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ON ASYMPTOTIC BEHAVIOUR OF CENTRAL DISPERSIONS
OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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1.1. Consider a differential equation

\( y'' = q(t) y, \quad q \in C^0[a, b], \quad b \leq \infty \)

where \( C^n[a, b] \) (\( n \) being a non-negative integer) is the set of all continuous functions having continuous derivatives up to and including the order \( n \) on \([a, b]\). In all the paper we will deal only with oscillatory \((t \to b_-)\) differential equations (i.e. every non-trivial solution has infinitely many zeros on every interval of the form \([t_0, b]\), \( t_0 \in [a, b] \)).

Let \( y \) be a non-trivial solution of \((q)\) vanishing at \( t \in [a, b] \). If \( \varphi(t) \) is the first zero of \( y \) lying to the right from \( t \), then \( \varphi \) is called the basic central dispersion of the 1-st kind of \((q)\) (briefly, dispersion of \((q)\)). The function \( \varphi \) has the following properties (see \([2] \) § 13):

1) \( \varphi \in C^3[a, b], \)
2) \( \varphi'(t) > 0 \) on \([a, b], \)
3) \( \varphi(t) > t \) on \([a, b], \)
4) \( \lim_{t \to b_-} \varphi(t) = b, \)
5) \( - \frac{1}{2} \frac{\varphi''}{\varphi'} + \frac{3}{4} \left( \frac{\varphi''}{\varphi'} \right)^2 + q(\varphi) \varphi'^2 = q(t), \quad t \in [a, b], \)
6) \( \varphi'(t) = \frac{q(t_1)}{q(t_2)}, \quad t < t_1 < t_2 < \varphi(t). \)

1.2. In our later considerations we will generalize some of the following results which have been proved by the author of \([3]\) (see also \([1]\)):
Theorem 1. Let \( q \in C^n[a, \infty) \), \( p \in C^n[a, \infty) \) (\( n \geq 0 \) being an integer) and let \( q \in C^1[a, \infty) \) if \( n = 0 \). Let \( \limsup_{t \to \infty} q(t) < 0 \), \( \liminf_{t \to \infty} q(t) > -\infty \), \( \lim_{t \to \infty} (q(t) - p(t)) = 0 \), \( q'(t) = 0 \) and if \( n > 0 \) let \( \lim_{t \to \infty} q^{(k)}(t) = 0 \), \( \lim_{t \to \infty} p^{(k)}(t) = 0 \) for \( k = 1, 2, \ldots, n \).

If \( \varphi, \tilde{\varphi} \) are the dispersions of \( q \) and \( y^* = p(t)y \), respectively, then
\[
\lim_{t \to \infty} (\varphi(t) - \tilde{\varphi}(t)) = 0, \quad k = 1, 2, \ldots, n + 3.
\]

Lemma 1. Let \( q \in C^0[a, b) \), \( \limsup_{t \to b^-} q(t) < 0 \). Let \( \varphi \) be the dispersion of the differential equation \( q \). Then there exists a number \( k > 0 \) such that
\[
\varphi(t) - t \leq k, \quad t \in [a, b).
\]

2. Lemma 2. Let \( \varphi \) be the dispersion of an oscillatory \((t \to b_-)\) differential equation \( q \), \( q \in C^1[a, b) \), \( \limsup_{t \to b^-} q(t) < 0 \). Let
\[
\lim_{t \to b^-} \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi(t) - t) = 0.
\]

Then
\[
\lim_{t \to b^-} \varphi'(t) = 1,
\]
\[
\lim_{t \to b^-} \varphi''(t) = \lim_{t \to b^-} \varphi'''(t) = 0.
\]

Proof. It follows from the assumption \( \limsup_{t \to b^-} q(t) = c < 0 \) and from Lemma 1 that there exist numbers \( t_0 \in [a, b) \), \( K > 0 \) such that we have
\[
\varphi(t) - t \leq K, \quad t \in [a, b),
\]
\[
q(t) \leq \frac{c}{2}, \quad t \in [t_0, b).
\]

