Časopis pro pěstování matematiky, roč. 98 (1973), Praha

ON THE MINIMUM NUMBER OF VERTICES AND EDGES IN A GRAPH WITH A GIVEN NUMBER OF SPANNING TREES

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(Received May 31, 1971)

By a graph we shall mean a finite connected undirected graph without loops and multiple edges (for notions and results of graph theory see, for example, [1] or [2]). If p, q and r are integers such that $1 \leqq p \leqq q \leqq r$ and $2 \leqq q$ then by $D(p, q, r)$ we shall denote the graph with cyclomatic number 2 and with no separating vertex and such that its two vertices of degree 3 are connected to each other by arcs ([2]) of length p, q and r; the graph $D(p, q, r)$ has of course $p + q + r - 1$ vertices, $p + q + r$ edges and $pq + qr + pr$ spanning trees.

In the following, by x we shall denote a positive integer other than 2. By $\alpha(x)$ we denote the smallest number y_1 such that there is a graph having y_1 vertices and x spanning trees; by $\beta(x)$ we denote the smallest number y_2 such that there is a graph having y_2 edges and x spanning trees. Obviously $\alpha(x) \leqq \beta(x) \leqq x$, for any $x \geqq 3$. The function α has been studied by J. SEDLÁČEK [3], who also gave an impulse to the rise of the present paper.

The very simple generalization of one of the procedures used in [3] for the estimate of the function α leads to the following estimate of the function β which is given by graphs with at least one separating vertex: if x_1 and x_2 are integers and $x_1, x_2 \geqq 3$, then

$$\beta(x_1 x_2) \leqq \beta(x_1) + \beta(x_2)\,. \tag{1}$$

By making use of the graph $D(1, 2, (x - 2)/3)$ and a graph with no separating edge and with two circuits of length 3 and $x/3$, J. Sedláček [3] found an upper estimate of the function α for almost all $x \equiv 2, 3 \pmod 3$. By using the same graphs it is quite readily possible to find an estimate of the function β for the same values of the argument:

$$\text{if} \quad x \equiv 2 \pmod 3\,, \quad x \geqq 8\,, \quad \text{then} \quad \beta(x) \leqq (x + 7)/3\,; \tag{2}$$

$$\text{if} \quad x \equiv 3 \pmod 3\,, \quad x \geqq 9\,, \quad \text{then} \quad \beta(x) \leqq (x + 9)/3\,. \tag{3}$$

Estimate (3) of course also follows from estimate (1). Upper estimates of the function β (and hence also the function α) for almost all $x \equiv 1 \pmod 3$ are given by the following lemma.

Lemma. *It holds that:*

(4) *if* $x \equiv 1 \pmod{30}$, $x \geqq 91$, *then* $\beta(x) \leqq (x + 269)/30$;

(5) *if* $x \equiv 16 \pmod{30}$, $x \geqq 106$, *then* $\beta(x) \leqq (x + 254)/30$;

(6) *if* $x \equiv 4 \pmod{30}$, $x \geqq 64$, *then* $\beta(x) \leqq (x + 206)/30$;

(7) *if* $x \equiv 19 \pmod{30}$, $x \geqq 79$, *then* $\beta(x) \leqq (x + 221)/30$;

(8) *if* $x \equiv 7 \pmod{15}$, $x \geqq 37$, *then* $\beta(x) \leqq (x + 98)/15$;

(9) *if* $x \equiv 10 \pmod{15}$, $x \geqq 40$, *then* $\beta(x) \leqq (x + 110)/15$;

(10) *if* $x \equiv 13 \pmod{15}$, $x \geqq 43$, *then* $\beta(x) \leqq (x + 92)/15$.

Proof. By G_1 we denote the graph with 10 vertices $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5, c_0$ and 11 edges $c_0a_1, a_1a_2, a_2a_3, a_3a_4, a_4c_0, c_0b_1, b_1b_2, b_2b_3, b_3b_4, b_4b_5, b_5c_0$; G_1 obviously has 30 spanning trees. By G_2 we denote the graph with 6 vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and with 8 edges $a_1a_2, a_2a_3, a_3a_1, b_1b_2, b_2b_3, b_3b_1, a_1b_1, a_3b_3$; G_2 obviously has 30 spanning trees. By G_3 we denote the graph with 7 vertices $a_1, a_2, b_1, b_2, b_3, b_4, c_0$ and 8 edges $c_0a_1, a_1a_2, a_2c_0, c_0b_1, b_1b_2, b_2b_3, b_3b_4, b_4c_0$; G_3 obviously has 15 spanning trees. We now construct graphs $G_4, \ldots, G_{10}$ such that in any one of the graphs G_i, $i = 1, 2, 3$, we select vertices v and w, and then complete the respective graph G_i by $j - 1$ vertices and j edges so that the vertices v and w are connected to each other by an arc of length j of which every inner vertex is different from all vertices of the graph G_i. We obtain the graph $G_4, \ldots, G_{10}$, by selecting i, v, w and j as follows (j is, of course, always an integer):

