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ON SOLUTIONS OF NONAUTONOMOUS LINEAR DELAYED DIFFERENTIAL EQUATIONS, WHICH ARE DEFINED AND EXPONENTIALLY BOUNDED FOR $t \to -\infty$

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Dedicated to the memory of my teacher Prof. VOJTĚCH JARNÍK

Let M_n be the space of square matrices of order n, R – the real line, R^+ – the positive halfline (closed), R^- – the negative halfline, $A : R^- \to M_n$, $B : R^- \to M_n$ locally integrable. For $y \in R^n$ denote by $|y|$ the Euclidean norm of y and for $C \in M_n$ put $|C| = \sup_{|y| \leqq 1} |Cy|$.

For $\gamma \in R^+$ let $\mathscr{X}(\gamma)$ be the set of such solutions $x : R^- \to R^n$ of

$$\frac{dx}{dt}(t) = A(t)\,x(t) + B(t)\,x(t-1) \tag{1}$$

that

$$\sup_{t \leqq 0} e^{\gamma t}|x(t)| < \infty\,. \tag{2}$$

Obviously $\mathscr{X}(\gamma)$ is a linear manifold.

Theorem 1. *Assume that $|B|^2$ is locally integrable and that*

$$\sup_{t \leqq 0} \int_{t-1}^{t} |A(\tau)|\,d\tau < \infty\,, \quad \sup_{t \leqq 0} \int_{t-1}^{t} |B(\tau)|^2\,d\tau < \infty\,. \tag{3}$$

Then the dimension of $\mathscr{X}(\gamma)$ is finite. Moreover, there exists $\Theta : (R^+)^3 \to R^+$ such that if

$$\sup_{t \leqq 0} \int_{t-1}^{t} |A(\tau)|\,d\tau \leqq a\,, \quad \sup_{t \leqq 0} \int_{t-1}^{t} |B(\tau)|^2\,d\tau \leqq b^2\,, \tag{4}$$

then

$$\dim \mathscr{X}(\gamma) \leqq \Theta(a, b, \gamma)\,. \tag{5}$$

Note 1. $\Theta(a, b, \gamma)$ may be calculated (of course not the best one). Thus it may be shown that

$$\text{(6)} \qquad \dim \mathscr{X}(\gamma) \leqq n\,, \qquad \text{if} \quad e^{(n+1)\gamma}[1 + 4e^{2a} \max(1, b^2)]^{n/2} e^a b < 1$$

$$\text{(7)} \qquad \dim \mathscr{X}(\gamma) \leqq n + 1\,, \quad \text{if} \quad e^{(n+2)\gamma}[1 + 4e^{2a} \max(1, b^2)]^{n/2} e^{2a} b^2 < 1$$

$$\text{(8)} \qquad \text{if} \quad e^a b \geqq 1 \quad \text{and} \quad e^\gamma(1 + ae^a)\, b \to \infty\,,$$

then

$$\Theta(a, b, \gamma) \approx \frac{2ne}{\pi^2} e^{2\gamma}(1 + ae^a)^2\, b^2$$

(i.e. to any $\varepsilon > 0$ there exists such a $\varrho > 0$ that

$$\left|\Theta(a, b, \gamma)\, \pi^2 (2ne)^{-1} e^{-2\gamma}(1 + ae^a)^{-2} b^{-2} - 1\right| \leqq \varepsilon$$

provided that $e^a b \geqq 1$ and $e^\gamma(1 + ae^a)\, b \geqq \varrho$.

Note 2. Theorem 1 is related to applications of Theory of Invariant Manifolds to Delayed Differential Equations (cf. [1], [2], [3]). Let us review some results, which may be obtained for (1). For this purpose extend A and B to R putting $A(t) = 0 = B(t)$ for $t > 0$.

