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On H-closed extensions of topological spaces.

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Like many other notions in General Topology, the H-closed spaces are due to Alexandroff and Urysohn [1][1]. In their paper they stated two problems concerning H-closed spaces: (i) is a space every closed subspace of which is H-closed necessarily compact? (ii) may any Hausdorff space P be imbedded as a dense subset in a H-closed space R? A step towards the solution of (ii) was made by Tychonoff [2] showing that P may be imbedded in a H-closed space R (without being, in general, necessarily dense in R). In his important paper [3] M. H. Stone solved both (i) and (ii) and showed moreover that there exists a H-closed space $R \supset P$ which is a strict extension (loc. cit., definition 14) of P and has the same character (i. e. the minimal power of an open base) as P. Another notion introduced by Stone in connection with his algebraic considerations is the semiregularity which is less restrictive than regularity but simplifies the theory of H-closed extensions considerably. In his paper M. H. Stone uses an elaborate algebraic theory. A part of his results was proved in a similar but more direct way by Fomin [4].

In his paper [2] A. Tychonoff showed that any completely regular space may be imbedded in a compact space (as a matter of fact, he showed more, namely that there exists an universal compact space of character $\aleph$ for any infinite cardinal $\aleph$). A further important result is due to E. Čech [5] who proved that any completely regular space P possesses a compact extension[2] βP such that $\overline{P} = \beta P$ and that every bounded continuous real function on P may be extended to a continuous real function on βP; the space βP is uniquely determined by these properties. Later on, H. Wallman [6] proved that every topological space P may be imbedded

[1]) The numbers in brackets refer to the list at the end of this paper.
[2]) If P is a subspace of a space $R \supset P$, then we say that P is *imbedded* in R or that R is an *extension* of P.

as a dense set in a compact space ωP. The space ωP is a Hausdorff space if and only if P is normal; in this case $\omega P = \beta P$. The space ωP possesses the same homology theory as P.

The question arises whether a H-closed extension of a similar kind exists for Hausdorff spaces. This problem is solved in the author's paper [7]. Every Hausdorff space P possesses a H-closed extension τP such that $(i) \overline{P} = \tau P$; (ii) every mapping f of P into a Hausdorff space S such that $\overline{f(P)} = S$ may be extended to a mapping of a subspace $P' \subset \tau P$ onto S. The space τP is uniquely determined by the properties (i) and (ii).

The extensions βP, ωP, τP have been so far characterized either by their „construction" (e. g. the „ideal" points of ωP correspond to certain collections — the so called maximal basic sets — of closed subsets of P) or by certain properties of mappings of P, viz. by possibility of their continuous extending. This cannot be considered as a wholly satisfactory descriptive characterization of the extensions βP, ωP, τP. Such a characterization for ωP was not given till recently by Čech and Novák [8]. The space ωP is characterized by P being imbedded in ωP both combinatorially, which means that $\prod_1^n \overline{F}_i = 0$ whenever F_i are relatively closed in P and $\prod_1^n F_i = 0$, and regularly, which means that every closed set $\Phi \subset \omega P$ may be represented as intersection of a family of sets $\overline{F}$, $F \subset P$.

In the present paper I intend to give an analogous descriptive characterization of the space τP and three other types of H-closed extension which are obtained by imposing different conditions concerning relative semiregularity.

In § 1 of the present paper semiregularity of a point relatively to a set is defined and examined. It is shown that M. H. Stone's strict extension and E. Čech's regular imbedding are equivalent notions and may be both expressed in terms of relative semiregularity. Certain modifications (we call them SR-modifications) are considered, transforming a given Hausdorff space into a space satisfying appropriate relative semiregularity conditions. A modification of this kind occurs implicitly already in the author's paper [7].

In § 2 hypercombinatorial and paracombinatorial imbedding are examined which are closely related to Čech's combinatorial imbedding. Whereas however there is a difference between n-combinatorial ($n = 2, 3, \ldots$), combinatorial and combinatorial in the strong sense imbedding (Čech and Novák [8]), the analogous notions coincide for hypercombinatorial and paracombinatorial imbedding as shown in (2,1) and (2,4).

In § 3 a descriptive characterization of the extensions τP, $\tau' P$, σP, $\sigma' P$ is given. The space τP occurs already in [7]. The spaces σP and $\sigma' P$ may be obtained as Fomin's [4] spaces $\sigma(P)$ by taking for the basic collection $\{G\}$ the family of all open sets for σP and the family of all regularly open sets for $\sigma' P$.

There remain several unsolved problems.

1. I do not know what conditions a space P must satisfy in order that it might be imbedded combinatorially in a H-closed space. If such an extension exists it need not be unique. Example: Let P_0 be the space of all pairs of ordinals (ξ, η), $\xi \leqq \omega_0$, $\eta \leqq \omega_1$, $(\xi, \eta) \neq (\omega_0, \omega_1)$ with the usual topology. The set $\omega P_0 - P_0$ contains exactly two points and by cancelling any of them a H-closed space is obtained. The two spaces are not topologically equivalent but P_0 is combinatorially imbedded in either of them. — It seems probable that an extension of this kind is possible and unique if and only if P is normal (in that case it coincides, of course, with ωP).

2. It is perhaps of some interest to examine the conditions under which several of the spaces τP, $\tau' P$, σP, $\sigma' P$, ωP, βP coincide. It is known only that $\omega P = \beta P$ if and only if P is normal. The conditions for the other equivalences should be far more restrictive.

3. If completely regular spaces P_1 and P_2 satisfy the first countability axiom, then $\beta P_1 = \beta P_2$ implies $P_1 = P_2$ (= denotes topological equivalence here). It could be of some interest to find sufficiently broad conditions under which a similar implication holds for τP and the other H-closed extensions.

§ 1.

