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A HAMILTONIAN CYCLE AND A 1-FACTOR IN THE FOURTH POWER OF A GRAPH

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By a graph we shall mean a finite undirected graph with no loops or multiple edges (a graph in the sense of [1] or [3]). If G is a graph, then the vertex set of G and the edge set of G will be denoted by $V(G)$ and $E(G)$, respectively, and if $u, v, w \in V(G)$, then the degree of u in G and the distance between v and w in G will be denoted by $\deg_G u$ and $d_G(v, w)$, respectively.

If G is a graph and n is a positive integer, then the n-th power G^n of G is the graph defined as follows:

$$V(G^n) = V(G) \quad \text{and} \quad E(G^n) = \{vv';\, v, v' \in V(G) \quad \text{and} \quad 1 \leqq d_G(v, v') \leqq n\}\,.$$

We now mention some results concerning regular factors and hamiltonian properties of powers of connected graphs.

Theorem A [5]. *Let n be a positive integer, and let G be a connected graph of order $p \geqq n$. Assume that if n is even, then p is also even. Then G^n has an $(n-1)$-factor.*

For $n = 2$, this theorem was proved in [2] and [8]. For $n = 3, 4, 5$, stronger results are known.

Theorem B [7]. *If G is a nontrivial connected graph, then G^3 is hamiltonian-connected.*

Theorem C [4]. *If G is a connected graph of even order $\geqq 4$, then G^4 has a 3-factor F such that each component of F is a copy of K_4 or $K_3 \times K_2$.*

Theorem D [6]. *Let G be a connected graph of order $\geqq 5$. Then there exist a hamiltonian cycle C of G^3 and a hamiltonian cycle C' of G^5 such that C and C' are edge-disjoint.*

In the present paper we shall prove the following theorem:

Theorem 1. *Let G be a connected graph of even order $\geqq 4$. Then there exits a hamiltonian cycle C of G^3 and a 1-factor F of G^4 such that C and F are edge-disjoint.*

We say that an ordered pair (T', r') is a rooted tree if T' is a tree and $r' \in V(T')$. If (T', r') is a rooted tree, then we say that r' is its root. The root of a rooted tree will be drawn as $\otimes$ in the figures throughout the paper. We say that rooted trees (T', r') and (T'', r'') are isomorphic if T' and T'' are isomorphic and there exists an

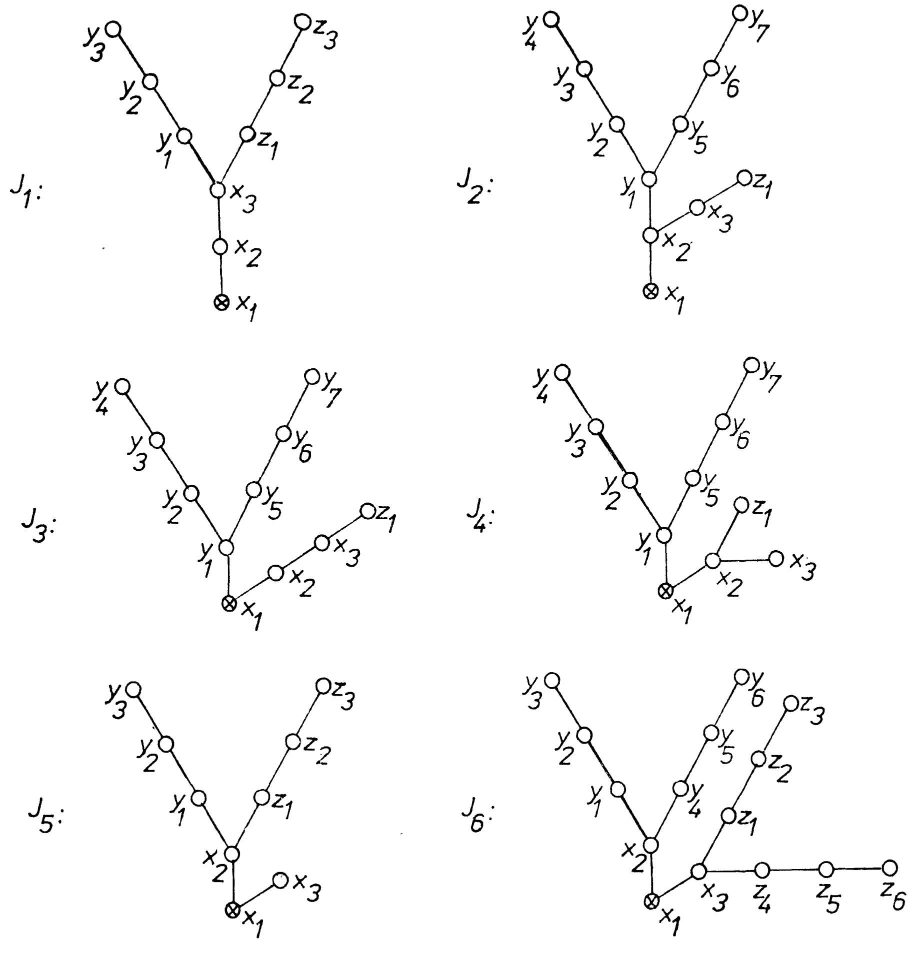

Fig. 1.