Proposition. *Assume that A fulfils* (4), *that B instead of* (3) *and* (4) *fulfils*

$$\text{(9)} \qquad \sup_t \int_{t-1}^{t} |B(\tau)|\, d\tau \leqq \beta$$

and that there exists $L > 0$ such that

$$\text{(10)} \qquad e^a(e^a + L)^2\, b \leqq L\,,$$

$$\text{(11)} \qquad e^a(e^a + 1)(e^a + L)\, b < 1\,.$$

Denote by U a fundamental matrix of

$$\text{(12)} \qquad \frac{dx}{dt}(t) = A(t)\, x(t)\,.$$

Then there exists $Q : R \to M_n$, continuous, $|Q(t)| \leqq L$ for $t \in R$ such that every solution of

$$\text{(13)} \qquad \frac{dx}{dt}(t) = (A(t) + B(t)[U(t - 1)\, U^{-1}(t) + Q(t)])\, x(t)$$

fulfils (1). *Moreover, solutions of* (13) *belong to $\mathscr{X}(\gamma)$ with $\gamma = a + \log[1 + \beta(e^a + L)]$, so that* $\dim \mathscr{X}(\gamma) \geqq n$.

As $\int_{t-1}^{t} |B(\tau)| \, d\tau \leqq (\int_{t-1}^{t} |B(\tau)|^2)^{1/2}$, Proposition may be applied if B fulfils (4) and if (10) and (11) hold, β being replaced by b.

Fix a and choose L, e.g. $L = e^a$. Find such a b that (10) and (11) are fulfilled for β being replaced by b and that the inequality in (6) is fulfilled with $\gamma \geqq a + \log [1 + b(e^a + L)]$. Then it may be concluded that $\dim \mathscr{X}(\gamma) = n$ (provided that A and B fulfil (4)).

Theorem 1 will be deduced from Theorem 2, which will be formulated below. If X, Y are linear spaces, $X \subset Y$ the codimension of X with respect to Y will be denoted by $\operatorname{codim}(X \mid Y)$ or $\operatorname{codim} X$ if no confusion can arise. If Y is a Hilbert space, then $\langle x, y \rangle$ will be the scalar product of $x, y \in Y$, $\|y\|$ will be the norm of y and if $C : Y \to Y$ is linear and continuous, then $\|C\| = \sup_{\|y\| \leqq 1} \|Cy\|$.

Let H be a Hilbert space, k_j integers, $r_j \in R^+$, $j = 0, 1, 2, \ldots$ such that

$$\lim_{j \to \infty} r_j = 0\,, \quad 0 = k_0 < k_1 < k_2 < \ldots \tag{14}$$

Denote by $\Omega = \Omega\begin{pmatrix} r_0, & r_1, & r_2, & \ldots \\ k_0, & k_1, & k_2, & \ldots \end{pmatrix}$ the set of bounded linear operators $Q : H \to H$ which fulfil the following condition:

(15) there exist linear subspaces $H^{(j)}$ of H such that $H^{(0)} = H$, $H^{(j)} \supset H^{(j+1)}$, $\operatorname{codim}(H^{(j)} \mid H) \leqq k_j$ and $\|Qx\| \leqq r_j \|x\|$ for $x \in H^{(j)}$, $j = 0, 1, 2, \ldots$

(Subspaces $H^{(j)}$ may depend on $Q \in \Omega$.)

Note 3. If $T \in \Omega\begin{pmatrix} r_0, & r_1, & r_2, & \ldots \\ k_0, & k_1, & k_2, & \ldots \end{pmatrix}$, then T is completely continuous. In order to show it, let $H^{(j)}$ be the linear subspaces of H which correspond to T according to (15). Denote by $Y^{(j)}$ the orthogonal complements of $H^{(j)}$ and define a linear operator $U^{(j)} : H \to H$ by $U^{(j)} y = Ty$ for $y \in Y^{(j)}$, $U^{(j)} z = 0$ for $z \in H^{(j)}$. By $\mathfrak{N}(U)$ denote the null-space of a linear operator U. Obviously $\mathfrak{N}(U^{(l)}) \supset H^{(j)}$ for $l \leqq j$ and it may be seen that the following conditions are fulfilled:

$$\operatorname{codim}\left(\prod_{l=0}^{j} \mathfrak{N}(U^{(l)})\right) \leqq k_j\,, \tag{i}$$

$$\|1 - U^{(j)}\| \leqq r_j\,, \quad j = 0, 1, 2, \ldots \tag{ii}$$

Therefore T is completely continuous. On the other hand, if $T : H \to H$ is a linear operator and if there exist finitedimensional operators $U^{(j)} : H \to H$ such that conditions (i), (ii) are fulfilled, then $T \in \Omega\begin{pmatrix} r_0, & r_1, & r_2, & \ldots \\ k_0, & k_1, & k_2, & \ldots \end{pmatrix}$.

