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A NOTE ON PERMUTABILITY IN VARIETIES

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Summary. It is well-known that a variety is permutable if and only if the free algebra with three generators is permutable. The paper contains characterizations of varieties whose free algebras with two generators are permutable.

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Denote by Con A the congruence lattice of an algebra A. A is called *permutable* if $\theta \,.\, \Phi = \Phi \,.\, \theta$ for each two congruences $\theta, \Phi \in \operatorname{Con} A$. A variety $\mathscr{V}$ is *permutable* if each $A \in \mathscr{V}$ has this property. The permutability is very important because

(i) for any $\theta, \Phi \in \operatorname{Con} A$, the relational product $\theta \,.\, \Phi$ is a congruence on A if and only if $\theta \,.\, \Phi = \Phi \,.\, \theta$;

(ii) if A is permutable, the lattice operation of join in Con A has the simplest form, namely $\theta \vee \Phi = \theta \,.\, \Phi$.

Especially, (ii) enables us to give simple characterizations of some important properties of permutable algebras and varieties.

Denote by $F_{\mathscr{V}}(x_1, \ldots, x_n)$ the free algebra of a variety $\mathscr{V}$ with n free generators $x_1, \ldots, x_n$. A. I. Mal'cev [4] gave a simple characterization of permutable varieties:

Mal'cev Theorem. *For a variety $\mathscr{V}$, the following conditions are equivalent:*

(1) *$\mathscr{V}$ is permutable;*

(2) *$F_{\mathscr{V}}(x, y, z)$ is permutable;*

(3) *there exists a ternary term $t(x, y, z)$ such that $t(x, y, y) = x$ and $t(x, x, y) = y$.*

The ternary term $t(x, y, z)$ from (3) is called the *Mal'cev term.* Using Mal'cev Theorem, one can easily show that every variety of groups, quasigroups or Boolean algebras is permutable, i.e. it has permutable $F_{\mathscr{V}}(x, y, z)$. On the other hand, no nontrivial lattice variety has this property. Since lattices play an important role in algebraic investigations, we have a natural problem: How to characterize varieties of algebras having permutable algebras with two generators (but not necessarily ones with three generators). The aim of this paper is to give some properties and examples of such varieties.

Theorem 1. *Let $\mathscr{V}$ be a variety. The following conditions are equivalent:*
(1) *Each $A \in \mathscr{V}$ with two generators is permutable.*
(2) *For every three binary terms r, s, q there exists a binary term p such that for each $A \in \mathscr{V}$ and every x, y of A we have:*
if $s(x, y) = q(x, y)$ then $r(x, y) = p(x, y)$ and
if $r(x, y) = q(x, y)$ then $s(x, y) = p(x, y)$.

Proof. (1) $\Rightarrow$ (2): Let $a, b, c \in F_{\mathscr{V}}(x, y)$. Then

$$\langle a, c\rangle \in \theta(a, b) \,.\, \theta(b, c)\,,$$

thus (1) implies $\langle a, c\rangle \in \theta(b, c) \,.\, \theta(a, b)$. Since $a, b, c \in F_{\mathscr{V}}(x, y)$, there exist binary terms r, s, q such that $a = r(x, y)$, $b = q(x, y)$, $c = s(x, y)$. By the last relation, there must exist a binary term $p(x, y) \in F_{\mathscr{V}}(x, y)$ with

$$\langle r(x, y), p(x, y)\rangle \in \theta(q(x, y), s(x, y))\,,$$
$$\langle p(x, y), s(x, y)\rangle \in \theta(r(x, y), q(x, y))\,,$$

whence (2) follows.