Then according to (1) 6, we obtain for \( t \geq t_0 \) that
\[
|q(\varphi)(1 - \varphi')| = |q(\varphi)\left|1 - \frac{q(t_1)}{q(t_2)}\right| = \left|\frac{q(\varphi)}{q(t_2)}\right| |q(t_2) - q(t_1)| \leq
\]
\[
\leq \frac{|q(t_2)| + M_1(t)}{|q(t_2)|} \cdot |q'(\xi)| \cdot (t_2 - t_1) \leq \left(1 + \frac{2}{c} M_2\right) \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi(t) - t)
\]
holds where \( t < t_1 < t_2 < \varphi(t) \), \( \xi \in (t_1, t_2) \), \( \eta \in (t_2, \varphi(t)) \),
\[
M_1(t) = |q'(t)| (\varphi(t) - t) \leq \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi - t) \xrightarrow{t \to b^-} 0,
\]
\[
M_2 = \max_{t \in (a, b)} M_1(t).
\]
Hence it follows
\begin{equation}
\lim_{t \to b^-} \left| q(\phi) (1 - \phi') \right| = 0
\end{equation}
and thus \( \lim_{t \to b^-} \phi'(t) = 1 \). So the first part of the statement is proved.

According to (2) and (1), we have
\begin{align*}
\lim_{t \to b^-} \left| -\frac{1}{2} \phi'' \phi' + \frac{3}{2} \phi'^2 \right| &= \lim_{t \to b^-} \left| q(t) - q(\phi) \phi'^2 \right| = \\
&= \lim_{t \to b^-} \left| (q(t) - q(\phi)) + q(\phi) (1 - \phi'^2) \right| = \lim_{t \to b^-} |q(t) - q(\phi)| = \\
&= \lim_{t \to b^-} |q'(\xi) (t - \phi)| \leq \lim_{t \to b^-} \max_{x \in [t, \phi(t)]} |q'(x)| (\phi - t) = 0
\end{align*}
where \( \xi \in (t, \phi) \). So
\begin{equation}
\lim_{t \to b^-} \left| -\frac{1}{2} \phi'' \phi' + \frac{3}{2} \phi'^2 \right| = 0.
\end{equation}

Suppose \( \lim_{t \to b^-} \phi'' = c > 0 \). Then \( \lim_{t \to b^-} \phi'(t) = \pm \infty \) but this is in contradiction with the proved part of the lemma. Assume that \( \lim_{t \to b^-} \phi'' \) does not exist. Let \( M = \{ t \in [a, b), \phi''(t) = 0 \} \). Then the set \( M \) contains every local maximum of the function \( \phi'' \) and the point \( t = b \) is an accumulation point of \( M \). According to (3),
\begin{equation}
\lim_{t \to b^-} \phi''(t) = 0
t_{\in M}
\end{equation}
holds and hence we have \( \lim_{t \to b^-} \phi''(t) = 0 \). But this is in contradiction with our assumption.

Thus \( \lim_{t \to b^-} \phi''(t) = 0 \) and the rest of the statement follows from (3).

**Lemma 3.** Let \((q), (\bar{q})\) be oscillatory \((t \to b_-)\) differential equations such that \(q \in C^0[a, b), \bar{q} \in C^1[a, b), \limsup_{t \to b_-} \bar{q}(t) < 0\) and \(\lim_{t \to b_-} (q(t) - \bar{q}(t)) = 0\). Let \(\phi(\bar{q})\) be the dispersion of \((q)\) (\((\bar{q})\)) and let
\begin{equation}
\lim_{t \to b_-} \max_{x \in [t, \phi(t)]} \left| \bar{q}'(x) \right| (\phi(t) - t) = 0
\end{equation}
where \(\phi(t) = \max (\phi(t), \bar{q}(t))\). Then
\begin{align*}
\lim_{t \to b_-} \phi'(t) &= 1, \\
\lim_{t \to b_-} \phi''(t) &= \lim_{t \to b_-} \phi'''(t) = 0.
\end{align*}
Proof. By virtue of (1) 6, we have:

\[
\bar{q}(\tilde{\varphi}(t)) (\varphi'(t) - \tilde{\varphi}'(t)) = \bar{q}(\tilde{\varphi}(t)) \cdot \left( \frac{q(t_1)}{\bar{q}(t_2)} - \frac{\bar{q}(t_3)}{\bar{q}(t_4)} \right) = \\
\bar{q}(\tilde{\varphi}(t)) \cdot \left( \frac{q(t_1)}{\bar{q}(t_2)} \bar{q}(t_4) - \frac{\bar{q}(t_3)}{\bar{q}(t_4)} q(t_2) \right) = \\
\bar{q}(\tilde{\varphi}(t)) \cdot \left[ \frac{q(t_4)}{\bar{q}(t_2)} (q(t_1) - \bar{q}(t_1)) + \frac{\bar{q}(t_3)}{\bar{q}(t_4)} (\bar{q}(t_2) - q(t_2)) + \\
\bar{q}(t_1) (\bar{q}(t_4) - \bar{q}(t_2)) - \bar{q}(t_2) (\bar{q}(t_3) - \bar{q}(t_1)) \right],
\]

where

\[
t < t_1 < t_2 < \varphi(t), \quad t < t_3 < t_4 < \tilde{\varphi}(t).
\]