Note 4. If $T : H \to H$ is linear and completely continuous, then there exist k_j, r_j fulfilling (14) such that $T \in \Omega\begin{pmatrix} r_0, & r_1, & r_2, & \ldots \\ k_0, & k_1, & k_2, & \ldots \end{pmatrix}$. Otherwise there exists $\varepsilon > 0$ such that for any linear subspace $V \subset H$ such that $\operatorname{codim}(V \mid H) < \infty$ there exists $v \in V$, $\|Tv\| \geqq \varepsilon\|v\| > 0$. By induction there exists a sequence $v_j \in H$, $j = 1, 2, \ldots$ such that $\|v_j\| = 1$, $\|Tv_j\| \geqq \varepsilon$, $(Tv_j, Tv_k) = 0$ for $j \neq k$. Followingly $\|Tv_j - Tv_k\| \geqq \varepsilon$ for $j \neq k$ and T is not completely continuous.

For $\varrho \geqq 1$, $m = 1, 2, 3, \ldots$ find s that $k_s < m \leqq k_{s+1}$ $s = 0, 1, 2, \ldots$ and put

$$S(\varrho, m) = \varrho^m r_0^{k_1} r_1^{k_2 - k_1} \ldots r_{s-1}^{k_s - k_{s-1}} r_s^{m - k_s} .$$

Obviously $S(\varrho, m) \to 0$ with $m \to \infty$. Let $\vartheta(\varrho)$ be the smallest (nonnegative) integer such that $S(\varrho, \vartheta(\varrho) + 1) < 1$.

Let $Q_i \in \Omega$ for $i = -1, -2, -3, \ldots$ Denote by $Z(\varrho)$, $\varrho \geqq 1$ the set of such sequences $\{x_i\}_{i=0}^{-\infty}$, $x_i \in H$ that

$$(16) \qquad Q_i x_i = x_{i+1}, \quad i = -1, -2, \ldots$$

$$(17) \qquad \sup_{i \leqq 0} \varrho^i \|x_i\| < \infty .$$

$Z(\varrho)$ is obviously a linear manifold.

Theorem 2. $\dim Z(\varrho) \leqq \vartheta(\varrho)$ *for* $\varrho \geqq 1$.

Corollary.

(18) *if* $\varrho r_0 < 1$, *then* $\vartheta(\varrho) = 0$, *i.e.* $\dim Z(\varrho) = 0$;

(19) *if* $\varrho r_0 \geqq 1$, $\varrho^{k_1+1} r_0^{k_1} r_1 < 1$ *then* $\vartheta(\varrho) = k_1$, *i.e.* $\dim Z(\varrho) \leqq k_1$;

(20) *if* $k_2 > k_1 + 1$, $\varrho^{k_1+1} r_0^{k_1} r_1 \geqq 1$, $\varrho^{k_1+2} r_1^{k_1} r_2^2 < 1$, *then* $\vartheta(\varrho) = k_1 + 1$, *i.e.* $\dim Z(\varrho) \leqq k_1 + 1$ *etc.*

Let $G : R^m \to R^m$ be linear. Choose $\{e_1, \ldots, e_m\}$, $\{f_1, \ldots, f_m\}$ – orthonormal bases in R^m and put

$$(21) \qquad (Ge_i, f_j) = g_{j,i}$$

i.e. $G \sum_i \lambda_i e_i = \sum_i (\sum_j g_{j,i} \lambda_i) f_j$. It is easy to see that $\det g_{j,i}$ does not depend on the choice of orthonormal bases $\{e_1, \ldots, e_m\}$, $\{f_1, \ldots, f_m\}$; put $\det G = \det g_{j,i}$.