(2) $\Rightarrow$ (1): Let $A \in \mathscr{V}$ have two generators, say x, y, and let $\theta, \Phi \in \operatorname{Con} A$ and $a, c \in A$. Suppose $\langle a, c\rangle \in \theta \,.\, \Phi$. Then $\langle a, b\rangle \in \theta$ and $\langle b, c\rangle \in \Phi$ for some $b \in A$ and there exist binary terms r, s, q such that $a = r(x, y)$, $b = q(x, y)$, $c = s(x, y)$. By (2), there exists a binary term $p(x, y)$ satisfying

$$\langle a, p(x, y)\rangle = \langle r(x, y), p(x, y)\rangle \in \theta(q(x, y), s(x, y)) = \theta(b, c) \subseteq \Phi\,,$$
$$\langle p(x, y), c\rangle = \langle p(x, y), s(x, y)\rangle \in \theta(r(x, y), q(x, y)) = \theta(a, b) \subseteq \Phi\,;$$

thus $\langle a, c\rangle \in \Phi \,.\, \theta$, which proves the permutability of A. $\square$

Example 1. Every variety $\mathscr{V}$ of lattices has permutable members with two generators.

Evidently, the only binary terms in $\mathscr{V}$ are $x, y, x \vee y, x \wedge y$. Without loss of generality suppose

$$r(x, y) = x\,, \quad q(x, y) = x \vee y\,, \quad s(x, y) = x \wedge y\,.$$

Then we can put $p(x, y) = y$, which yields the implications:

if $s(x, y) = q(x, y)$, i.e. $x \wedge y = x \vee y$, then $x = y$ which gives

$$r(x, y) = p(x, y)\,;$$

if $r(x, y) = q(x, y)$, i.e. $x = x \vee y$, then $y \leqq x$ which gives

$$s(x, y) = x \wedge y = y = p(x, y)\,.$$

The proof for any other choice of terms $r, s, q \in \{x, y, x \vee y, x \wedge y\}$ is analogous.

Example 2. A nontrivial variety of semilattices has no permutable members with two generators.

Chose e.g. $r(x, y) = x$, $s(x, y) = y$, $q(x, y) = x \vee y$ and suppose there exists (x, y) satisfying (2) of Theorem 1. Then

$$\text{if} \quad s(x, y) = q(x, y), \quad \text{i.e.} \quad y = x \vee y, \quad \text{then} \quad x \leqq y \quad \text{which gives}$$
$$r(x, y) = p(x, y), \quad \text{i.e.} \quad x = p(x, y);$$
$$\text{if} \quad r(x, y) = q(x, y), \quad \text{i.e.} \quad x = x \vee y, \quad \text{then} \quad y \leqq x \quad \text{which gives}$$
$$s(x, y) = p(x, y), \quad \text{i.e.} \quad y = p(x, y).$$

This implies $p \notin \{x, y, x \vee y\}$, but $x, y, x \vee y$ are the only binary terms in the join semilattice.

The condition (2) of Theorem 1 is in the form of an implication which is not a suitable form for some applications. We will proceed to finding a more suitable sufficient condition which can be also necessary in some important cases.

Theorem 2. *Let a variety* $\mathscr{V}$ *satisfy the following condition*:
$(*)$ *for every binary terms* r, q, s *there exist* 4-*ary terms* w, u *such that*

$$r(x, y) = w(q(x, y), s(x, y), x, y)$$
$$s(x, y) = u(q(x, y), r(x, y), x, y)$$
$$w(s(x, y), q(x, y), x, y) = u(r(x, y), q(x, y), x, y).$$

Then each $A \in \mathscr{V}$ *with two generators is permutable.*

Proof. Let $A \in \mathscr{V}$ have two generators x, y, let $\theta, \Phi \in \operatorname{Con} A$ and suppose $\langle a, b\rangle \in \theta \,.\, \Phi$ for some a, b of A. Then $a = r(x, y)$ and $b = s(x, y)$ for some binary terms r, s and there exists $c = q(x, y) \in A$ with

$$\langle r(x, y), q(x, y)\rangle \in \theta \quad \text{and} \quad \langle q(x, y), s(x, y)\rangle \in \Phi\,.$$