isomorphism from T' onto T'' which maps r' onto r''. Let T be a tree. Similarly as in [6], by a terminal subtree of T we shall mean a rooted tree (T', r') with the properties that T' is a subtree of T and for each $v \in V(T' - r')$, $\deg_{T'} v = \deg_T v$.

The following notions will be useful for us.

Let T be a nontrivial tree, and let u and v be adjacent vertices of T. Then $T - uv$ is a forest with exactly two components. We denote by $T(u, v)$ or $T(v, u)$ the component of $T - uv$ which contains u or v, respectively.

Let $m \geqq 0$ and $n \geqq 1$ be integers, and let $u_0, \ldots, u_m, w_1, \ldots, w_n$ be mutually distinct vertices.

We denote by B_{mn} the path with

$$V(B_{mn}) = \{u_m, \ldots, u_0, w_1, \ldots, w_n\} \text{ and}$$

$$E(B_{mn}) = \{u_j u_{j-1};\ m \geqq j > 0\} \cup \{u_0 w_1\} \cup \{w_k w_{k+1};\ 1 \leqq k \leqq n-1\}.$$

We define the following sets of rooted trees:

$$D_{mn} = (B_{mn}, u_0),$$

$$D_{mn*} = (B_{mn} - w_{n-1}w_n + w_{n-2}w_n, u_0), \text{ for } n \geqq 3;$$

$$D_{*mn} = (B_{mn} - u_{m-1}u_m + u_{m-2}u_m, u_0), \text{ for } m \geqq 2;$$

$$D_{*mn*} = (B_{mn} - u_{m-1}u_m - w_{n-1}w_n + u_{m-2}u_m + w_{n-2}w_n, u_0),$$
$$\text{for } m \geqq 2,\ n \geqq 3, \text{ and}$$

$$D_{mn**} = (B_{mn} - w_{n-1}w_n + w_{n-3}w_n, u_0), \text{ for } n \geqq 4.$$

Denote

$$D = \{D_{*21}, D_{*22}, D_{22}, D_{23*}, D_{23}, D_{*31}, D_{31}, D_{*33*}, D_{*33}, D_{33}, D_{04**}, D_{04*}, D_{04}\},$$

$$\mathscr{D}' = \mathscr{D} - \{D_{33}\},$$

$$\mathscr{J} = \{J_1, \ldots, J_6\}, \text{ where } J_1, \ldots, J_6 \text{ denote the rooted trees in Fig. 1.}$$

Lemma 1. *Let T be a tree of order $\geqq 5$. Then there exists a terminal subtree (T_0, r_0) of T such that either (T_0, r_0) is isomorphic to one of the elements of $\mathscr{D}'$, or (T_0, r_0) is isomorphic to D_{33} and $\deg_T r_0 \geqq 4$, or (T_0, r_0) is isomorphic to one of the elements of $\mathscr{J}$.*

Proof. If $|V(T)| = 5$, the statement of the lemma is correct. Assume that $|V(T)| \geqq 6$. Then there exist adjacent vertices u and v such that $|V(T(u, v))| \geqq 5$ and $|V(T(w, u))| \leqq 4$ for every vertex $w \neq v$ such that $uw \in E(T)$; cf. the proof of Lemma 1 in [5]. It is easy to see that there exist a subtree T_1 of $T(u, v)$ and $r_1 \in V(T)_1$ such that (T_1, r_1) is a terminal subtree of T and (T_1, r_1) is isomorphic to one of the elements of $\mathscr{D}$. If there exists a terminal subtree (T_0, r_0) of T such that either (T_0, r_0) is isomorphic to D_{33} and $\deg_T r_0 \geqq 4$ or (T_0, r_0) is isomorphic to an element of $\mathscr{D}'$, then the statement of the lemma is correct. We shall assume that for every terminal subtree (T', r') of T, if (T', r') is isomorphic to an element of $\mathscr{D}$, then (T', r') is isomorphic to D_{33} and $\deg_T r' < 4$. Then $|V(T)| \geqq 10$. There exist adjacent vertices x and y of T such that $|V(T(x, y))| \geqq 8$ and $|V(T(z, x))| \leqq 7$ for every vertex $z \neq y$ such that $xz \in E(T)$. We denote by T^* the subtre of T induced by $V(T(x, y)) \cup \{y\}$.

