\}, u \rangle$ defined by the rule $t_U(\{y_i\}) = t_T(X_i)$, $i = = 1, \ldots, n$, is a reduced good context.*

Proof. Apparently $U$ is a reduced context. Since $T$ is a reduced good context, $X_j \nsubseteq \bigcup_{\substack{i=1 \\ i \neq j}}^{n} X_i$ implies $t_T(X_j) \nsupseteq t_T(\bigcup_{\substack{i=1 \\ i \neq j}}^{n} X_i)$, see Theorem 1 (3). Further $t_T(X_j) = t_U(\{y_j\})$,

$$t_T\Big(\bigcup_{\substack{i=1\\ i\neq j}}^{n} X_i\Big) = \bigcap_{\substack{i=1\\ i\neq j}}^{n} t_T(X_i) = \bigcap_{\substack{i=1\\ i\neq j}}^{n} t_U(\{y_i\}) = t_U(\{y_i;\ i \leqq n\} \setminus \{y_j\}),$$

and so we conclude that $t_U(\{y_j\}) \not\supseteq t_U(\{y_i;\ i \leqq n\} \setminus \{y_j\})$ for any $j = 1, \ldots, n$. Theorem 1 (2) completes the proof.

**Theorem 3.** *Let $T = \langle G, M, r\rangle$ be a reduced good context. Then there exists a uniquely determined irredundant subset system $\{X_m;\ m \in M\}$ in $G$ such that $t_0(X_m) = t(\{m\})$ for any $m \in M$.*

Proof. Take an arbitrary element $m \in M$ and consider the subset $t(\{m\}) \subseteq G$. Applying the context of inequality $\langle G, G, \neq\rangle$ we get the subset $s_0(t(\{m\})) \subseteq G$. Put $X_m = s_0(t(\{m\}))$. Clearly $t_0(X_m) = t_0(s_0(t(\{m\}))) = t(\{m\})$. It remains to verify that $\{X_m;\ m \in M\}$ is an irredundant subset system in $G$. Suppose on the contrary that $X_m \subseteq \bigcup_{\substack{i \in M\\ i \neq m}} X_i$ for some $m \in M$. Then $t_0(X_m) \supseteq t_0(\bigcup_{\substack{i \in M\\ i \neq m}} X_i) = \bigcap_{\substack{i \in M\\ i \neq m}} t_0(X_i)$ follows from Lemma 1. It was already shown in the first part of this proof that $t_0(X_m) = t(\{m\})$, $m \in M$, and so we have $t(\{m\}) \supseteq \bigcap_{\substack{i \in M\\ i \neq m}} t(\{i\}) = t(\bigcup_{\substack{i \in M\\ i \neq m}} \{i\}) = t(M \setminus \{m\})$. However, the inclusion obtained does not hold, see Theorem 1 (2), a contradiction.

**Corollary 2.** *For a context $T = \langle G, M, r\rangle$ we have:*
*(1) if $|G| < |M|$ then $T$ is not a reduced good context;*
*(2) if $|G| = |M|$ then $T$ is a reduced good context iff $T \cong \langle G, G, \neq\rangle$.*

Proof. (1) Combine Lemma 3 (a) with Theorem 3.
(2) Combine Lemma 3 (b) with Theorem 3.
The following example shows how a given reduced good context $T = \langle G, M, r\rangle$ can be obtained from the context of inequality $\langle G, G, \neq\rangle$.

Example 2. One can easily verify that the context $T = \langle\{g_1, \ldots, g_5\}, \{m_1, m_2, m_3\}, r\rangle$ represented by the incidence matrix on the left hand side is a reduced good context.

| $r$ | $m_1$ | $m_2$ | $m_3$ |
|---|---|---|---|
| $g_1$ | 0 | 1 | 1 |
| $g_2$ | 0 | 0 | 1 |
| $g_3$ | 1 | 0 | 1 |
| $g_4$ | 1 | 1 | 0 |
| $g_5$ | 1 | 1 | 1 |

| $\neq$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
|---|---|---|---|---|---|
| $g_1$ | 0 | 1 | 1 | 1 | 1 |
| $g_2$ | 1 | 0 | 1 | 1 | 1 |
| $g_3$ | 1 | 1 | 0 | 1 | 1 |
| $g_4$ | 1 | 1 | 1 | 0 | 1 |
| $g_5$ | 1 | 1 | 1 | 1 | 0 |