Put $d = w(s(x, y), q(x, y), x, y)$. Then $d \in A$ and, by $(*)$, we obtain

$$\langle a, d\rangle = \langle r(x, y), w(s(x, y), q(x, y), x, y)\rangle =$$
$$= \langle w(q(x, y), s(x, y), x, y), w(s(x, y), q(x, y), x, y\rangle \in \Phi\,,$$
$$\langle d, b\rangle = \langle w(s(x, y), q(x, y), x, y), s(x, y)\rangle =$$
$$= \langle u(r(x, y), q(x, y), x, y), u(q(x, y), r(x, y), x, y)\rangle \in \theta\,,$$

thus $\langle a, b\rangle \in \Phi \,.\, \theta$. □

Example 3. Let $\mathscr{V}$ be a variety of rectangular bands, i.e. a variety of semigroups satisfying $x^2 = x$ and $yxy = x$. Then each $A \in V$ with two generators is permutable.

It is evident that $\mathscr{V}$ has only four binary terms, namely x, y, xy, yx. Suppose e.g. $r(x, y) = xy$, $s(S, y) = yx$, $q(x, y) = x$. Then for $w(a, b, x, y) = ay$, $u(a, b, x, y) = ya$ we have

$$w(q(x, y), s(x, y), x, y) = xy = r(x, y)\,,$$

$$u(q(x, y), r(x, y), x, y) = yx = s(x, y),$$

$$w(s(x, y), q(x, y), x, y) = yx \cdot y = y = y \cdot xy = u(r(x, y), q(x, y), x, y).$$

The assertion can be shown analogously for another choice of

$r, s, q \in \{x, y, xy, yx\}$, e.g. for $r(x, y) = x$, $s(x, y) = y$, $q(x, y) = xy$

we have $w(a, b, y) = ax$, $u(a, b, x, y) = ya$,

for $r(x, y) = x$, $s(x, y) = xy$, $q(x, y) = y$ we have

$w(a, b, x, y) = bx$, $u(a, b, x, y) = ba$,

for $r(x, y) = yx$, $s(x, y) = x$, $q(x, y) = xy$ we have

$w(a, b, x, y) = yax$, $u(a, b, x, y) = ax$, etc.

Let us introduce the following concepts. By a *tolerance* on an algebra A we mean a reflexive and symmetrical binary relation on A which is compatible with all operations of A, i.e., it is a subalgebra of $A \times A$. Clearly, every congruence on A is a tolerance on A but not vice versa, see e.g. [2], [5]. The set of all tolerances on A forms a complete lattice with respect to set inclusion, see [2]. Thus for any elements a, b of A there exists a tolerance $T(a, b)$ which is the least tolerance containing the pair $\langle a, b\rangle$. An algebra A is *tolerance trivial* if each tolerance on A is a congruence. A variety $\mathscr{V}$ is tolerance trivial if each $A \in \mathscr{V}$ has this property. By [1], $\mathscr{V}$ is tolerance trivial if and only if $\mathscr{V}$ is permutable. An algebra A is *principal tolerance trivial* if each tolerance of the form $T(a, b)$ is a congruence on A, i.e., if $T(a, b) = \theta(a, b)$ for each a, b of A. Some characterizations of principal tolerance trivial algebras are in [1].

Lemma. *Every distributive lattice is principal tolerance trivial (but not tolerance trivial in the general case).*

For the proof, see e.g. [1] or [3].

Theorem 3. *Let $\mathscr{V}$ be a variety in which $F_{\mathscr{V}}(x, y)$ is principal tolerance trivial. The following conditions are equivalent:*

(1) *each $A \in \mathscr{V}$ with two generators is permutable;*

(2) *$\mathscr{V}$ satisfies $(*)$ of Theorem 2.*

Proof. By Theorem 2, it remains only to prove (1) $\Rightarrow$ (2). Let $r(x, y)$, $s(x, y)$, $q(x, y)$ be binary terms in $\mathscr{V}$. Clearly

$$\langle r(x, y), q(x, y)\rangle \in \theta(r(x, y), q(x, y)) = \theta_1,$$

$$\langle q(x, y), s(x, y)\rangle \in \theta(q(x, y), s(x, y)) = \theta_2,$$

i.e. $\langle r(x, y), s(x, y)\rangle \in \theta_1 \cdot \theta_2$ in $\operatorname{Con} F_{\mathscr{V}}(x, y)$. Since $F_{\mathscr{V}}(x, y)$ is permutable, we have $\langle r(x, y), s(x, y)\rangle \in \theta_2 \cdot \theta_1$, i.e. there exists an element $p \in F_{\mathscr{V}}(x, y)$ such that

$$\langle r(x, y), p\rangle \in \theta(q(x, y), s(x, y)) \quad \text{and} \quad \langle p, s(x, y)\rangle \in \theta(r(x, y), q(x, y)).$$

