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ON GENERALIZED WEINGARTEN SURFACES

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Following the ideas of A. ŠVEC [1], I am going to present further generalizations of the H- and K-theorems.

1.

Theorem 1. Let \( G \subset \mathbb{R}^2 \) be a bounded domain, \( M : G \cup \partial G \to E^3 \) a surface with a net of lines of curvature, \( v_1 \) and \( v_2 \) the unit tangent vector fields of these lines, \( k_1 \) and \( k_2 \) the corresponding principal curvatures. Let \( M(\partial G) \) consist of umbilical points. Further, suppose

\[
\begin{align*}
(1.1) & \quad K \geq 0, \\
(1.2) & \quad (k_1 - k_2)(v_1v_1 - v_2v_2)H \geq 0
\end{align*}
\]

on \( M \). Then \( M(G \cup \partial G) \) is a part of a sphere.

Proof. On \( M \), consider a field of tangent orthonormal moving frames \( \{m; v_1, v_2, v_3\} \). Then

\[
\begin{align*}
(1.3) & \quad dm = \omega^1 v_1 + \omega^2 v_2, \\
& \quad dv_1 = \omega^1_1 v_2 + \omega^1_3 v_3, \\
& \quad dv_2 = -\omega^2_1 v_1 + \omega^2_3 v_3, \\
& \quad dv_3 = -\omega^3_1 v_1 - \omega^3_2 v_2
\end{align*}
\]

with the usual integrability conditions. We have

\[
\begin{align*}
(1.4) & \quad \omega^1_3 = a\omega^1, \quad \omega^3_2 = c\omega^2 \\
(1.5) & \quad da = \alpha \omega^1 + \beta \omega^2, \\
& \quad (a - c) \omega^2_1 = \beta \omega^1 + \gamma \omega^2, \\
& \quad dc = \gamma \omega^1 + \delta \omega^2;
\end{align*}
\]
\begin{align}
(1.6) \quad d\alpha - 3\beta \omega_1^2 &= A\omega_1 + B\omega^2, \\
& \quad d\beta + (\alpha - 2\gamma) \omega_1^3 = B\omega_1 + (C + aK) \omega^2, \\
& \quad d\gamma + (2\beta - \delta) \omega_1^2 = (C + cK) \omega_1 + D\omega^2, \\
& \quad d\delta + 3\gamma \omega_1^2 = D\omega_1 + E\omega^2 \\
\end{align}

and

\begin{align}
(1.7) \quad v_1 a &= \alpha, \quad v_2 a = \beta, \quad v_1 c = \gamma, \quad v_2 a = \delta ; \\
(1.8) \quad v_1 H &= \frac{1}{2}(\alpha + \gamma), \quad v_2 H = \frac{1}{4}(\beta + \delta); \\
(1.9) \quad (a - c) v_1 \alpha &= 3\beta^2 + A(a - c), \\
& \quad (a - c) v_1 \beta &= 3\beta \gamma - \alpha \beta + B(a - c), \\
& \quad (a - c) v_1 \gamma &= \beta(\delta - 2\beta) + (C + cK)(a - c), \\
& \quad (a - c) v_1 \delta &= D(a - c) - 3\beta \gamma, \\
& \quad (a - c) v_2 \alpha &= 3\beta \gamma + B(a - c), \\
& \quad (a - c) v_2 \beta &= 2\gamma^2 - \alpha \gamma + (C + aK)(a - c), \\
& \quad (a - c) v_2 \gamma &= \gamma(\delta - 2\beta) + D(a - c), \\
& \quad (a - c) v_2 \delta &= E(a - c) - 3\gamma^2; \\
(1.10) \quad (a - c) v_1 v_1 H &= \frac{1}{4}(\beta + \delta) \beta + \frac{1}{2}(a - c)(A + C + cK), \\
& \quad (a - c) v_1 v_2 H &= -\frac{1}{4}(\alpha + \gamma) \beta + \frac{1}{2}(a - c)(B + D), \\
& \quad (a - c) v_2 v_1 H &= \frac{1}{4}(\beta + \delta) \gamma + \frac{1}{2}(a - c)(B + D), \\
& \quad (a - c) v_2 v_2 H &= -\frac{1}{4}(\alpha + \gamma) \gamma + \frac{1}{2}(a - c)(C + E + aK). \\
\end{align}