Lemma. *Let* $G : R^m \to R^m$ *be linear. Let* V_i, $i = 0, 1, 2, \ldots, l$ *be linear subspaces of* R^m, $R^m = V_0 \supset V_1 \supset \ldots \supset V_l$, $\operatorname{codim}(V_i \mid R^m) = k_i$, $r_i \geqq 0$, $i = 0, 1, 2, \ldots, l$ *and assume that*

$$(22) \qquad |Gx| \leqq r_i |x| \quad for \quad x \in V_i, \quad i = 0, 1, 2, \ldots, l .$$

Then

$$(23) \qquad |\det G| \leqq r_0^{k_1} \cdot r_1^{k_2 - k_1} \ldots r_{l-1}^{k_l - k_{l-1}} \cdot r_l^{m - k_l} .$$

Proof. For $u, v \in R^m$ let (u, v) denote the scalar product. Find an orthonormal basis $e_1, \ldots, e_m \in R^m$ such that $e_{k_s+1}, \ldots, e_{k_{s+1}} \in V_s$ for $s = 0, 1, \ldots, l$. Let G' be adjoint to G. Obviously $\det G = \det G'$ and – by the usual identification of R^m with its adjoint – $(\det G)^2 = \det G'G = \det((G'Ge_i, e_j))$. $((G'Ge_i, e_j))$ is a positive semidefinite matrix and by Hadamard inequality (cf. [4], II, (10,3) or [5], IX, §5)

$$\det((G'Ge_i, e_j)) \leqq \prod_{i=1}^{m} (G'Ge_i, e_i) = \prod_{i=1}^{m} (Ge_i, Ge_i) = r_0^{2k_1} r_1^{2(k_2-k_1)} \ldots r_l^{2(m-k_l)}$$

and (23) holds.

Proof of Theorem 2. Take at first the special case $\varrho = 1$ and put $m = \vartheta(1) + 1$. If Theorem 2 is false, there exist $\{x_i^{(j)}\}_{i \leqq 0} \in Z(1), j = 1, 2, \ldots, m$ linearly independent. If $x_i^{(j)}, j = 1, 2, \ldots, m$ are linearly dependent for some $i < 0$, then $x_r^{(j)}, j = 1, 2, \ldots, m$ are linearly dependent for any $r \geqq i$ with the same constants. Hence it can be shown that there exists such a $p \leqq 0$ that $x_i^{(j)}, j = 1, 2, \ldots, m$ are linearly independent for any $i \leqq p$.

For $i \leqq -1$ let $H_i^{(j)}$ be linear subspaces of H such that (15) is fulfilled (with $Q = Q_i$). Find s such that $k_s < m \leqq k_{s+1}$. Let $V_i^{(0)}$ be spanned by $x_i^{(j)}, j = 1, 2, \ldots, m$, $i \leqq p$. Obviously $\dim V_i^{(0)} = m$ and $\operatorname{codim}(V_i^{(0)} \cap H_i^{(j)} \mid V_i^{(0)}) \leqq k_j$, $j = 1, 2, \ldots$ Choose linear spaces $V_i^{(j)}, j = 1, 2, \ldots, s$ such that $V_i^{(j-1)} \supset V_i^{(j)}$, $V_i^{(j)} \subset V_i^{(0)} \cap H_i^{(j)}$ and $\operatorname{codim}(V_i^{(j)} \mid V_i^{(0)}) = k_j$, $j = 1, 2, \ldots, s$. $Q_i|_{V_i^{(0)}}$ maps $V_i^{(0)}$ onto $V_{i+1}^{(0)}$ for $i < p$ and by Lemma and by the choice of m

$$\left|\det(Q_i|_{V^{(0)}})\right| \leqq r_0^{k_1} r_1^{k_2-k_1} \ldots r_s^{m-k_s} = \varkappa < 1 .$$

Let Λ_i, $i \leqq p$ be the simplex with the vertices $0, x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(m)}$ and let λ_i be its volume. Obviously $Q_i(\Lambda_i) = \Lambda_{i+1}$ and therefore $\varkappa\lambda_i \geqq \lambda_{i+1}$. Hence $\lambda_i \to \infty$ with $i \to -\infty$ and this is impossible, as $\{x_i^{(j)} \mid j = 1, 2, \ldots, m,\ i = 0, -1, -2, \ldots\}$ is a bounded set. Theorem 2 holds in the special case $\varrho = 1$.