Since $F_{\mathscr{V}}(x, y)$ is principal tolerance trivial, by Theorem 13 in [2] there exist binary algebraic functions φ, ψ over $F_{\mathscr{V}}(x, y)$ such that

$$r(x, y) = \varphi(q(x, y), s(x, y)),$$
$$p = \varphi(s(x, y), q(x, y)).$$

and

$$p = \psi(r(x, y), q(x, y)),$$
$$s(x, y) = \psi(q(x, y), r(x, y)).$$

Since $F_{\mathscr{V}}(x, y)$ has only two free generators x, y, there exist 4-ary terms w, u such that

$$\varphi(a, b) = w(a, b, x, y) \quad \text{and} \quad \psi(a, b) = u(a, b, x, y),$$

whence (2) is evident.

Example 4. Let $\mathscr{V}$ be a variety of lattices. The free algebra $F_{\mathscr{V}}(x, y)$ is a four element lattice, hence it is distributive. By Lemma, $F_{\mathscr{V}}(x, y)$ is principal tolerance trivial. Since $x, y, x \vee y, x \wedge y$ are the only binary terms in $\mathscr{V}$, suppose (without loss of generality) e.g.

$$r(x, y) = x \wedge y, \quad s(x, y) = x \vee y, \quad q(x, y) = x.$$

Then for $w(a, b, x, y) = a \wedge y$, $u(a, b, x, y) = a \vee y$ we obtain

$$w(q(x, y), s(x, y), x, y) = q(x, y) \wedge y = x \wedge y = r(x, y),$$
$$u(q(x, y), r(x, y), x, y) = q(x, y) \vee y = x \vee y = s(x, y),$$
$$w(s(x, y), q(x, y), x, y) = s(x, y) \wedge y = (x \vee y) \wedge y = y,$$
$$u(r(x, y), q(x, y), x, y) = r(x, y) \vee y = (x \wedge y) \vee y = y.$$

The proof for any other choice of r, s, q is analogous.

Example 5. Evidently, in every permutable variety $\mathscr{V}$, algebras with two generators are also permutable. Moreover, $\mathscr{V}$ is tolerance trivial (see [1]) and hence $F_{\mathscr{V}}(x, y)$ is tolerance trivial, thus also principal tolerance trivial. Hence $\mathscr{V}$ must satisfy (2) of Theorem 3. Let $t(x, y, z)$ be a Mal'cev term of $\mathscr{V}$. For given $r(x, y), s(x, y), q(x, y)$ put

$$w(a, b, x, y) = u(a, b, x, y) = t(r(x, y), b, s(x, y)).$$

Evidently, we obtain

$$w(q(x, y), s(x, y), x, y) = t(r(x, y), s(x, y), s(x, y)) = r(x, y),$$
$$w(q(x, y), r(x, y), x, y) = t(r(x, y), r(x, y), s(x, y)) = s(x, y),$$
$$w(s(x, y), q(x, y), x, y) = t(r(x, y), q(x, y), x, y) = u(r(x, y), q(x, y), x, y).$$

Remark 1. Following Example 5, we can ask if a suitable condition for permutability of two generated algebras in $\mathscr{V}$ can be formulated in the form similar to that of Example 5. We show that it is impossible: suppose $\mathscr{V}$ is a variety with permutable algebras with two generators and suppose there exists a ternary term $t(x, y, z)$ such that

$$t(a(x, y), b(x, y), b(x, y)) = a(x, y) ,$$
$$t(a(x, y), a(x, y), b(x, y)) = b(x, y) .$$

Let $\mathscr{V}$ be a variety of lattices which is non-trivial. Put $a(x, y) = x \wedge y$, $b(x, y) = x \vee y$. Since $a(x, y) \leqq b(x, y)$ and the lattice terms are monotone, we obtain

$$b(x, y) = t(a(x, y), a(x, y), b(x, y)) \leqq t(a(x, y), b(x, y), b(x, y)) = a(x, y)$$

which implies $x = y$, i.e. $\mathscr{V}$ is trivial, which is a contradiction.