For

\begin{align}
(1.11) \quad f &= 2(H^2 - K) = \frac{1}{2}(a - c)^2, \\
\end{align}

define its covariant derivatives $f_0, f_{ij}$ by

\begin{align}
(1.12) \quad df &= f_1 \omega^1 + f_2 \omega^2; \\
& \quad df_1 - f_2 \omega_1 = f_{11} \omega^1 + f_{12} \omega^2, \\
& \quad df_2 + f_1 \omega_1 = f_{12} \omega^1 + f_{22} \omega^2. \\
\end{align}
Then
\begin{align*}
(1.13) \quad f_{11} &= (c^2 - ac) K + (\alpha - \gamma)^2 + 4\beta^2 + (a - c)(A - C), \\
&\quad f_{22} = (a^2 - ac) K + (\beta - \delta)^2 + 4\gamma^2 + (a - c)(C + E), \\
&\quad f_{12} = (\alpha - \gamma)(\beta - \delta) + 4\beta\gamma + (a - c)(B - D).
\end{align*}
Now, set
\begin{align*}
(1.14) \quad S &= (v_1 + v_2) H; \\
&\quad v_1v_1H + v_1v_2H = v_1S, \\
&\quad v_2v_1H + v_2v_2H = v_2S,
\end{align*}
i.e.,
\begin{align*}
(1.15) \quad \beta^2 + \beta\delta + K(ac - c^2) - \beta\gamma - \alpha\beta + (a - c) A + (a - c) B + \\
&\quad + (a - c) C + (a - c) D - 2(a - c) v_1S = 0, \\
&\quad \beta\gamma + \gamma\delta - \gamma^2 - \alpha\gamma + (a^2 - ac) K + (a - c) B + (a - c) C + \\
&\quad + (a - c) D + (a - c) E - 2(a - c) v_2S = 0.
\end{align*}
Eliminating $A, B, C, D, E$ from (1.13) and (1.15), we get
\begin{align*}
(1.16) \quad f_{11} + f_{22} &= 4Kf + 2(a - c)(v_1S - v_2S) + \alpha^2 - 3\alpha\gamma + \\
&\quad + 4\gamma^2 + 4\beta^2 - 3\beta\delta + \delta^2 + \alpha\beta + 2\beta\gamma + \gamma\delta.
\end{align*}
Now,
\begin{align*}
(1.17) \quad v_1S - v_2S &= v_1v_1H + v_1v_2H - v_2v_1H - v_2v_2H, \\
(1.18) \quad (a - c)(v_1S - v_2S) &= (a - c)(v_1v_1H - v_2v_2H) - \frac{1}{2}(\beta + \delta) \gamma - \\
&\quad - \frac{1}{2}(\alpha + \gamma) \beta,
\end{align*}
and (1.16) turns out to be
\begin{align*}
(1.19) \quad f_{11} + f_{22} - 4Kf &= 2(a - c)(v_1v_1 - v_2v_2) H + (\alpha - \frac{3}{2}\gamma)^2 + \\
&\quad + (\delta - \frac{3}{4}\beta)^2 + \frac{1}{4}(\beta^2 + \gamma^2).
\end{align*}
This equation satisfies the conditions of the maximum principle because of (1.1) and (1.2). Thus $H^2 - K = 0$ on $M(\partial G)$ implies $H^2 - K = 0$ on $M(G)$. QED.

**Theorem 2.** Let $G \subset \mathbb{R}^2$ be a bounded domain, $M : G \cup \partial G \rightarrow E^3$ a surface with a net of lines of curvature, $v_1$ and $v_2$ be the fields of the unit tangent vectors of these lines, $k_1$ and $k_2$ be the corresponding principal curvatures. Let $M(\partial G)$ consist of umbilical points. On $M$, suppose
K > 0 , 
(1.21) \( (k_1 - k_2) (v_1 v_2 - v_2 v_2) K \geq 0 , \)
(1.22) \( \frac{4}{11} \leq \frac{k_2^2}{k_1^2} \leq \frac{11}{4} . \)

Then \( M(G \cup \partial G) \) is a part of a sphere.