If $\varrho > 1$, put $\tilde{Q}_i = \varrho Q_i$, $\tilde{r}_j = \varrho r_j$, $i = -1, -2, \ldots, j = 0, 1, 2, \ldots$ and for $\tilde{\varrho} \geqq 1$ denote by $\tilde{Z}(\tilde{\varrho})$ the set of such sequences $\{\tilde{x}_i\}_{i \geqq 0}$, $\tilde{x}_i \in H$ that $\tilde{Q}_i \tilde{x}_i = \tilde{x}_{i+1}$ for $i = -1, -2, \ldots$ and $\sup_{i \leqq 0} \tilde{\varrho}^i \|x_i\| < \infty$. If $\{x_i\}_{i \leqq 0} \in Z(\varrho)$, put $\tilde{x}_i = \varrho^i x_i$, $i = 0, -1, -2, \ldots$ Obviously $\{\tilde{x}_i\}_{i \geqq 0} \leqq \tilde{Z}(1)$. Therefore $\dim Z(\varrho) = \dim \tilde{Z}(1)$ and the proof of Theorem 2 may be finished by applying Theorem 2 in case $\tilde{\varrho} = 1$ to $\tilde{Z}(1)$.

Proof of Theorem 1. For $S \subset \langle -1, 0 \rangle$ Lebesgue measurable denote by $|S|$ the Lebesgue measure of S, let $v_1(S) = 1$ if $-1 \in S$, $v_1(S) = 0$ otherwise, let $v_2(S) = 1$ if $0 \in S$, $v_2(S) = 0$ otherwise and put $\mu(S) = |S| + v_1(S) + v_2(S)$. Let $H = L_{2,\mu}(\langle -1, 0 \rangle \to R^n)$, (i.e. elements of H are classes of μ-equivalent square integrable functions from $\langle -1, 0 \rangle$ to R^n). If $u, v \in R^n$, let (u, v) be the scalar product of u, v and for $x, y \in H$ define the scalar product by

$$\langle x, y \rangle = (x(-1), y(-1)) + \int_{-1}^{0} (x(t), y(t))\, dt + (x(0), y(0)) .$$

Let $U : R^- \to R^n$ be a fundamental matrix of

$$\frac{dx}{dt}(t) = A(t)\, x(t)\,.$$

Define $U_i : (-\infty, -i\rangle \to M_n$ by $U_i(t) = U(t + i)\, U^{-1}(i)$, $i = 0, -1, -2, \ldots$ and $Q_i : H \to H$ by

$$(24)\quad (Q_i y)(t) = U_i(t + 1)\, y(0) + U_i(t + 1) \int_0^{t+1} U_i^{-1}(\sigma)\, B(\sigma + i)\, y(\sigma - 1)\, d\sigma\,.$$

The estimate

$$(25)\qquad |U_i(t + 1)| \leqq e^a\,, \quad |U_i(t + 1)\, U_i^{-1}(\sigma)| \leqq e^a \quad \text{for} \quad i = -1, -2, \ldots,$$
$$t \in \langle -1, 0\rangle\,, \quad \sigma \in \langle 0, t + 1\rangle$$

follows from (4): Keep σ and i fixed and put $L(\tau) = U_i(\tau)\, U_i^{-1}(\sigma)$. Obviously $L(\tau) = I + \int_\sigma^\tau A(i + \zeta)\, L(\zeta)\, d\zeta$, I being the identity matrix and $L(\tau) = \lim_{j\to\infty} L_j(\tau)$ with $L_0(\tau) = I$, $L_{j+1}(\tau) = I + \int_\sigma^\tau A(i + \zeta)\, L_j(\zeta)\, d\zeta$, $j = 0, 1, 2, \ldots$ Put $\alpha(\tau) = \left|\int_\sigma^\tau |A(\zeta)|\, d\zeta\right|$. As $|I| = 1$, we obtain by induction that $|L_j(\tau)| \leqq e^{\alpha(\tau)}$ for $\tau \in (-\infty, -i\rangle$ and the second inequality in (25) holds. The first inequality in (25) is a special case of the second one for $\sigma = 0$.