**Theorem 4.** *Let $T = \langle G, M, r \rangle$ be a context. The following conditions are equivalent*:

(1) *$T$ is a reduced good context*;

(2) *the mapping $s$: $\mathbf{B}(G) \to \mathbf{B}(M)$ is onto*;

(3) $\mathfrak{B}(T) \cong \mathbf{2}^{|M|}$;

(4) *there is a subset $H \subseteq G$ such that $\langle H, M, r \cap H \times M \rangle \cong \langle M, M, \neq \rangle$.*

Proof. $T$ is a reduced good context iff the mapping $t$: $\mathbf{B}(M) \to \mathbf{B}(G)$ is injective, see Proposition 1. Then the equivalence $(1) \Leftrightarrow (2)$ is a consequence of the fact that the mappings $t$ and $s$ form a Galois connection.

$(2) \Leftrightarrow (3)$ is evident.

$(1) \Rightarrow (4)$: By Theorem 3 there is an irredundant subset system $\{X_m;\ m \in M\}$ in $G$ which enables us to derive the context $T$ from the context of inequality $\langle G, G, \neq \rangle$. Let $\{x_m;\ m \in M\} \subseteq G$ be a representative set of $\{X_m;\ m \in M\}$. Now it is a routine to verify that assertion (4) holds for $H = \{x_m;\ m \in M\}$.

$(4) \Rightarrow (1)$ follows directly from Corollary 1 (2).

Dually we have

**Theorem 5.** *Let $T = \langle G, M, r \rangle$ be a context. The following conditions are equivalent*:

(1) *the mapping $t$: $\mathbf{B}(M) \to \mathbf{B}(G)$ is onto*;

(2) *the mapping $s$: $\mathbf{B}(G) \to \mathbf{B}(M)$ is injective*;

(3) $\mathfrak{B}(T) \cong \mathbf{2}^{|G|}$;

(4) *there is a subset $N \subseteq M$ such that $\langle G, N, r \cap G \times N \rangle \cong \langle G, G, \neq \rangle$.*

**Corollary 3.** *Let $T = \langle G, M, r \rangle$ be a context. The following conditions are equivalent*:

(1) *the mappings $t, s$ are injective*;

(2) *the mappings $t, s$ are onto*;

(3) *the mapping $t$ is a bijection*;

(4) *the mapping $s$ is a bijection*;

(5) $|G| = |M|$ *and* $\mathfrak{B}(T) \cong \mathbf{2}^{|G|}$ $(= \mathbf{2}^{|M|})$;

(6) $|G| = |M|$ *and* $T \cong \langle G, G, \neq \rangle$ $(\cong \langle M, M, \neq \rangle)$;

(7) *the mappings $t, s$ are mutually inverse, i.e.* $p = s \circ t = 1_{\mathbf{B}(G)}$ *and* $q = t \circ s = 1_{\mathbf{B}(M)}$.

Proof. Immediate.

Remark 3. An implicit form of Theorem 4 (3) can be found in [5]. Notice that neither this condition nor Theorem 5 (3) can be weakened to

(3′) *$\mathfrak{B}(T)$ is a Boolean algebra*,

see the following counterexamples.

Example 3. Consider the contexts $T_i = \langle G, M, r_i \rangle$, $i = 1, 2$, where $G = \{g_1, g_2, g_3\}$, $M = \{m_1, m_2, m_3\}$ and where $r_1, r_2$ are represented by incidence matrices

(1)

| $r_1$ | $m_1$ | $m_2$ | $m_3$ |
|---|---|---|---|
| $g_1$ | 0 | 0 | 1 |
| $g_2$ | 0 | 1 | 0 |
| $g_3$ | 1 | 1 | 1 |

(2)

| $r_2$ | $m_1$ | $m_2$ | $m_3$ |
|---|---|---|---|
| $g_1$ | 1 | 1 | 1 |
| $g_2$ | 0 | 1 | 1 |
| $g_2$ | 1 | 0 | 1 |

One can easily verify that

(1) the mapping $t_{T_1}$ is not injective,

(2) the mapping $t_{T_2}$ is not onto,

but $\mathfrak{B}(T_1) \cong \mathfrak{B}(T_2) \cong \mathbf{2}^2$.