All spaces considered are Hausdorff spaces even if it is not explicitly stated. The signs $\Rightarrow$ and $\Leftrightarrow$ denote logical implication and equivalence.

Definitions. Let P be a space, $M \subset P$. A set $G \subset P$ is said to be *regularly open* (Kuratowski [9]) if $G = \operatorname{Int} \overline{G}$[3]) and is said to be *regularly open relatively to M* if $G = \operatorname{Int}(G + M\overline{G})$.

A point $x \in P$ is called *semiregular relatively to M* if whenever G is open and $x \in G$ there exists an open set H such that $x \in H \subset \operatorname{Int}(H + M\overline{H}) \subset G$.

If every point $x \in P$ is semiregular relatively to M, then P is said to be *semiregular relatively to M*. If a point $x \in P$ is semiregular relatively to P, then it is called simply *semiregular*. If

[3]) Int A is the interior of the set A, i. e. the set $P - \overline{P - A}$.

every point $x \in P$ is semiregular, then the space P is said to be *semiregular*. This definition is evidently equivalent with M. H. Stone's [3] definition of semiregularity.

(1.1) *If* G_i $(i = 1, \dots n)$ *are regularly open relatively to* M, *then* $\prod_1^n G_i$ *is so as well.*

Proof. Denoting $\prod_1^n G_i$ by H we have $H + M\overline{H} \subset \prod_1^n (G_i + M\overline{G}_i)$, whence $\mathrm{Int}\,(H + M\overline{H}) \subset \mathrm{Int} \prod_1^n (G_i + M\overline{G}_i) = \prod_1^n \mathrm{Int}\,(G_i + M\overline{G}_i) = \prod_1^n G_i = H$.

Clearly:

(1.2) *For any open* $G \subset P$ *the set* $\mathrm{Int}\,(G + M\overline{G})$ *is regularly open relatively to* M.

(1.3) *A point* $x \in P$ *is semiregular relatively to* M *if and only if it possesses fundamentel neighborhoods which are regularly open relatively to* M.

Proof. If x is semiregular relatively to M, then for any open G, $x \in G$, there exists an open H such that $x \in H \subset H_1 = \mathrm{Int}\,(H + M\overline{H}) \subset G$. By (1.2) H_1 is regularly open relatively to M. The other half of the lemma is obvious.

Definitions. Let P be a space, $Q \subset P$. Q is said to be *regularly imbedded* [8] in P if for any closed set $F \subset P$ and every $x \in P - F$ there exists $A \subset Q$ such that $F \subset \overline{A} \subset P - x$. P is said to be a *strict extension* [3] of Q if $\overline{Q} = P$ and for any open $G \subset P$ and every $x \in G$ there exists an open neighborhood H of x such that $\mathrm{Int}\,(H + A) \subset G$ whenever A is nowhere dense and $AQ = 0$.

(1.4) *Let* S *be dense in* P. S *is regularly imbedded in* P *if and only if* P *is semiregular relatively to* $P - S$.

Proof. I. Let P be semiregular relatively to $M = P - S$ and let $F \subset P$ be closed, $x \in P - F$. Then there exists an open set G such that $x \in G$, F. $\mathrm{Int}\,(G + M\overline{G}) = 0$. Setting $A = P - (G + M\overline{G}) = (S - G) + (M - \overline{G})$ we have $F \subset \overline{A} \subset P - x$. Since $\overline{M - \overline{G}} \subset \overline{P - \overline{G}} = \overline{S - \overline{G}}$ we have $\overline{A} = \overline{S - G}$. Hence the imbedding $S \subset P$ is regular. II. Let S be regularly imbedded in P and let $H \subset P$ be open, $x \in H$. There exists a set $A \subset S$ such that $P - H = F \subset \overline{A} \subset P - x$. Setting $G = P - \overline{A}$, $M = P - S$ we have $H \supset P - \overline{A} = \mathrm{Int}\,(P - S\overline{A}) = \mathrm{Int}\,(G + M) = \mathrm{Int}\,[G + M\overline{G} + (M - \overline{G})] = \mathrm{Int}\,(G + M\overline{G})$ which proves the theorem.

(1.5) *Let* $Q \subset P$, $\overline{Q} = P$. P *is a strict extension of* Q *if and only if* P *is semiregular relatively to* $P - Q$.

Proof. The implication: strict extension $\Rightarrow$ relative semiregularity follows at once from the above definition of strict extension by setting $A = (\overline{H} - H)(P - Q)$. Let P be semiregular relatively to $P - Q$ and let G be open, $x \in G$. There exists an open set H such that $x \in H \subset \operatorname{Int}[H + (\overline{H} - Q)] \subset G$. Now let $A \subset P$ be nowhere dense, $AQ = 0$. Then $\operatorname{Int}(H + A) \subset \operatorname{Int}(H + A\overline{H}) \subset \operatorname{Int}[H + (\overline{H} - Q)] \subset G$. Hence P is a strict extension of Q.

(1.4) and (1.5) imply:

(1.6) *Let* $S \subset P$, $\overline{S} = P$. P *is a strict extension of* S *if and only if* S *is regularly imbedded in* P.

Definition. Let P be a space, $Q \subset P$, $M \subset P$. Let $\mathfrak{G}$ denote the family of all open sets $G \subset P$ such that every $x \in QG$ possesses a neighborhood $H \subset G$ which is regularly open relatively to M. The space P' which is obtained by choosing $\mathfrak{G}$ as an open base will be called the *SR-modification of the space* P *on the set* Q *relatively to* M.

Since for open sets G and H $GH = 0 \Rightarrow \operatorname{Int}\overline{G}.\operatorname{Int}\overline{H} = 0$ and by (1.1) $G_1 \in \mathfrak{G}$, $G_2 \in \mathfrak{G} \Rightarrow G_1G_2 \in \mathfrak{G}$, any SR-modification of a Hausdorff space is a Hausdorff space again. If P' is a SR-modification of P, then the identical transformation $P \to P'$ is a mapping.