It follows from the relations

\[
\lim_{t \to b^-} \bar{q}(t) = c < 0, \quad \lim_{t \to b^-} (q(t) - \bar{q}(t)) = 0
\]

that there exists \( t_0 \in [a, b) \) such that

\[
|q(t)| \geq \frac{1}{2} |\bar{q}(t)| \geq \frac{|c|}{4}, \quad t \in [t_0, b).
\]

Then the following inequalities are valid for \( t \in [t_0, b) \) and \( t_5 \in [t, \varphi(t)) \) (by the Taylor Theorem):

\[
\left| \frac{\bar{q}(\tilde{\varphi}(t)) \cdot \bar{q}(t_5)}{\bar{q}(t_2) \cdot \bar{q}(t_4)} \right| \leq 2 \left| \frac{\bar{q}(\tilde{\varphi}(t))}{\bar{q}(t_2)} \cdot \frac{\bar{q}(t_5)}{\bar{q}(t_4)} \right| \leq 2 \cdot \frac{|\bar{q}(t_2)| + M_1(t)}{|\bar{q}(t_2)|} \cdot \frac{|\bar{q}(t_4)| + M_1(t)}{|\bar{q}(t_4)|} \leq \\
\leq 2 \cdot \left( 1 + 2 \cdot \frac{M_1(t)}{|c|} \right) \left( 1 + 2 \cdot \frac{M_1(t)}{|c|} \right) \leq 2 \cdot \left( 1 + 2 \cdot \frac{M_2}{|c|} \right)^2 = M_3 < \infty.
\]

Here

\[
M_1(t) = \max_{x \in [t, \varphi(t)]} |\bar{q}'(x)| (\varphi(t) - t) \xrightarrow{t \to b^-} 0, \quad M_2 = \max_{t \in [t_0, b)} M_1(t).
\]

Hence

\[
|\bar{q}(\tilde{\varphi}(t)) (\varphi'(t) - \tilde{\varphi}'(t))| \leq M_3 \cdot \left[ |q(t_1) - \bar{q}(t_1)| + |\bar{q}(t_2) - q(t_2)| + \\
+ |\bar{q}'(\xi_1)| (\varphi - t) + |\bar{q}'(\xi_2)| (\varphi - t) \right] \xrightarrow{t \to b^-} 0
\]

for \( \xi_1 \in (t_2, t_4), \xi_2 \in (t_1, t_3) \) and thus

\[
(4) \quad \lim_{t \to b^-} |\bar{q}(\tilde{\varphi}(t)) (\varphi'(t) - \tilde{\varphi}'(t))| = 0.
\]
This implies \( \lim_{t \to b^-} (\varphi'(t) - \bar{\varphi}'(t)) = 0 \) and by virtue of Lemma 2 we can see that
\[
\lim_{t \to b^-} \varphi'(t) = 1
\]
which proves the first part of the lemma.

The dispersions \( \varphi \) and \( \bar{\varphi} \) fulfil the non-linear differential equation (1) 5:
\[
- \frac{1}{2} \frac{\varphi''}{\varphi'} + \frac{3}{4} \frac{\varphi'^3}{\varphi''^2} = -q(\varphi) \varphi'^2 + q(t),
\]
\[
- \frac{1}{2} \frac{\bar{\varphi}''}{\bar{\varphi}'} + \frac{3}{4} \frac{\bar{\varphi}'^3}{\bar{\varphi}''^2} = -\bar{q}(\bar{\varphi}) \bar{\varphi}'^2 + \bar{q}(t).
\]

Subtracting and modifying these equations we get (by (4) and the proved part of Lemma 3):
\[
A = \left| - \frac{1}{2} \left( \frac{\varphi''}{\varphi'} - \frac{\bar{\varphi}''}{\bar{\varphi}'} \right) + \frac{3}{4} \left( \frac{\varphi'^3}{\varphi''^2} - \frac{\bar{\varphi}'^3}{\bar{\varphi}''^2} \right) \right| =
\]
\[
= |q(t) - \bar{q}(t) - q(\varphi) \varphi'^2 + \bar{q}(\bar{\varphi}) \bar{\varphi}'^2| =
\]
\[
= |(q(t) - \bar{q}(t)) - \bar{q}(\bar{\varphi}) (\varphi' - \bar{\varphi}') (\varphi' + \bar{\varphi}') - \varphi'^2[(q(\varphi) - \bar{q}(\varphi)) + (\bar{q}(\varphi) - \bar{q}(\bar{\varphi}))] | \leq |q(t) - \bar{q}(t)| +
\]
\[
+ |\bar{q}(\bar{\varphi}) (\varphi' - \bar{\varphi}') (\varphi' + \bar{\varphi}')| + \varphi'^2 |q(\varphi) - \bar{q}(\varphi)| +
\]
\[
+ \varphi'^2 \max_{x \in [t, \bar{\varphi}(t)]} |\bar{q}'(x)| (\bar{\varphi}(t) - t) \to 0.
\]