## 4. BOOLEAN CONCEPT LATTICES IN GENERAL

Our last Remark 3 motivates Theorem 6. For the proof, we need

**Lemma 4.** *Let $\mathfrak{B}(T) \cong \mathbf{2}^n$ for a context $T = \langle G, M, r \rangle$ and a positive integer $n$. If $Y_1, \ldots, Y_n$ are coatoms in $\langle \mathbf{C}(T), \subseteq \rangle \cong \mathfrak{B}(T)$ then $\{s_0(Y_i);\ i \leqq n\}$ is an irredundant subset system in $G$.*

Proof. Suppose on the contrary that $s_0(Y_i) \subseteq \bigcup_{\substack{j=1 \\ j \neq i}}^{n} s_0(Y_j)$ for some $i \leqq n$. Applying the mapping $t_0$ to this inclusion, we find that

$$Y_i = t_0(s_0(Y_i)) \supseteq t_0\Big(\bigcup_{\substack{j=1 \\ j \neq i}}^{n} s_0(Y_j)\Big) = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} t_0(s_0(Y_j)) = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Y_j\,.$$

Denote by $\vee$ the join in the Boolean algebra $\langle \mathbf{C}(T), \subseteq \rangle$. Then

$$Y_i = Y_i \vee Y_i \supseteq Y_i \vee \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Y_j = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} Y_i \vee Y_j = \bigcap_{\substack{j=1 \\ j \neq i}}^{n} G = G\,,$$

a contradiction.

**Theorem 6.** *Let $T = \langle G, M, r \rangle$ be a context, $n$ a positive integer. The following conditions are equivalent:*

(1) $\mathfrak{B}(T) \cong \mathbf{2}^n$;

(2) *there is a subset* $N \subseteq M$ *such that*
 (i) $|N| = n$,
 (ii) $\langle G, N, r \cap G \times N\rangle$ *is a reduced good context,*
 (iii) *for any element* $k \in M \setminus N$ *we have* $t(\{k\}) = t(K)$ *where* $K$ *is a uniquely determined subset of* $N$;

(2′) *there is a subset* $H \subseteq G$ *such that*
 (i) $|H| = n$,
 (ii) $\langle M, H, r^{-1} \cap M \times H\rangle$ *is a reduced good context,*
 (iii) *for any element* $f \in G \setminus H$ *we have* $s(\{f\}) = s(F)$ *where* $F$ *is a uniquely determined subset of* $G$;

(3) *there are subsets* $N \subseteq M$, $H \subseteq G$ *such that*
 (i) $|N| = |H| = n$,
 (ii) $\langle H, N, r \cap H \times N\rangle \cong \langle N, N, \neq\rangle$,
 (iii) *for any elements* $k \in M \setminus N$, $f \in G \setminus H$ *we have* $t(\{k\}) = t(K)$, $s(\{f\}) = s(F)$ *where* $K \subseteq N$ *and* $F \subseteq H$ *are uniquely determined subsets.*

Proof. (1) ⇒ (3): Since $\mathfrak{B}(T) \cong \mathbf{2}^n$ we have also $\langle \mathbf{C}(T), \subseteq\rangle \cong \mathbf{2}^n$. The coatoms of $\langle \mathbf{C}(T), \subseteq\rangle$ can be written in the form $t(\{m_1\}), \ldots, t(\{m_n\})$ for some $m_1, \ldots, m_n \in M$. Any subset $t(\{m\})$, $m \in M$, is $p$-closed and so $t(\{m\}) = \bigcap_{k \in I(m)} t(\{m_k\})$ for a uniquely determined subset $\emptyset \subseteq I(m) \subseteq \{1, \ldots, m\}$. Notice that $t(\{m\}) = G$ if $I(m) = \emptyset$. By Lemma 4, $\{s_0(t(\{m_i\})); i \leqq n\}$ is an irredundant subset system in $G$. Since $t_0(s_0(t(\{m_i\}))) = t(\{m_i\})$, $i = 1, \ldots, n$, the context $\langle G, N, r \cap G \times N\rangle$ ($N$ denotes the set $\{m_i; i \leqq n\}$) arises from the context of inequality $\langle G, G, \neq\rangle$ by the rule introduced in Theorem 2. Hence $\langle G, N, r \cap G \times N\rangle$ is a reduced good context.