We may consider any set $A \subset P$ either as a subset of P or as a subset of P'. If A is closed, open, ... if considered as a subset of P (or P') we shall say, for convenience, that A is closed, open, ... *in* P (or in P').

(1.7) *Let* P *be a space,* $Q \subset P$, $M \subset P$ *and denote by* P' *the SR-modification of* P *on* Q *relatively to* M. *Then* (i) *if* G *is open in* P, *then* G *has the same closure both in* P *and in* P'; (ii) *if* G *is regularly open in* P *relatively to* M, *then it is open in* P'; (iii) *if* G *is open in* P' *and regularly open in* P *relatively to a set* M_1, *then* G *is regularly open relatively to* M_1 *in the space* P' *as well.*

Proof. For any $A \subset P$ denote by $\bar{A}$, A^*, $\operatorname{Int} A$, $\operatorname{Int}^* A$ the closure and the interior of A in P and in P' respectively. Then clearly $\overline{G} \subset G^*$; if $x \in P - \overline{G}$, we have $HG = 0$, where $H = P - \overline{G}$, hence $\overline{H}G$ which implies, by the definition of SR-modification, $x \in P - G^*$. Hence $G^* = \overline{G}$.

If $G = \operatorname{Int}(G + \overline{G}M)$, then by (1.2) and by the definition of SR-modification G is open in P'. If $G = \operatorname{Int}(G + \overline{G}M_1)$ and G is open in P', then $G \subset \operatorname{Int}^*(G + G^*M_1) = \operatorname{Int}^*(G + \overline{G}M_1) \subset \subset \operatorname{Int}(G + \overline{G}M_1) = G$, hence G is regularly open in P' relatively to M_1.

(1.7) and (1.3) imply

(1.8) *Let P' be the SR-modification of P on Q relatively to M. Then every point $x \in Q$ is semiregular in P' relatively to M.*

(1.5), (i) implies

(1.9) *Let P' be a SR-modification of P. A set $G \subset P$ is regularly open in P' if and only if it is regularly open in P. If a point $x \in P$ is semiregular in P then it is so in P' as well.*

(1.10) *If a subspace $Q \subset P$ is semiregular, then its topology remains unchanged under an arbitrary SR-modification of P.*

Proof. Let $H \subset Q$ be a relative neighborhood of a point $x \in Q$ in Q. Since Q is semiregular there exists a set $G \subset Q$ such that $x \in G \subset H$ and G is regularly open in Q. Let G_0 be open in P, $G = QG_0$ and denote Int $\overline{G_0}$ by G_1. We have $G = Q - \overline{Q - Q\overline{G}} = Q.(P - \overline{Q(P - \overline{G})}) = Q.(P - \overline{P - \overline{G_0}}) = QG_1$ which proves the theorem since (1.2) and (1.5) imply that G_1 is open in P', P' denoting an arbitrary SR-modification of P.

(1.11) *Let $Q \subset P$, $M \subset P$. The topology of both $P_1 = P - \overline{Q}$ and $P_2 = P - M$ remains unchanged by the SR-modification of P on Q relatively to M.*

Proof. If G is open in P, then by (1.2) and (1.5) $H = \text{Int}\,(G + \overline{G}M)$ is open in P', P' denoting the SR-modification of P on Q relatively to M, and $HP_2 = GP_2$ which proves the theorem for P_2. For P_1, it follows immediately from the definition of SR-modification.

§ 2.

Definitions. Let Q be a dense subspace of a space P. Then Q is said to be

(i) *combinatorially imbedded* [8] in P if whenever $F_i \subset Q$ are relatively closed in Q_n

$$\prod_1^n F_i = 0 \Rightarrow \prod_1^n \overline{F_i} = 0 \quad (n = 2, 3, \dots);$$

(ii) *combinatorially imbedded in P in the strong sense* [8] if whenever F_1, F_2 are relatively closed in Q we have

$$\overline{F_1F_2} = \overline{F}_1\overline{F}_2;$$

(iii) *hypercombinatorially imbedded in P* if whenever $F_i \subset Q$ are relatively closed in Q we have

$$\prod_1^n F_i \text{ nowhere dense in } Q \Rightarrow \prod_1^n \overline{F_i} = \overline{\prod_1^n F_i} \; (n = 2, 3, \dots);$$

(iv) *paracombinatorially imbedded in* P if for any relatively open sets $G_i \subset Q$ we have

$$\prod_1^n G_i = 0 \Rightarrow \prod_1^n \overline{G}_i \subset Q. \quad (n = 2, 3, \ldots).$$

(2.1) *Let* Q *be dense in* P. Q *is hypercombinatorially imbedded in* P *if and only if one of the following equivalent conditions holds:*

(i) *whenever* F_1, F_2 *are relatively closed subsets of* Q *and* F_1F_2 *is nowhere dense in* Q *we have* $\overline{F}_1\overline{F}_2 = \overline{F_1F_2}$;

(ii) *if* $F_i \subset Q$ *are relatively closed in* Q, *then*

$$\prod_1^n \overline{F}_i - Q = \overline{\prod_1^n G_i} - Q,$$

where G_i *denotes the relative interior of* F_i *in* Q $(n = 1, 2, 3, \ldots)$.