For $x \in \mathscr{X}(\gamma)$, $i = 0, -1, -2, \ldots$ define $x_i \in H$ by $x_i(t) = x(i + t)$ and put $Px = \{x_i\}_{i \leqq 0}$. The following Lemma is easy to verify.

Lemma 3. *P is a linear bijection of $\mathscr{X}(\gamma)$ onto $Z(e^\gamma)$.*

In order to deduce Theorem 1 from Theorem 2 we have to find numbers $r_j \in R^+$ and integers k_j, $j = 0, 1, 2, \ldots$ $r_j \geqq r_{j+1}$, $\lim_{j\to\infty} r_j = 0$, $0 = k_0 < k_1 < k_2 < \ldots$ such that $Q_i \in \Omega\begin{pmatrix} r_0, r_1, \ldots \\ k_0, k_1, \ldots \end{pmatrix}$, $i = -1, -2, -3, \ldots$ r_j and k_j will dpend on a, b; we will denote the corresponding function ϑ by $\vartheta_{a,b}$ and we shall put $\Theta(a, b, \gamma) = \vartheta_{a,b}(e^\gamma)$. Obviously

$$(26)\qquad \|Q_i y\|^2 = |y(0)|^2 + $$
$$+ \int_{-1}^0 \left| U_i(t + 1)\, y(0) + \int_0^{t+1} U_i(t + 1)\, U_i^{-1}(\sigma)\, B(\sigma + i)\, y(\sigma - 1)\, d\sigma \right|^2 dt +$$
$$+ \left| U_i(1)\, y(0) + \int_0^1 U_i(1)\, U_i^{-1}(\sigma)\, B(\sigma + i)\, y(\sigma - 1)\, d\sigma \right|^2 \quad \text{for} \quad y \in H\,.$$

Using $(\alpha + \beta)^2 \leqq 2(\alpha^2 + \beta^2)$ we obtain from (25) and (26) that

$$(27)\qquad \|Q_i y\|^2 \leqq |y(0)|^2\, (1 + 4e^{2a}) + 4e^{2a} b^2 \int_{-1}^0 y^2(\sigma)\, d\sigma$$

and hence we may put

$$(28) \qquad r_0 = [1 + 4e^{2a} \max(1, b^2)]^{1/2} .$$

Define linear functionals $\varphi_k : H \to R^n$, $k = 1, 2, \ldots$ by

$$\begin{aligned}
\varphi_1(y) &= y(0) , \\
\varphi_2(y) &= \int_0^1 U_i^{-1}(\sigma)\, B(\sigma + i)\, y(\sigma - 1)\, d\sigma , \\
\varphi_3(y) &= \int_{-1}^0 (Q_i y)(t)\, dt , \\
\varphi_{2s}(y) &= \sqrt{2} \int_{-1}^0 (Q_i y)(t) \cos(2\pi(s-1)\, t)\, dt , \\
\varphi_{2s+1}(y) &= \sqrt{2} \int_{-1}^0 (Q_i y)(t) \sin(2\pi(s-1)\, t)\, dt , \quad s = 1, 2, 3, \ldots
\end{aligned}$$