Now let us turn to the Boolean algebra $\langle \mathbf{D}(T), \subseteq\rangle$. Let $s(\{g_i\})$, $i = 1, \ldots, n$, be coatoms in $\langle \mathbf{D}(T), \subseteq\rangle$ for some elements $g_1, \ldots, g_n \in G$. Then any subset $s(\{g\})$, $g \in G$, is generated by $\{s(\{g_i\}); i \leqq n\}$. Consequently, any subset $s(\{g\}) \cap N$, $g \in G$, is generated by $\{s(\{g_i\}) \cap N; i \leqq n\}$. Combining this with the fact that the context $\langle G, N, r \cap G \times N\rangle$ contains a subcontext isomorphic to $\langle N, N, \neq\rangle$, see Theorem 4 (4), we get that generating set $\{s(\{g_i\}) \cap N; i \leqq n\}$ is formed exactly by the coatoms of the Boolean algebra $\mathbf{B}(N)$. Hence $\langle H, N, r \cap H \times N\rangle \cong \langle N, N, \neq\rangle$ for $H = \{g_i; i \leqq n\}$. (The incidence matrix of the context $\langle G, N, r \cap G \times N\rangle$ can be used for a very transparent justification of the above conclusions.)

Part (iii) follows directly from the fact that the subsets $t(\{m\})$, $m \in N$, ($s(\{g\})$, $g \in H$) are coatoms in the Boolean algebra $\langle \mathbf{C}(T), \subseteq\rangle$ ($\langle \mathbf{D}(T), \subseteq\rangle$, respectively).

The implications (3) ⇒ (2) and (3) ⇒ (2′) are trivial.

(2) ⇒ (1): Let $X$ be an arbitrary subset of $M$. Then $X = (X \cap N) \cup (X \setminus N)$ and so $t(X) = t(X')$ for a subset $X' \subseteq N$. In this way we find that $\{t(X); X \subseteq M\} = \{t(X'); X' \subseteq N\}$. Since $\langle G, N, r \cap G \times N\rangle$ is a reduced good context, Theorem 4 (3) yields the isomorphism $\langle \{t(X'); X' \subseteq N\}, \subseteq\rangle \cong \mathbf{2}^n$. Altogether we have

$\langle \mathbf{C}(\boldsymbol{T}), \subseteq\rangle = \langle\{t(X);\ X \subseteq M\}, \subseteq\rangle = \langle\{t(X');\ X' \subseteq N\}, \subseteq\rangle \cong \mathbf{2}^n$, i.e. $\mathfrak{B}(\boldsymbol{T}) \cong \mathbf{2}^n$ which was to be proved.

Analogously $(2') \Rightarrow (1)$ can be proved. We omit the details.

## 5. SOME NUMERICAL DATA

**Definition 4.** We say that two contexts $\boldsymbol{T} = \langle G, M, r\rangle$ and $\boldsymbol{T}' = \langle G, M, r'\rangle$ are *essentially different* if there is no isomorphism $i$: $\boldsymbol{T} \underset{\rightarrow}{\cong} \boldsymbol{T}'$ of the form $i = \langle 1_G, \beta\rangle$ where $1_G$ is the identical mapping on $G$ and $\beta$: $M \to M$.

(In other words, the contexts $\boldsymbol{T}$ and $\boldsymbol{T}'$ are essentially different whenever the incidence matrix of $\boldsymbol{T}'$ cannot be obtained from the incidence matrix of $\boldsymbol{T}$ by a permutation of columns.)

Let us introduce the notation $g = |G|$ and $m = |M|$ throughout this section. Further, denote by

$GC(g, m)$ the number of all reduced good contexts $\langle G, M, r\rangle$, and by
$EGC(g, m)$ the number of all essentially different reduced good contexts $\langle G, M, r\rangle$.