Proof. If (i) holds and $x \in \overline{F}_1\overline{F}_2 - Q$, then setting $A_i = F_i - G_1G_2$ we have $x \in (\overline{G}_1\overline{G}_2 - Q) + (\overline{A}_1\overline{A}_2 - Q)$ which implies $x \in \overline{G_1G_2} - Q$ since $\overline{A_1A_2} - Q = 0$, A_i being nowhere dense. Hence (ii) holds for $n = 2$. Now let (ii) be true for $n = 2, 3, \ldots m$ and let F_i $(i = 1, 2, \ldots m+1)$ be relatively closed in Q. Then $\prod_1^m \overline{F}_i - Q = \overline{\prod_1^m G_i} - Q$ and setting $\Phi = Q \overline{\prod_1^m G_i}$, $\Gamma =$ relative interior of Φ in Q we have $\prod_1^m \overline{F}_i - Q = \overline{\Phi} - Q$, $\prod_1^{m+1} \overline{F}_i - Q = \overline{\Phi}\overline{F}_{n+1} - Q = \overline{\Gamma G_{m+1}} - Q$. Since the sets G_i are regularly open in Q, so is $\prod_1^m G_i$ by (1.1), therefore $\Gamma = \prod_1^m G_i$, hence $\prod_1^{m+1} \overline{F}_i - Q = \overline{\Gamma G_{m+1}} - Q = \overline{\prod_1^{m+1} G_i} - Q$. This yields by induction the implication (i) $\Rightarrow$ (ii) which proves the theorem since evidently (ii) $\Rightarrow$ hypercombinatorial imbedding $\Rightarrow$ (i).

The following obvious lemma is useful sometimes.

(2.2) *Whenever* $G_i \subset R$ *are open in* R *we have*

$$\prod_1^n \operatorname{Int} \overline{G}_i = \operatorname{Int} \prod_1^n \overline{G}_i = \operatorname{Int} \overline{\prod_1^n G_i}.$$

Proof. We have only to prove these equalities for $n = 2$. Evidently $\operatorname{Int} \overline{G}_1\overline{G}_2 = \operatorname{Int} \overline{G}_1 \operatorname{Int} \overline{G}_2$. Denoting this set by H we have $H \supset \operatorname{Int} \overline{G_1G_2}$, $\overline{G_1G_2} = \overline{\overline{G}_1G_2} \supset \overline{HG_2} = \overline{H\overline{G}_2} \supset \overline{H}$; hence $\operatorname{Int} \overline{G_1G_2} \supset H$, $H = \overline{G}_1\overline{G}_2$.

(2.3) *Hypercombinatorial imbedding is both paracombinatorial and combinatorial in the strong sense.*

Proof. Let Q be hypercombinatorially imbedded in P. Let G_i $(i = 1, \dots n)$ be relatively open in Q, $\prod_1^n G_i = 0$. Denoting $Q\overline{G_i}$ by A_i we have $\prod_1^n \overline{G_i} = \prod_1^n \overline{A_i} = \prod_1^n A_i \subset Q$ since by (2.2) $\prod_1^n A_i$ is nowhere dense in Q. Hence the imbedding $Q \subset P$ is paracombinatorial. Now let F_1, F_2 be relatively closed in Q and denote by H_i the relative interior of F_i. Then by (2.1) $\overline{F_1}\overline{F_2} - Q = \overline{H_1}\overline{H_2} - Q$, hence $\overline{F_1}\overline{F_2} = \overline{F_1 F_2}$ which proves the theorem.

(2.4) *Let Q be dense in P. Q is paracombinatorially imbedded in P if and only if one of the following conditions holds:*

(i) *whenever G_1, G_2 are relatively open in Q and $G_1G_2 = 0$ we have $\overline{G_1}\overline{G_2} \subset Q$;*

(ii) *for any choice of relatively open $G_i \subset Q$ we have*

$$\prod_1^n \overline{G_i} - \overline{\prod_1^n G_i} \subset Q \quad (n = 2, 3, \dots).$$

Proof. Let the implication (*) $\prod_1^n G_i = 0 \Rightarrow \prod_1^n \overline{G_i} \subset Q$ hold for $n = 2, \dots m$. Then we have, for arbitrary relatively open $G_i \subset Q$, $\prod_1^m (G_i - A) = 0$, where $A = \overline{\prod_1^m G_i}$, whence $\prod_1^m \overline{G_i} - \overline{\prod_1^m G_i} = \prod_1^m \overline{G_i} - A \subset \prod_1^m \overline{G_i - A} \subset Q$. If $H_i \subset Q$ are relatively open, $\prod_1^{m+1} H_i = 0$, then $\prod_1^{m+1} \overline{H_i} \subset (\prod_1^m \overline{H_i} - \overline{\prod_1^m H_i}) + \overline{\prod_1^m H_i}.\overline{H_{m+1}} \subset Q$, hence (*) holds for $n = m + 1$. This yields, by induction, (i) $\Rightarrow$ (ii) which proves the theorem, since clearly (ii) $\Rightarrow$ paracombinatorial imbedding $\Rightarrow$ (i).

(2.5) *A paracombinatorial imbedding $Q \subset P$ is hypercombinatorial if and only if every relatively closed set $F \subset Q$ which is nowhere dense in Q is closed in P.*

Proof. We have only to prove that the condition is sufficient. Let F_1, F_2 be relatively closed in Q, F_1F_2 nowhere dense in Q. Denoting the relative interior of F_i by G_i we have $G_1G_2 = 0$, $\overline{F_i} - Q = (\overline{F_i - G_i} - Q) + (\overline{G_i} - Q) = \overline{G_i} - Q$, hence $\overline{F_1}\overline{F_2} - Q = \overline{G_1}\overline{G_2} - Q = 0$ which implies by (2.1) that the imbedding is hypercombinatorial.

The following theorems (2.6), (2.7), (2.8) are well known; (2.6) is due to Alexandroff and Urysohn [1], (2.7) and (2.8) are given in [7].

Definition. A Hausdorff space P is called *H-closed* if P is closed in any Hausdorff space in which it is imbedded.