Proof. Let us keep the notation of the proof of the previous theorem. Then

\[ v_1 K = a\gamma + c\alpha , \quad v_2 K = a\delta + c\beta ; \]

\[ (a - c) v_1 v_1 K = a[\beta(\delta - 2\beta) + K(ac - c^2)] + 3c\beta^2 + \]

\[ + a(a - c) C + c(a - c) A + 2(a - c) \alpha \gamma , \]

\[ (a - c) v_2 v_2 K = c[\gamma(2\gamma - \alpha) + K(a^2 - ac)] - 3a\gamma^2 + \]

\[ + c(a - c) C + a(a - c) E + 2(a - c) \beta \delta , \]

\[ (a - c) v_1 v_2 K = c\beta(2\gamma - \alpha) - 3a\beta\gamma + c(a - c) B + \]

\[ + a(a - c) D + (a - c) (\alpha \delta - \beta \gamma) , \]

\[ (a - c) v_2 v_1 K = a\gamma(\delta - 2\beta) + 3c\beta\gamma + c(a - c) B + \]

\[ + a(a - c) D + (a - c) (\alpha \delta + \beta \gamma) . \]

Set

\[ (v_1 + v_2) K = S . \]

Then

\[ (v_1 v_1 + v_1 v_2) K = v_1 S , \]

\[ (v_2 v_1 + v_2 v_2) K = v_2 S . \]

From (1.24)

\[ a(\beta \delta - 2\beta^2) + aK(ac - c^2) + 3c\beta^2 + 2(a - c) \alpha \gamma - \]

\[ - 3a\beta\gamma + 2c\beta\gamma - c\alpha \beta + (a - c) (\alpha \delta + \beta \gamma) - \]

\[ - (a - c) v_1 S + (ca - c^2) A + (ca - c^2) B + (a^2 - ac) C + \]

\[ + (a^2 - ac) D = 0 , \]

\[ a(\gamma \delta - 2\beta \gamma) + 3c\beta \gamma + (a - c) (\alpha \delta + \beta \gamma) - 3a\gamma^2 + \]

\[ + 2c\gamma^2 - c\alpha \gamma + Kc(a^2 - ac) + 2(a - c) \beta \delta - (a - c) v_2 S + \]

\[ + (ca - c^2) B + (ca - c^2) C + (a^2 - ac) D + (a^2 - ac) E = 0 . \]
Eliminating $A, B, C, D, E$ from (1.13) and (1.27), we get

\[(1.28)\quad c_{f_{11}} + a f_{22} - 2K(a + c)f = (a - c)(v_1 - v_2)S +
+ \{(a + c)\beta\gamma + c\alpha\beta + a\gamma\delta\} + (3a + c)\beta^2 +
+ (a + 2c)\beta\delta - (2a + c)\alpha\gamma + (a + 3c)\gamma^2 +
+ c\alpha^2 + a\delta^2 .
\]

From (1.24) and (1.26),

\[(1.29)\quad c_{f_{11}} + a f_{22} - 2K(a + c)f = (a - c)(v_1v_1 - v_2v_2)K +
+ a \left[ \left\{ \delta - \left( \frac{1}{2} + \frac{c}{a} \right) \beta \right\}^2 + \left( \frac{11}{4} - \frac{c^2}{a^2} \right) \beta^2 \right] +
+ c \left[ \left\{ \alpha - \left( \frac{1}{2} + \frac{a}{c} \right) \gamma \right\}^2 + \left( \frac{11}{4} - \frac{a^2}{c^2} \right) \gamma^2 \right] .
\]

This equation satisfies the conditions of the maximum principle because of (1.20) to (1.22). Again, $H^2 - K = 0$ on $\partial G$ implies $H^2 - K = 0$ on $G$. QED.

Remark. Let us replace (1.21) and (1.22) by the condition $K = ac = \text{const.} > 0$. Then

\[(1.30)\quad v_1K = c\alpha + a\gamma = 0 , \quad v_2K = c\beta + a\delta = 0 .
\]

Put

\[(1.31)\quad \alpha = pa , \quad \beta = qa , \quad \gamma = -pc , \quad \delta = -qc .
\]

The equation (1.29) turns out to be

\[(1.32)\quad c_{f_{11}} + a f_{22} - 2K(a + c)f = p^2(3a^2c + 2ac^2 + 3c^3) +
+ q^2(3ac^2 + 2a^2c + 3a^3) = (cp^2 +aq^2)(3a^2 + 2ac + 3c^2) ,
\]

and we get the proof of the K-theorem.