From [1] we recall the well-known *Stirling numbers of the second kind*, namely, the symbol

$S_2(g, m)$ designating the number of all partitions with $m$ blocks on the set $G$.

Now we can state

**Theorem 7.** *For any positive integers* $1 \leqq m \leqq g$ *we have*

(1) $EGC(g, m) = \sum_{k=m}^{g} \binom{g}{k} S_2(k, m)\,(2^m - m)^{g-k}$,

(2) $GC(g, m) = m!\ EGC(g, m)$,

(3) $EGC(g, 2) =$ *the number of all 2-element antichains in* $\mathbf{2}^g$ *for* $2 \leqq g$.

Proof. We use the incidence matrix representation. Let $\mathscr{T}, \mathscr{M}$ denote the incidence matrix of the context $\boldsymbol{T} = \langle G, M, r\rangle$, $\langle M, M, \neq\rangle$, respectively.

(1) By Theorem 4 (4) each row from $\mathscr{M}$ is contained in $\mathscr{T}$ with possible repetitions. The remaining $g - k$, $m \leqq k \leqq g$, rows of $\mathscr{T}$ are arbitrary but not from $\mathscr{M}$. Now compute: there are $\binom{g}{k}$ possibilities how to pick out $k$ rows from $g$ rows of $\mathscr{T}$. There are $S_2(k, m)$ possibilities how to distribute $m$ rows from $\mathscr{M}$ into $k$ already selected places in $\mathscr{T}$. Finally, there are exactly $2^m - m$ rows which do not belong to $\mathscr{M}$. So there are $(2^m - m)^{g-k}$ possibilities how to construct the remaining $g - k$ rows of $\mathscr{T}$. The formula (1) follows.

(2) See Definition 4.

(3) See Remark 1.

We close with two tables. They give the values of $EGC(g, m)$, $GC(g, m)$, respectively, for the integers $1 \leqq m \leqq g \leqq 8$.

| $EGC(g, m)$ | $m=$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $g=1$ | 1 | | | | | | | |
| 2 | 3 | 1 | | | | | | |
| 3 | 7 | 9 | 1 | | | | | |
| 4 | 15 | 55 | 26 | 1 | | | | |
| 5 | 31 | 285 | 425 | 70 | 1 | | | |
| 6 | 63 | 1351 | 5590 | 2945 | 177 | 1 | | |
| 7 | 127 | 6069 | 64701 | 96530 | 18284 | 427 | 1 | |
| 8 | 255 | 26335 | 688506 | 2716581 | 1439718 | 104202 | 996 | 1 |

| $GC(g, m)$ | $m=$ 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $g=1$ | 1 | | | | | | | |
| 2 | 3 | 2 | | | | | | |
| 3 | 7 | 18 | 6 | | | | | |
| 4 | 15 | 110 | 156 | 24 | | | | |
| 5 | 31 | 570 | 2550 | 1680 | 120 | | | |
| 6 | 63 | 2702 | 33540 | 70680 | 21240 | 720 | | |
| 7 | 127 | 12138 | 388206 | 2316720 | 2194080 | 307440 | 5040 | |
| 8 | 255 | 52670 | 4131036 | 65197944 | 172766160 | 75025440 | 5019840 | 40320 |

**Problem.** *Find a recursive formula for* $EGC(g, m)$, $1 \leqq m \leqq g$.


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## Souhrn

### BOOLEOVY SVAZY POJMŮ A DOBRÉ KONTEXTY

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Svazy pojmů zavedl a vyšetřoval R. Wille. Dobré kontexty definovali M. Novotný a Z. Pawlak při výzkumu černých skřínek. Článek se zabývá úzkou souvislostí mezi Booleovými svazy pojmů a dobrými kontexty.



## Резюме

### БУЛЕВЫ КОНЦЕПТУАЛЬНЫЕ СТРУКТУРЫ И ХОРОШИЕ КОНТЕКСТЫ

JAROMÍR DUDA

Статья рассматривает связь между концептуальными структурами и хорошими контекстами.



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