(2.6) *A Hausdorff space P is H-closed if and only if every open covering $\{G\}$ contains a finite subcollection $\{G_i\}$ such that $\sum_1^n \overline{G}_i = P$.*

Proof. Let P be H-closed and let $\{G\}$ be an open covering. Let $R = P + \alpha$ and let the point α possess fundamental neighborhoods $P - \sum_1^n \overline{G}_i + \alpha$. Then R is a Hausdorff space, P is imbedded in R, therefore closed. Hence there exist G_i such that $P - \sum_1^n \overline{G}_i = 0$.

If P is not H-closed, there exists a Hausdorff space $R \supset P$ such that P is not closed in R. Let $\alpha \in \overline{P} - P$. The family $\{G\}$ of all $G = P - \overline{H}$ where H is a neighborhood of α in R is an open covering of P. For arbitrary $G_i \in \{G\}$, $G_i = P - \overline{H}_i$ $(i = 1, \dots n)$, we have $\overline{G}_i \subset R - H_i$, $\sum_1^n \overline{G}_i \subset R - \prod_1^n H_i$, $P - \sum_1^n \overline{G}_i \supset P \cdot \prod_1^n H_i \neq 0$, since $\prod_1^n H_i$ is a neighborhood of α.

(2.7) *If P is H-closed and $G \subset P$ is open, then $Q = \overline{G}$ is H-closed.*

Proof. Let $\{H\}$ be an open covering of the space P_1. Then the collection consisting of the set $P - Q$ and of all $P - \overline{Q - H}$ is an open covering of P, hence there exist H_i $(i = 1, \dots n)$ such that $\sum_1^n \overline{\Gamma_i} + \overline{P - Q} = P$, where $\Gamma_i = P - \overline{Q - H_i}$, therefore $\sum_1^n \overline{\Gamma_i} \supset$ $\supset P - \overline{P - Q} \supset G$, $\sum_1^n \overline{\Gamma_i} \supset \overline{G} = Q$. Since $\Gamma_i Q = H_i$ we obtain $\sum_1^n \overline{H}_i \supset \sum_1^n \overline{\Gamma_i G} = \overline{\overline{G \sum_1^n \Gamma_i}} = \overline{G} = Q$ which proves the theorem.

(2.8) *If a H-closed space P is continuously mapped on a Hausdorff space R, then R is H-closed.*

Proof. Denote by f a mapping of P onto R. Let $\{G\}$ be an open covering of R. Then $\{f^{-1}(G)\}$ is an open covering of P, hence there exist G_i such that $\sum_1^n \overline{f^{-1}(G_i)} = P$, whence $\sum_1^n \overline{G}_i = R$. Therefore by (2.6) R is closed.

(2.9) *If a collection* $\{G\}$ *of open subsets of a H-closed space* R *has the finite intersection property* (i. e. $\prod_1^n G_i \neq 0$ for any choice of $G_i \in \{G\}$), *then the intersection of all* $\overline{G}$ *is non-empty.*

Proof. If $\Pi\overline{G}$ were empty, then the collection $\{R - \overline{G}\}$ would be an open covering of R, hence by (2.6) there would exist G_i such that $\sum_1^n \overline{R - \overline{G}_i} = R$, whence $\prod_1^n G_i = 0$ which is not possible.

(2.10) *Let* P *be H-closed and let* f *be a* $1 - 1$ *mapping of* P *onto a Hausdorff space* R. *Then* $G \subset P$ *is regularly open if and only if* $f(G) \subset R$ *is regularly open.*

Proof. Denote $f(G)$ by H. The set $f(\overline{G})$ is closed by (2.7) and (2.8); hence $f(\overline{G}) = \overline{H}$, $R - \overline{H} = f(P - \overline{G})$ and since $f(\overline{P - \overline{G}})$ is closed by (2.7) and (2.8) we have $\overline{R - \overline{H}} = f(\overline{P - \overline{G}})$, therefore $\text{Int}\,\overline{H} = f\,(\text{Int}\,\overline{G})$ which proves the lemma.

(2.11) *Let* Q *be paracombinatorially imbedded in a H-closed space* P. *Let* f *be a* $1 - 1$ *mapping of* P *onto* R. *Then the imbedding* $f(Q) \subset R$ *is paracombinatorial.*

Proof. Let H_1, H_2 be relatively open in $S = f(Q)$, $H_1H_2 = 0$ and denote $f^{-1}(H_i)$ by G_i. If $f(a) = b \in \overline{H}_1 - S$, then $a \in \overline{G}_1 - Q$ (cf. the proof of 2.10), hence $a \in P - \overline{G}_2$, $b \in R - f(\overline{G}_2)$. Since $f(\overline{G}_2)$ is closed we have $b \in R - \overline{H}_2$, therefore $\overline{H}_1\overline{H}_2 \subset S$ which proves the lemma.

Example 1. Q denotes the plane; A, B, C denote the set of all $(x, y) \in Q$ such that $y > 0$, $y = 0$, $y < 0$ respectively. $P_1 = \omega Q$ is Wallman's [6] compact space. The imbedding $Q \subset P_1$ is combinatorial in the strong sense [8], but is not paracombinatorial since A and C are relatively open in Q, $AC = 0$, but $\overline{A}\overline{C} - Q \supset \overline{B} - - B \neq 0$.

(2.12) *If a normal space* Q *is paracombinatorially imbedded in* P, *then the imbedding is combinatorial.*

Proof. If $F_i \subset Q$ are relatively closed in Q and $\prod_1^n F_i = 0$, then there exist[4]) relatively open sets $G_i \subset Q$ such that $G_i \supset F_i$, $\prod_1^n G_i = 0$, hence $\prod_1^n \overline{G}_i \subset Q$, $\prod_1^n \overline{F}_i \subset Q$, $\prod_1^n \overline{F}_i = \prod_1^n F_i = 0$.