2.

Let us consider the surfaces with nets of lines of curvature (for notation, see our Theorems) for which they are functions $P, Q, T : M \to \mathcal{A}$ such that

\[(2.1)\quad P v_1H + Q v_2H + T = 0 .
\]
Following the remark to Theorem 2 in [1], we wish to establish the class of operators (2.1) such that we might be able to prove by means of the maximum principle that each surface satisfying (2.1) is a part of a sphere.

Without loss of generality, (2.1) may be written as

\[ v_1 H + Rb_2 H = S. \]

Applying \( v_1 \) and \( v_2 \) to (2.2) and using (1.10), we get the equations of the form

\[
\begin{align*}
(a - c)(A + RB + C + RD) &= \Phi_1(a, C, x, \beta, \gamma, \delta), \\
(a - c)(B + RC + D + RE) &= \Phi_2(a, C, x, \beta, \gamma, \delta).
\end{align*}
\]

Now, our task is to eliminate \( A, \ldots, E \) from (2.3) and (1.13). For this, the rank of the matrix (of the coefficients at \( A, \ldots, E \))

\[
\begin{pmatrix}
1 & R & 1 & R & 0 \\
0 & 1 & R & 1 & R \\
1 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}
\]

should be <5. This implies \( R = \pm 1 \), and our operators are given by

\[ (v_1 \pm v_2) H - S = 0. \]

Similarly, for the operators of the form

\[ P'v_1 K + Q'v_2 K + T' = 0, \]

our class of "convenable" operators is given again by

\[ (v_1 \pm v_2) K - S' = 0. \]

3.

We might give a generalization of Theorems 1 and 2, this being, of course, not as sharp in the suppositions.

**Theorem 3.** Let \( G \subset \mathbb{R}^2 \) be a bounded domain, \( M : G \cup \partial G \rightarrow E^3 \) a surface with a net of lines of curvature, \( v_1 \) and \( v_2 \) the unit tangent vector fields of these lines, \( k_1 \) and \( k_2 \) be the corresponding principal curvatures. Let \( M(\partial G) \) consist of umbilical points; further, let us suppose
on $M$, and let there be a function $F: M \to \mathbb{R}^2$ satisfying

\[(3.3) \quad (k_1 - k_2)(v_1 v_1 - v_2 v_2) F(H, K) \geq 0,\]

\[(3.4) \quad F_H \geq 0, \quad F_K \geq 0,\]

\[(3.5) \quad (k_1 - k_2)(v_1 + v_2) H \cdot \{F_{HH}(v_2 - v_1) H + F_{HK}(v_2 - v_1) K\} \geq 0,\]

\[(3.6) \quad (k_1 - k_2)(v_1 + v_2) H \cdot \{F_{KH}(v_2 - v_1) H + F_{KK}(v_2 - v_1) K\} \geq 0\]

on $M(G \cup \partial G)$. Then $M(G \cup \partial G)$ is a part of a sphere.

Proof. The function $S$ be defined by

\[(3.7) \quad (v_1 + v_2) F(H, K) + S = 0.\]

\[(3.8) \quad \{F_{HH} v_1 H + F_{HK} v_1 K\} \cdot (v_1 + v_2) H + (v_1 v_1 + v_1 v_2) H \cdot F_H + \]
\[+ \{F_{KH} v_1 H + F_{KK} v_1 K\} \cdot (v_1 + v_2) K + (v_1 v_1 + v_1 v_2) K \cdot F_K + \]
\[+ v_1 S = 0,\]

\[\{F_{HH} v_2 H + F_{HK} v_2 K\} \cdot (v_1 + v_2) H + (v_2 v_1 + v_2 v_2) H \cdot F_H + \]
\[+ \{F_{KH} v_2 H + F_{KK} v_2 K\} \cdot (v_1 + v_2) K + \]
\[+ (v_2 v_1 + v_2 v_2) K \cdot F_K + v_2 S = 0.\]