If $\deg_T x = 2$, then (T^*, y) is a terminal subtree of T and it is isomorphic to J_1. Let $\deg_T x \neq 2$. Then $\deg_T x \geqq 3$. It is easy to see that there exists a subtree T_2 of $T(x, y)$ such that (T_2, x) is a terminal subtree of T which is isomorphic to an element of $\{J_2', J_3, J_4, J_5, J_6\}$, where J_2' denotes the rooted tree in Fig. 2. If there exists a terminal subtree (T_0, r_0) of T which is isomorphic to an element of $\mathscr{J} - \{J_2\}$, then the statement of the lemma is correct .Assume that no terminal subtree of T is isomorphic to an element of $\mathscr{J} - \{J_2\}$. Then $(T(x, y), x)$ is isomorphic to J_2' and $\deg_T x = 3$. This means that (T^*, y) is a terminal subtree of T which is isomorphic to J_2.

Thus the lemma is proved.

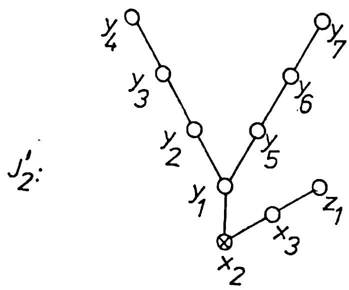

Fig. 2.

If G is a graph, then we denote by $\mathscr{H}(G)$ and $\mathscr{F}(G)$ the set of hamiltonian cycles of G and the set of 1-factors of G, respectively.

Lemma 2. *Let S be a tree which contains a terminal subtree (T_0, r_0) isomorphic to D_{03}. Let f be an isomorphism mapping D_{03} onto (T_0, r_0). Assume that there exist $C' \in \mathscr{H}(S^3)$ and $F' \in \mathscr{F}(S^4)$ such that $E(C') \cap E(F') = \emptyset$. Then there exist $\tilde{C} \in \mathscr{H}(S^3)$ and $\tilde{F} \in \mathscr{F}(S^4)$ such that either*

$$E(\tilde{C}) \cap E(\tilde{F}) \subset \{f(w_1)f(w_2), f(w_2)f(w_3)\} \subset E(\tilde{C})$$

or

$$E(\tilde{C}) \cap E(\tilde{F}) \subset \{f(w_1)f(w_3), f(w_2)f(w_3)\} \subset E(\tilde{C}).$$

Proof. For the sake of simplicity we shall assume that $(T_0, r_0) = D_{03}$. Then $r_0 = u_0$. If $\{w_1w_2, w_2w_3\} \subset E(C')$ or $\{w_1w_3, w_2w_3\} \subset E(C')$, we put $\tilde{C} = C'$ and $\tilde{F} = F'$. Let $\{w_1w_2, w_2w_3\} - E(C') \neq \emptyset$ and $\{w_1w_3, w_2w_3\} - E(C') \neq \emptyset$. We denote by **C** one of the two orientations of the cycle C'. Let **E** denote the set of all directed arcs of the directed cycle **C**. For every $i \in \{1, 2, 3\}$ there exist $a_i, b_i \in V(S)$ such that $(a_i, w_i), (w_i, b_i) \in$ **E**. We now distinguish two cases.

1. Let $w_2w_3 \in E(C')$. Without loss of generality let $(w_2, w_3) \in$ **E**. Since $w_1w_2, w_1w_3 \notin \notin E(C')$, we have $\{a_1, a_2, b_1, b_3\} \cap \{w_1, w_2, w_3\} = \emptyset$. If $a_1a_2 \notin E(F')$, then we put

$$\tilde{C} = C' - w_1a_1 - w_2a_2 + w_1w_2 + a_1a_2 \quad \text{and} \quad \tilde{F} = F'.$$

If $a_1a_2 \in E(F')$, then we put

$$\tilde{C} = C' - w_1b_1 - w_3b_3 + w_1w_3 + b_1b_3,$$

$$\tilde{F} = F' \quad \text{if} \quad b_1b_3 \notin E(F'),$$

$$\tilde{F} = F' - a_1a_2 - b_1b_3 + a_1b_1 + a_2b_3 \text{ if } b_1b_3 \in E(F').$$