Example 2. Denote by I the discrete space of natural numbers. Choose a point $a \in \beta I - I$, βI denoting Čech's [5] compact

[4]) This is a well known property of normal spaces.

space. P_2 is the set βI with the topology defined in the following way: the points $n \in I$ are isolated; the fundamental neighborhoods of a are the same as in βI; any point $x \in \beta I - I - a$ possesses fundamental neighborhoods $GI + x$, G being a neighborhood of x in βI. Denote $P_2 - a$ by Q. Whenever G_i are open in Q and $a \in \overline{G}_1\overline{G}_2$ we have $a \in \overline{G_1 I}$, $a \in \overline{G_2 I}$, hence $G_1 G_2 I \neq 0$. Therefore the imbedding $Q \subset P_2$ is paracombinatorial.

Now choose for every infinite $A \subset I$ a point $x(A) \in \overline{A} - A$, $x(A) \neq a$, and denote by F the set of all $x(A)$. Then F is closed in Q, $a \in \overline{F}$ (since a clearly belongs to the closure of $\overline{F}$ in βI) and the power of F does not exceed the power c of the family of all subsets of I. Now $\Phi = Q - I - F$ is closed in Q and for any infinite $A \subset I$ we have $\overline{A}\Phi \neq 0$ since (Pospíšil [10]) $\overline{A}$ has the power 2^c. Hence $a \in \overline{\Phi}$, $a \in \overline{F}\overline{\Phi}$ which implies that the imbedding $Q \subset P_2$ is not combinatorial, not even 2-combinatorial [8].

Example 3. Choose again a point $a \in \beta I - I$. The space P_3 consists of the points x_{mn}, x_m, z $(m, n = 1, 2, \ldots)$. The points x_{mn} are isolated; every point x_m possesses fundamental neighborhoods U_{mG} consisting of x_m and all x_{mn}, $n \in G$, G running over all neighborhoods of a in βI. The point z possesses fundamental neighborhoods $U_{\{G_k\}}$ consisting of z and of the points x_m and x_{mn} such that $n \in G_m$, $m \in G_0$, where $\{G_k\}$ runs over all sequences of neighborhoods of a in βI. It is easy to show that P_3 is regular, hence, being countable, normal.

Denote $P_3 - z$ by Q. If $F \subset Q$ is relatively closed, $z \in \overline{F}$, then denoting by F^* the set of all $x_n \in F$ we have easily $z \in \overline{F^*}$.

Since the imbedding of the set A of all x_n in $A + z$ is clearly combinatorial, this proves that the imbedding $Q \subset P_3$ is combinatorial.

Now let G_1, G_2 be open in Q, $z \in \overline{G}_1\overline{G}_2$. Then $Q\overline{G}_1\overline{G}_2 \neq 0$ (since the imbedding is combinatorial), hence either there exists a $x_{mn} \in \overline{G}_1\overline{G}_2$ which implies $x_{mn} \in G_1G_2$, $G_1G_2 \neq 0$, or there exists a $x_m \in G_1G_2$ which implies again $G_1G_2 \neq 0$ since, for a given m, the imbedding of the set B_m of all x_{mn} in $B_m + x_m$ is clearly paracombinatorial. Therefore the imbedding $Q \subset P_3$ is paracombinatorial.

The set A is relatively closed and nowhere dense in Q, $z \in \overline{A} - A$. Hence Q is not hypercombinatorially imbedded in P_3.

§ 3.

(3.1) *Any Hausdorff space P may be hypercombinationally imbedded in a H-closed space $R = \tau P$ such that P is open in τP and the subspace $\tau P - P$ is discrete. This imbedding is essentially unique,* *i. e. if a space $R_1 \supset P$ possesses the same properties, then there exists a topological transformation of R onto R_1 which is identity on P.*

If P is paracombinatorially imbedded in a Hausdorff space S, then there exists a $1 - 1$ mapping f of a set $T \subset \tau P$, $T \supset P$, onto S such that $f(x) = x$ for $x \in P$; if S is H-closed, then $T = \tau P$.

Proof. I. Suppose that P is not H-closed since otherwise the theorem is trivial. A collection $\mathfrak{A}$ of open sets $A \subset P$ will be called an α-collection if (i) $A \in \mathfrak{A} \Rightarrow A \neq 0$; (ii) $A_1 \in \mathfrak{A}$, $A_2 \in \mathfrak{A} \Rightarrow A_1A_2 \in \mathfrak{A}$; (iii) the intersection of all $\bar{A}$, $A \in \mathfrak{A}$, is void. A maximal α-collection will be called a β-collection. By Zorn's theorem every α-collection is contained in a β-collection.

The space $\tau P = R$ consists of the points $\tau_{\mathfrak{B}}$ each of them corresponding to a β-collection $\mathfrak{B}$ and of all $x \in P$. Fundamental neighborhoods of the points $x \in P$ in R are their neighborhoods in P. Every $\tau_{\mathfrak{B}}$ possesses fundamental neighborhoods $B + \tau_{\mathfrak{B}}$, where $B \in \mathfrak{B}$. Clearly R is a Hausdorff space, P is open in R and the subspace $R - P$ is discrete.

Now let $\mathfrak{G}$ be an open covering of R and denote by $\mathfrak{A}$ the collection of the sets $P - \sum\limits_1^n \bar{G}_i$, $G_i \in \mathfrak{G}$. $\mathfrak{A}$ is no α-collection since otherwise there would exist a β-collection $\mathfrak{B} \supset \mathfrak{A}$ and we would have the implications $G \in \mathfrak{G} \Rightarrow P - \bar{G} \in \mathfrak{A} \Rightarrow \tau_{\mathfrak{B}} \in \overline{P - \bar{G}} = R - \bar{\bar{G}} \subset R - G$ which is impossible. Evidently $\mathfrak{A}$ possesses the properties (ii) and (iii) of an α-collection; hence $\mathfrak{A}$ does not possess the property (i), i. e. $0 \in \mathfrak{A}$ which proves by (2.6) that R is H-closed.