Put $H_i^{(0)} = H$, $H_i^{(j)} = \{ y \in H \mid \varphi_l(y) = 0,\ l = 1, 2, \ldots, 2j - 1 \}$, $j = 1, 2, \ldots$ $i = -1, 2, \ldots$; therefore we may define

$$(29) \qquad k_j = n(2j - 1) .$$

If $y \in H_i^{(1)}$, then

$$(Q_i y)(t) = \int_0^{t+1} U_i(t + 1)\, U_i^{-1}(\sigma)\, B(\sigma + i)\, y(\sigma - 1)\, d\sigma ;$$

hence $\|Q_i y\| \leqq e^a b \|y\|$ and we may put

$$(30) \qquad r_1 = e^a b .$$

It follows from the Fourier expansion of $Q_i y$ that

$$(31) \qquad \|Q_i y\|^2 = \sum_{l=2j}^{\infty} (\varphi_l(y))^2 \quad \text{for} \quad y \in H_i^{(j)} , \quad j = 2, 3, \ldots, \quad i = -1, -2, \ldots$$

As $(Q_i y)(-1) = 0 = (Q_i y)(0)$ for $y \in H_i^{(2)}$, it follows that

$$\varphi_{2s}(y) = \sqrt{2} \int_{-1}^0 \int_0^{1+t} U_i(1 + t)\, U_i^{-1}(\sigma)\, B(\sigma + i)\, y(\sigma - 1)\, d\sigma \cos 2\pi(s - 1)\, t\, dt =$$

$$= \frac{-\sqrt{2}}{2\pi(s-1)} \int_{-1}^0 \Big[B(i + 1 + t)\, y(t) + A(i + 1 + t) \int_0^{1+t} U_i(1 + t)\, U_i^{-1}(\sigma)\, B(\sigma + i) \,.$$

$$.\, y(\sigma - 1)\, d\sigma \Big] \sin 2\pi(s - 1)\, t\, dt , \quad y \in H_i^{(2)} , \quad s = 2, 3, \ldots , \quad i = -1, -2, \ldots$$

Hence (cf. (25))

$$|\varphi_{2s}(y)| \leqq \frac{\sqrt{2}}{2\pi(s-1)} [1 + ae^a] \, b\|y\|$$

and similarly

$$|\varphi_{2s+1}(y)| \leqq \frac{\sqrt{2}}{2\pi(s-1)} [1 + ae^a] \, b\|y\| .$$

It follows from (31) that

$$\|Q_i y\|^2 \leqq \frac{1}{2\pi^2} [1 + ae^a]^2 \, b\|y\|^2 \, 2 \sum_{l=j}^{\infty} \frac{1}{(l-1)^2} \leqq$$

$$\leqq \frac{1}{\pi^2(j-2)} [1 + ae^a]^2 \, b\|y\|^2 , \quad y \in H_i^{(j)} , \quad j = 3, 4, \ldots , \quad i = -1, -2, \ldots$$

and we may put (cf. (30))

$$r_2 = e^a b , \quad r_j = \frac{1}{\pi} [1 + ae^a] \, b(j-2)^{-1/2} , \quad j = 3, 4, \ldots \tag{32}$$

The assumptions of Theorem 2 are fulfilled and Theorem 1 is proved completely (cf. Lemma 3).

(6) and (7) in Note 1 follow from (18) and (19) in Corollary and (28), (29), (30) and (32). Let us indicate, how (8) may be obtained. For $m = 1, 2, 3, \ldots$ define the iteger $t(m)$ by

$$(2t(m) + 1)\, n < m \leqq (2t(m) + 3)\, n . \tag{33}$$

As $e^a b \geqq 1$, it follows that $S(e^\gamma, m) \geqq 1$ for $m = 1, 2, \ldots, 5n$, (cf. (28), (29), (30) and (32)). $S(e^\gamma, m)$ may be given the following form for $m > 5n$

$$\begin{aligned} S(e^\gamma, m) = {} & (\pi^{-1} e^\gamma (1 + ae^a)\, b)^{m-5n} \,.\, e^{5n\gamma} (1 + 4e^{2a} \max(1, b^2))^{n/2} \,. \\ & .\, (e^a b)^{4n} \,.\, ((t(m) - 2)!)^{-n} \,.\, (t(m) - 1)^{-(m-(2t(m)+1)n)/2} . \end{aligned} \tag{34}$$

Let η be such an integer that

$$S(e^\gamma, \eta) \geqq 1 > S(e^\gamma, \eta + 1) . \tag{35}$$

It is easy to see that $\eta \geqq 6$ and that η is unique. $\eta = \Theta(a, b, \gamma)$ by definition of ϑ and Θ.