From (1.8), (1.10), (1.23) and (1.24),

\[(3.9) \quad (a - c) \{F_{HH}(\frac{1}{2} \alpha + \frac{1}{2} \gamma) + F_{HK}(a \gamma + c \alpha)\} (\frac{1}{2} \alpha + \frac{1}{2} \gamma + \frac{1}{2} \beta + \frac{1}{2} \delta) + \]
\[+ \frac{1}{2} F_H[\beta^2 + \beta \delta - \beta \gamma - c \alpha + (a - c)(A + C + c K + B + D)] + \]
\[+ (a - c) \{F_{KH}(\frac{1}{2} \alpha + \frac{1}{2} \gamma) + F_{KK}(a \gamma + c \alpha)\} (a \gamma + c \alpha + a \delta + c \beta) + \]
\[+ F_K[\alpha \beta \delta - 2 \beta^2 + c K(a - c)] + (a - c) \gamma \alpha + 3 c \beta^2 + (a - c) \alpha \gamma + \]
\[+ a(a - c) C + c(a - c) A + a(a - c) D + c(a - c) B - 3 a \beta \gamma + \]
\[+ (a - c) a \delta + 2 c \beta \gamma - c \alpha \beta + (a - c) \beta \gamma] + (a - c) v_1 S = 0,\]
\[(a - c) \{F_{HH}(\frac{1}{2} \beta + \frac{1}{2} \delta) + F_{HK}(a \delta + c \beta)\} (\frac{1}{2} \alpha + \frac{1}{2} \beta + \frac{1}{2} \gamma + \frac{1}{2} \delta) + \]
\[ + \frac{1}{2} F_{H}[\beta \gamma + \gamma \delta + (a - c)(B + D) - \gamma^2 - \gamma \alpha + (a - c)(C + aK + E)] + (a - c) \left\{ F_{KH}(\frac{1}{2} \beta + \frac{1}{2} \delta) + F_{KK}(a \delta + c \beta) \right\} \left( a \gamma + c \alpha + a \delta + c \beta \right) + F_{K}(a \gamma \delta - 2a \beta \gamma + (a - c) \beta \gamma + 3c \beta \gamma + (a - c) \alpha \delta + a(a - c) D + c(a - c) B + a(a - c) E + c(a - c) C - 3a \gamma^2 + (a - c) \beta \delta + 2c \gamma^2 - c \gamma \alpha + c(a^2 - ac) K + (a - c) \beta \delta \right\} + (a - c) v_2 S = 0. \]

Multiplying the first two equations (1.13) by \((-\frac{1}{2} F_{H} - cF_{K})\) and \((-\frac{1}{2} F_{H} - aF_{K})\) resp., and using (3.9), we can eliminate \(A, B, C, D\) and \(E\), and we get

\[ (3.10) \quad (\frac{1}{2} F_{H} + cF_{K}) f_{11} + (\frac{1}{2} F_{H} + aF_{K}) f_{22} - \left\{ \frac{1}{2} F_{H} K + 2F_{KH} \right\} 2f = (a - c)(v_1 v_1 - v_2 v_2) F + \frac{1}{2} F_{H} \left[ 3(\beta - \frac{1}{2} \delta)^2 + 3(\gamma - \frac{1}{2} \alpha)^2 + \beta^2 + \gamma^2 + \frac{1}{2} \alpha^2 + \frac{1}{2} \delta^2 \right] + F_{K} \left[ a \left\{ \delta - \left( \frac{1}{2} + \frac{c}{a} \right) \beta \right\} ^2 + a \left( \frac{11}{4} - \frac{c^2}{a^2} \right) \beta^2 + c \left( \alpha - \left( \frac{1}{2} + \frac{a}{c} \right) \gamma \right) ^2 + c \left( \frac{11}{4} - \frac{a^2}{c^2} \right) \gamma^2 \right] + (a - c) F_{HH}(v_1 + v_2) H \cdot (v_2 - v_1) H + (a - c) F_{HK}(v_1 + v_2) H \cdot (v_2 - v_1) K + (a - c) F_{KH}(v_1 + v_2) K \cdot (v_2 - v_1) H + (a - c) F_{KK}(v_1 + v_2) K \cdot (v_2 - v_1) K. \]

The result follows.

**Bibliography**


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