Taking into account Lemma 2 (for \( q \equiv \bar{q} \) we have \( \lim_{t \to b^-} \varphi''(t) = \lim_{t \to b^-} \bar{\varphi}''(t) = 0 \)) we can see from this that
\[
\lim_{t \to b^-} A \varphi'^2 = \lim_{t \to b^-} \left| - \frac{1}{2} \frac{\varphi''}{\varphi'} \cdot \varphi' + \frac{3}{4} \varphi'^2 \right| = 0
\]
holds. The relation (5) is the same as the relation (3) and therefore we can prove in the same way as in Lemma 1 that
\[
\lim_{t \to b^-} \varphi''(t) = \lim_{t \to b^-} \varphi'''(t) = 0.
\]

So the statement of the lemma is proved.

**Theorem 2.** Let \( q, \bar{q} \) be oscillatory \( t \to \infty \) differential equations such that
\( q \in C^0[a, \infty), \bar{q} \in C^1[a, \infty), \limsup \bar{q}(t) < 0, \lim (q(t) - \bar{q}(t)) = 0, \lim \bar{q}'(t) = 0. \)

Let \( \varphi \) be the dispersion of \( q \). Then
\[
\lim_{t \to \infty} \varphi(t) = 1, \quad \lim_{t \to \infty} \varphi''(t) = \lim_{t \to \infty} \varphi'''(t) = 0.
\]
Proof. Let \( \varphi(t) \) be the dispersion of \((q)\). It follows from Lemma 1 that there exists a constant \( M > 0 \) such that \( \varphi(t) - t \leq M, \varphi(t) - t \leq M, t \in [a, \infty) \).

Thus

\[
\lim_{t \to \infty} \max_{x \in [t, \varphi(t)]} |q'(x)| (\varphi(t) - t) = 0
\]

where \( \varphi(t) = \max (\varphi(t), \tilde{\varphi}(t)) \). This together with Lemma 3 implies the statement of the theorem.

**Theorem 3.** Let \((q), (\tilde{q})\) be oscillatory \((t \to \infty)\) differential equations such that \( q, \tilde{q} \in C^{0}[a, \infty), q \in C^{1}[a, \infty), \lim_{t \to \infty} q(t) = 0, \lim_{t \to \infty} q(t) = -\infty, |q'(t)| \leq \text{const.}\) for \( t \in [a, \infty) \). Let \( \varphi \) be the dispersion of \((q)\). Then

\[
\lim_{t \to \infty} (\varphi(t) - t) = 0, \quad \lim_{t \to \infty} \varphi'(t) = 1, \quad \lim_{t \to \infty} \varphi''(t) = \lim_{t \to \infty} \varphi'''(t) = 0.
\]

Proof. Let \( C < 0 \) be an arbitrary number. As \( \lim_{t \to \infty} q(t) = -\infty \), there exists a number \( t_1, t_1 \in [a, \infty) \) such that \( q(t) < C, t \in [t_1, \infty) \). From the Sturm Comparison Theorem for the equations \((q)\) and \( y'' = C \cdot y \) we obtain

\[
0 < \varphi(t) - t \leq \frac{\pi}{\sqrt{-C}}, \quad t \in [t_1, \infty).
\]

Hence \( \lim_{t \to \infty} (\varphi(t) - t) = 0 \). We can prove similarly that \( \lim_{t \to \infty} (\tilde{\varphi}(t) - t) = 0 \) where \( \tilde{\varphi} \) is the dispersion of \((\tilde{q})\). Thus

\[
\lim_{t \to \infty} \max_{x \in [t, \tilde{\varphi}(t)]} |\tilde{q}'(x)| (\tilde{\varphi}(t) - t) = 0
\]

where \( \tilde{\varphi}(t) = \max (\varphi(t), \tilde{\varphi}(t)) \) and the statement of the theorem follows from Lemma 3.

References


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