On the other hand, if $\mathscr{V}$ is a variety of lattices, for every binary term $q(x, y)$ there exists a ternary term $t(x, y, z)$ such that

$$t(x, q(x, y), q(x, y)) = x ,$$
$$t(q(x, y), q(x, y), y) = y .$$

Explicitly,

for $q(x, y) = x \wedge y$ we have $t(a, b, c) = a \vee b \vee c$,

for $q(x, y) = x \vee y$ we have $t(a, b, c) = a \wedge b \wedge c$,

for $q(x, y) = x$ we have $t(a, b, c) = c$,

for $q(x, y) = y$ we have $t(a, b, c) = a$.

We can find a similar condition which holds also in other varieties with permutable 2-generated algebras:

Corollary. *Let $\mathscr{V}$ be a variety in which $F_{\mathscr{V}}(x, y)$ is principal tolerance trivial and permutable. For each binary term $q(x, y)$ there exist 4-ary terms p, t such that*

$$x = p(x, y, y, q(x, y)) ,$$
$$y = t(x, x, y, q(x, y)) ,$$
$$p(x, q(x, y), y, y) = t(x, q(x, y), y, x) .$$

Proof. If we put $r(x, y) = x$, $s(x, y) = y$ and

$$p(x, a, y, b) = w(b, a, x, y) , \quad t(x, a, y, b) = u(b, a, x, y) \quad \text{in (2)}$$

of Theorem 3, we obtain Corollary. $\square$

Example 6. If $\mathscr{V}$ is a variety of lattices and $q(x, y) = x \vee y$, then $p(a, b, c, d) = a \wedge d$, $t(a, b, c, d) = a \wedge c$. Evidently, we obtain

$$p(x, y, y, q(x, y)) = (x \vee y) \wedge x = x ,$$
$$t(x, x, y, q(x, y)) = (x \vee y) \wedge y = y ,$$
$$p(x, q(x, y), y, y) = x \wedge y = t(x, q(x, y), y, x) .$$

Similarly, for $q(x, y) = x \wedge y$ the terms p, t are dual. For $q(x, y) = x$ we have $p(a, b, c, d) = d$, $t(a, b, c, d) = c$, for $q(x, y) = y$ we have $p(a, b, c, d) = a$, $t(a, b, c, d) = d$.

Remark 2. Analogously we can investigate varieties in which algebras with one generator are permutable. Clearly every idempotent variety $\mathscr{V}$ has this property since a one generated algebra is a one element algebra (hence the variety of all semilattices has this property). The varieties of BCK-algebras or of implication algebras have this property as well since their algebras with one generator have only two elements. In general, such varieties can be characterized by conditions similar to those given above, only the binary terms are substituted by unary ones.

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Souhrn

POZNÁMKA O PERMUTABILITĚ VE VARIETÁCH

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Je známo, že varieta je permutabilní, právě když má tuto vlastnost volná algebra s třemi generátory. V práci jsou charakterizovány variety, v nichž jsou permutabilní volné algebry s dvěma generátory.

Резюме

ЗАМЕЧАНИЕ О ПЕРЕСТАНОВОЧНОСТИ КОНГРУЭНЦИЙ В МНОГООБРАЗИЯХ АЛГЕБР

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Известно, что многообразие имеет перестановочные конгруенции тогда и только тогда, когда это свойство имеет свободная алгебра с тремья образующими. В работе характеризированы многообразия, в которых свободные алгебры с двумья образующими имеют перестановочные конгруэнции.

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