Let φ be the smallest integer greater than $\pi^{-1} e^\gamma (1 + ae^a)\, b$. Applying Stirling formula ($s! = (s/e)^s . (2\pi s)^{1/2} \, \psi_1(s)$, $\psi_1(s) \to 1$ with $s \to \infty$) to (34) we obtain (cf.

(33)) that $S(e^{\gamma}, \varphi) > 1$ so that

$$\eta \geqq \pi^{-1} e^{\gamma}(1 + ae^{a})\, b \tag{36}$$

(the right hand side in (36) being sufficiently large).

(34) implies that

$$S(e^{\gamma}, \eta)\, S^{-1}(e^{\gamma}, \eta + 1) = (t(\eta + 1) - 2)^{1/2}\, \pi e^{-\gamma}(1 + ae^{a})^{-1}\, b^{-1}$$

and by (35)

$$1 \leqq S(e^{\gamma}, \eta) \leqq (t(\eta + 1) - 2)^{1/2}\, \pi e^{-\gamma}(1 + ae^{a})^{-1}\, b^{-1}\,. \tag{37}$$

(36), (37) and (33) imply that

$$(S(e^{\gamma}, \eta))^{1/\eta} \to 1 \quad \text{with} \quad e^{\gamma}(1 + ae^{a})\, b \to \infty\,. \tag{38}$$

By Stirling formula $(s!)^{1/s} = (s/e)\, \psi_2(s)$, $\psi_2(s) \to 1$ with $s \to \infty$. Observe that

$$((t(m) - 2)!)^{-n/m} = \left(\frac{2ne}{m}\right)^{1/2} \psi_3(m)\,, \quad \psi_3(m) \to 1 \quad \text{with} \quad m \to \infty \tag{39}$$

and (as $0 < m - (2t(m) + 1)\, n \leqq 2n$)

$$(t(m) - 1)^{-(m-(2t(m)+1)n)/2m} \to 1 \quad \text{with} \quad m \to \infty\,. \tag{40}$$

Obviously

$$(1 + ae^{a})^{-2}\, b^{-2} \leqq (1 + 4e^{2a} \max(1, b^{2}))\,(1 + ae^{a})^{-2}\, b^{-2} \leqq 1$$
$$(1 + ae^{a})\, b \geqq e^{a}b \geqq 1$$

and

$$[((1 + ae^{a})\, b)^{-1}]^{((1+a^{a})b)^{-1}} \geqq e^{-e^{-1}}\,.$$

Therefore (cf. (36))

$$((1 + ae^{a})^{-1}\, b^{-1})^{1/\eta} \to 1 \quad \text{with} \quad e^{\gamma}(1 + ae^{a})\, b \to \infty$$

and

$$[(1 + 4e^{2a} \max(1, b^{2}))\,(1 + ae^{a})^{-2}\, b^{-2}]^{n/2\eta} \to 1 \quad \text{with} \quad e^{\gamma}(1 + ae^{a})\, b \to \infty\,. \tag{41}$$

As $(e^{-a} + a)^{-1} = e^{a}b(1 + ae^{a})^{-1}\, b^{-1} \leqq 1$, it may be shown (in a similar way as (41)) that

$$[e^{a}b(1 + ae^{a})^{-1}\, b^{-1}]^{4n/\eta} \to 1 \quad \text{with} \quad e^{\gamma}(1 + ae^{a})\, b \to \infty\,. \tag{42}$$

Substituting (34) in (38) and making use of the Stirling formula, (36), (39)–(42) we obtain that

$$\pi^{-1} e^{\gamma}(1 + ae^{a})\, b(2ne)^{1/2}\, \eta^{-1/2} \to 1 \quad \text{with} \quad e^{\gamma}(1 + ae^{a})\, b \to \infty\,,$$

which is equivalent to (8).

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