Let F_1 and F_2 be relatively closed subsets of P and let $\tau_{\mathfrak{B}} \in \bar{F}_1\bar{F}_2$. Then $P - F_i$ non $\in \mathfrak{B}$ $(i = 1, 2)$ and since $\mathfrak{B}$ is a maximal α-collection there exists a set $B_i \in \mathfrak{B}$ such that $B_i(P - F_i) = 0$. Hence $B_i \subset F_i$, $B_1B_2 \in \mathfrak{B}$, $0 \neq B_1B_2 \subset F_1F_2$, therefore F_1F_2 is not nowhere dense. This proves by (2.1) that the imbedding $P \subset R$ is hypercombinatorial.

II. Now let P be paracombinatorially imbedded in a space S. For every $y \in S - P$ denote by $\mathfrak{B}(y)$ the collection of all open $A \subset P$ such that $y \in \bar{A}$. The intersection of all $P\bar{A}$, $A \in \mathfrak{B}(y)$, is void. If $A_1 \in \mathfrak{B}(y)$, $A_2 \in \mathfrak{B}(y)$, then $y \in \bar{A}_1\bar{A}_2$, hence by (2.4) $y \in \overline{A_1A_2}$, $A_1A_2 \in \mathfrak{B}(y)$. Therefore $\mathfrak{B}(y)$ is an α-collection. If $B \subset P$ is relatively open and $A \in \mathfrak{B}(y) \Rightarrow BA \neq 0$, then clearly $y \in \bar{B}$, whence $B \in \mathfrak{B}(y)$. Hence $\mathfrak{B}(y)$ is a β-collection.

For any $y \in S - P$ we set $\tau_y = \tau_{\mathfrak{B}(y)}$ and denote by T the set consisting of the points τ_y and of all $x \in P$. Clearly $y \neq y' \Rightarrow \tau_y \neq \tau_{y'}$. We set $f(\alpha_y) = y$ and $f(x) = x$ for $x \in P$; thus f is a $1 - 1$ transformation, $f(T) = S$. If G is an open neighborhood of a point $y \in S - P$, then $y \in \overline{GP}$, $GP \in \mathfrak{B}(y)$, $GP + \tau_y$ is a neighborhood of $\alpha_y = f^{-1}(y)$ in R. If G is an open neighborhood (in S) of a point $x \in P$, then $f^{-1}(G) \supset GP$ and GP is a neighborhood of x in R. Hence f is a continuous mapping.

If S is H-closed, let $x = \tau_{\mathfrak{B}} \in R - P$. Denote by C the intersection of closures (in S) of all $B \in \mathfrak{B}$. Then by (2.7) and (2.9) $C \neq 0$ since S is H-closed. If $y \in C$, then $B \in \mathfrak{B} \Rightarrow y \in \overline{B}$, whence $\mathfrak{B} \subset \mathfrak{B}(y)$, therefore $\mathfrak{B} = \mathfrak{B}(y)$ since $\mathfrak{B}$ is a β-collection. This implies $x = \alpha_y$, whence $T = R$.

III. Let P be hypercombinatorially, hence by (2.3) paracombinatorially, imbedded in a H-closed space R_1 such that P is open in R_1 and $R_1 - P$ is discrete. There exists, by II., a $1 - 1$ mapping f of R onto R_1 such that $f(x) = x$ for $x \in P$. Let $z = \tau_{\mathfrak{B}} \in R - P$; then $B \in \mathfrak{B} \Rightarrow y = f(z) \in \overline{f(B)} = \overline{B}$. The imbedding $P \subset R_1$ being hypercombinatorial, $B \in \mathfrak{B} \Rightarrow y$ non $\in \overline{P - B}$; hence $R_1 - (\overline{P - B})$ is a neighborhood of y for any $B \in \mathfrak{B}$. Since $R_1 - P$ is discrete, $y + P$ is a neighborhood of y, hence so is $(y + P)\,[R_1 - (\overline{P - B})] = y + B = f(z + B)$. Therefore f is a topological transformation. This completes the proof.

Remark. It is immediately seen from the first part of this proof that τP is identical with the H-closed extension described in [7], 2.1.

(3.2) *Any Hausdorff space P may be paracombinatorially imbedded in a H-closed space $\tau' P$ such that P is open in $\tau' P$ and every point $x \in \tau' P - P$ is semiregular. The imbedding $P \subset \tau' P$ is essentially unique, and the SR-modification of τP on the set $\tau P - P$ may be taken as $\tau' P$.*

Proof. Denote by $\tau' P = R$ the SR-modification of τP on the set $\tau P - P$. Then by (1.11) P is imbedded in R. By (2.11) the imbedding is paracombinatorial and by (1.8) every $x \in R - P$ is semiregular. R is H-closed by (2.8).

If a space $R_1 \subset P$ possesses the above properties, then by (3.1) there exists a $1 - 1$ mapping f of R onto R_1 such that $f(x) = x$ for $x \in P$. Both in R_1 and in R the family consisting of all open sets contained in P and of all regularly open sets is an open base, since P is open and every point of its complement is semiregular. This implies by (2.10) that f is a topological transformation, i. e. the imbedding $P \subset R$ is essentially unique.