$$G_4\colon\ i = 1,\quad v = a_2,\quad w = b_1,\quad j = (x - 61)/30 \geqq 1;$$
$$G_5\colon\ i = 1,\quad v = a_2,\quad w = b_2,\quad j = (x - 76)/30 \geqq 1;$$
$$G_6\colon\ i = 2,\quad v = a_1,\quad w = b_2,\quad j = (x - 34)/30 \geqq 1;$$
$$G_7\colon\ i = 2,\quad v = a_1,\quad w = a_2,\quad j = (x - 19)/30 \geqq 2;$$
$$G_8\colon\ i = 3,\quad v = a_1,\quad w = b_1,\quad j = (x - 22)/15 \geqq 1;$$
$$G_9\colon\ i = 3,\quad v = a_1,\quad w = a_2,\quad j = (x - 10)/15 \geqq 2;$$
$$G_{10}\colon\ i = 3,\quad v = a_1,\quad w = b_2,\quad j = (x - 28)/15 \geqq 1.$$

There is little difficulty in seeing that the numbers of edges of the graphs $G_4, \ldots, G_{10}$ give successively estimations (4)–(10).

Theorem 1. *If* $x = 1$, *then* $\alpha(x) = 1$, $\beta(x) = 0$; *if* x *is one of the numbers* 3, 4, 5, 6, 7, 10, 13, 22, *then*

(11) $$\alpha(x) = \beta(x) = x;$$

if $x = 8$, *then* $\alpha(x) = 4$, $\beta(x) = 5$; *if* $x = 9$, *then* $\alpha(x) = 5$, $\beta(x) = 6$. *Otherwise*

$$(12) \qquad \alpha(x) < \beta(x) \leqq \frac{x + 1}{2}.$$

Proof. The cases $x \leqq 10$ are easily verifiable; the value of the function α for $x \leqq 9$ have been given by J. Sedláček [3]. From (2) it follows that (12) holds for $x = 11$. The graph $D(2, 2, 2)$ leads to estimate (12) for $x = 12$. There is no graph with cyclomatic number 2 which has 13 spanning trees, and any graph with a greater cyclomatic number has more than 13 spanning trees; hence (11) holds for $x = 13$. There is no graph with cyclomatic number 2 or 3 which has 22 spanning trees, and any graph with a greater cyclomatic number has more than 22 spanning trees; hence (11) holds for $x = 22$. If $x \geqq 106$ it is possible to use exactly one of the estimates (2)–(10) for it; this one estimate then leads to estimate (12).

Now, let us assume that $14 \leqq x < 106$, $x \neq 22$. In so far as it is possible to use for such an x any of estimates (1)–(10), we obtain estimate (12) for it. There remain the cases $x = 19, 31, 34, 46$ and 61; for these x it is possible to obtain estimate (12) by graphs $D(1, 3, 4)$, $D(1, 3, 7)$, $D(1, 4, 6)$, $D(2, 3, 8)$ and $D(3, 4, 7)$ in turn. The proof is complete.

Now we shall turn to other relationship between the functions α and β.

Theorem 2. *Let z be an integer such that $z \geqq 11$ and $z \neq 13, 22$. Then there is no graph having simultaneously $\alpha(z^{z-2})$ vertices, $\beta(z^{z-2})$ edges and z^{z-2} spanning trees.*

Proof. The only graph having $\alpha(z^{z-2})$ vertices and z^{z-2} spanning trees is the complete graph having z vertices; it has $z(z-1)/2$ edges. From (1) and (12) it follows that $\beta(z^{z-2}) \leqq (z-2)\,\beta(z) \leqq (z-2)(z+1)/2 < z(z-1)/2$. The proof is complete.

References

[1] *R. G. Busacker, T. L. Saaty:* Finite Graphs and Networks: An Introduction with Applications, McGraw-Hill, New York 1965.

[2] *O. Ore:* Theory of Graphs, Amer. Math. Soc. Colloq. Publ. *38*, Providence 1962.

[3] *J. Sedláček:* On the minimal graph with a given number of spanning trees, Canad. Math. Bull. *13* (1970), 515–517.

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