(3.3) *Any Hausdorff space P may be imbedded both regularly and hypercombinatorially in a H-closed space σP. This imbedding is essentially unique, and the SR-modification of τP relatively to $\tau P - P$ may be taken as σP.*

Proof. Denote by $\sigma P = R$ the SR-modification of τP relatively to $\tau P - P$. Then by (1.11) P is imbedded in R; the imbedding is regular by (1.8) and (1.4) and paracombinatorial by (2.11); R is H-closed by (2.8). If $F \subset P$ is closed, then $G = \tau P - F$ is open in τP and, for any $x \in G$, the set $H = GP + x$ is open in τP and $H + (\overline{H} - P) \subset G$, $\overline{H}$ denoting the closure of H in τP, hence G is open in R. Therefore F is closed in R. This implies by (2.5) that the imbedding $P \subset R$ is hypercombinatorial.

Now let P be both regularly and hypercombinatorially imbedded in a H-closed space R_1. By (3.1) there exists a $1 - 1$ mapping f of τP onto R_1 such that $f(x) = x$ for $x \in P$. Since the imbedding $P \subset R$ is regular, the family $\mathfrak{G}$ consisting of all $R - \overline{P - G}$, $G \subset P$ relatively open, is an open base of R. Since the imbedding $P \subset R$ is hypercombinatorial, $R - \overline{P - G} = G + (\overline{G} - P)$. Similarly, the family $\mathfrak{G}_1$ consisting of all sets $R_1 - \tilde{F} = G + (\tilde{G} - P)$, where $F = P - G$, G is open in P, and $\tilde{G}$, $\tilde{F}$ denote the closures of G, F in R_1, is an open base of R_1. Now it is easily seen that $f(\overline{G}) = \tilde{G}$ for any relatively open $G \subset P$ (since the closure of G in τP is H-closed it must be equal both to $\overline{G}$ and $f^{-1}(\tilde{G})$). Therefore we have the equivalence $H \in \mathfrak{G} \Leftrightarrow f(H) \in \mathfrak{G}_1$. Hence f is a topological transformation.

(3.4) *Any Hausdorff space P may be both regularly and paracombinatorially imbedded in a H-closed space $\sigma' P$ such that every point $x \in \sigma' P - P$ is semiregular. The imbedding is essentially unique and the SR-modification of $\tau' P$ relatively to $\tau' P - P$ may be taken as $\sigma' P$.*

Proof. Denote by $\sigma' P = R$ the SR-modification of $\tau' P$ relatively to $\tau' P - P$. Then by (1.11) P is imbedded in R and the imbedding is paracombinatorial by (2.11) and regular by (1.8) and (1.4). The points $x \in R - P$ are semiregular by (1.8). R is H-closed by (2.8).

Now let P be both regularly and paracombinatorially imbedded in R_1 and let every point $x \in R_1 - P$ be semiregular. By (3.1) there exists a $1 - 1$ mapping f of R onto R_1. For every relatively closed set $F \subset P$ the set $\overline{F} - F$ ($\overline{F}$ denoting the closure in R) consists of all points $y \in R - P$ such that $GF \neq 0$ for any regularly open $G \subset R$ containing y. Since the same holds for R_1 we have by (2.10) $f(\overline{F}) = \tilde{F}$, where F is relatively closed in P and $\tilde{F}$ denotes closure in R_1. Since P is regularly imbedded both in R and R_1 we have $f(\overline{A}) = \tilde{A}$ for any $A \subset R$. Hence f is a topological transformation.

(3.5) *Any semiregular Hausdorff space P may be paracombinato-*

rially imbedded in a semiregular H-closed space R. The imbedding is essentially unique and we may set $R = \sigma'P$.

Proof. Let R be the SR-modification of τP on τP relatively to τP. Then by (1.10) P is imbedded in R and the imbedding is paracombinatorial by (2.11). R is H-closed by (2.8) and semiregular by (1.8). Hence by (1.4) P is regularly imbedded in R. This implies by (3.4) the topological equivalence $R = \sigma'P$ and the essential uniqueness of R.

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O H-uzavřených obalech topologických prostorů.

(Obsah předešlého článku.)

Nechť P je AHF-prostor. Říkáme, že bod $x \in P$ je *poloregulární*, když ke každému okolí H bodu x existuje otevřená množina G taková, že $x \in G \subset \operatorname{Int} \overline{G} \subset H$.

Nechť $Q \subset P$. Říkáme, že množina Q je *regulárně vnořena* do P, když každá uzavřená množina $F \subset P$ je průnikem některých množin tvaru $\overline{A}$, $A \subset Q$.

Říkáme, že množina Q je *hyperkombinatoricky vnořena* do P, když $\overline{Q} = P$ a pro libovolné uzavřené v Q množiny $F_i \subset Q$ platí: je-li $\prod_1^n F_i$ řídká v Q, pak $\prod_1^n \overline{F}_i = \prod_1^n F_i$.

Říkáme, že množina Q je *parakombinatoricky vnořena* do P, když $\overline{Q} = P$ a pro libovolné otevřené v Q množiny $G_i \subset Q$ platí: $\prod_1^n G_i = 0 \Rightarrow \prod_1^n \overline{G_i} \subset Q$.

Nazýváme AHF-prostor P *H-uzavřeným*, je-li P množina uzavřená v libovolném AHF-prostoru R, do něhož je prostor P vnořen.

Hlavním výsledkem práce jsou tyto věty:

Každý AHF-prostor P lze hyperkombinatoricky vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru τP takového, že množina P je otevřená v τP a všechny body prostoru $\tau P - P$ jsou isolované (v $\tau P - P$).

Každý AHF-prostor P lze parakombinatoricky vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru $\tau' P$ takového, že množina P je otevřená v $\tau' P$ a každý bod $x \in \tau' P - P$ je poloregulární.

Každý AHF-prostor P lze hyperkombinatoricky a regulárně vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru σP.

Každý AHF-prostor P lze parakombinatoricky a regulárně vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru $\sigma' P$ takového, že každý bod $x \in \sigma' P - P$ je poloregulární.