2. Let $w_2w_3 \notin E(C')$. Then $u_0w_3, w_1w_3 \in E(C')$. Without loss of generality let $(u_0, w_3), (w_3, w_1) \in \mathbf{E}$. We put

$$\tilde{C} = C' - w_2a_2 - u_0w_3 + w_2w_3 + u_0a_2.$$

If $u_0a_2 \notin E(F')$, we put $\tilde{F} = F'$. Let $u_0a_2 \in E(F')$. There exists $\bar{y} \in V(S)$ such that $w_3\bar{y} \in E(F')$. Since $u_0 \neq w_3 \neq a_2$, we have $\bar{y} \notin \{u_0, a_2\}$. If $\bar{y}u_0 \in E(\tilde{C})$, we put

$$\tilde{F} = F' - u_0a_2 - w_3\bar{y} + u_0w_3 + a_2\bar{y}.$$

If $\bar{y}u_0 \notin E(\tilde{C})$, we put

$$\tilde{F} = F' - u_0a_2 - w_3\bar{y} + u_0\bar{y} + w_3a_2, \text{ if } w_3a_2 \notin E(\tilde{C}),$$

$$\tilde{F} = F' - u_0a_2 - w_3\bar{y} + u_0w_3 + a_2\bar{y}, \text{ if } w_3a_2 \in E(\tilde{C}).$$

In all cases $\tilde{C}$ and $\tilde{F}$ have the desired properties.

Lemma 3. *Let S be a tree which contains a terminal subtree (T_0, r_0) isomorphic to D_{03*}. Let f be an isomorphism mapping D_{03*} onto (T_0, r_0). Assume that there exist $C' \in \mathscr{H}(S^3)$ and $F' \in \mathscr{F}(S^4)$ such that $E(C') \cap E(F') = \emptyset$. Then there exist $\tilde{C} \in \mathscr{H}(S^3)$ and $\tilde{F} \in \mathscr{F}(S^4)$ such that*

$$|E(\tilde{C}) \cap \{f(w_1)f(w_2), f(w_2)f(w_3), f(w_1)f(w_3)\}| = 2$$

and

$$E(\tilde{C}) \cap E(\tilde{F}) \subset \{f(w_1)f(w_2), f(w_2)f(w_3), f(w_1)f(w_3)\}.$$

Proof. For the sake of simplicity we shall assume that $(T_0, r_0) = D_{03*}$. We denote $U = E(C') \cap \{w_1w_2, w_2w_3, w_1w_3\}$. Obviously $|U| \leqq 2$. If $|U| = 2$, we put $\tilde{C} = C'$ and $\tilde{F} = F'$. Let $|U| \leqq 1$. We denote by $\mathbf{C}$ one of the two orientations of the cycle C'. Let $\mathbf{E}$ denote the set of all directed arcs of the directed cycle $\mathbf{C}$. For every $i \in \{1, 2, 3\}$, there exist $a_i, b_i \in V(S)$ such that $(a_i, w_i), (w_i, b_i) \in \mathbf{E}$. For arbitrary $j, k \in \{1, 2, 3\}$, the following implications hold:

if $a_j, a_k \notin \{w_1, w_2, w_3\}$, then $d_S(a_j, a_k) \leqq 3$;

if $b_j, b_k \notin \{w_1, w_2, w_3\}$, then $d_S(b_j, b_k) \leqq 3$;

if $a_j, b_k \notin \{w_1, w_2, w_3\}$ and $j \neq k$, then $d_S(a_j, b_k) \leqq 3$.

We distinguish two cases.

1. Let $|U| = 1$. There exist $i, j \in \{1, 2, 3\}$ such that $(w_i, w_j) \in \mathbf{E}$. We denote by k the only element of the set $\{1, 2, 3\} - \{i, j\}$. The fact that $|U| = 1$ implies that $a_i, b_j, a_k, b_k \notin \{w_1, w_2, w_3\}$. If $a_i a_k \notin E(F')$ we put

$$\tilde{C} = C' - w_i a_i - w_k a_k + w_i w_k + a_i a_k \text{ and } \tilde{F} = F'.$$

If $a_i a_k \in E(F')$ and $b_j b_k \notin E(F')$, we put

$$\tilde{C} = C' - w_j b_j - w_k b_k + w_j w_k + b_j b_k \text{ and } \tilde{F} = F'.$$

Let $a_i a_k \in E(F')$ and $b_j b_k \in E(F')$. Then a_i, a_k, b_j, b_k are distinct vertices. We put

$$\tilde{C} = C' - w_j b_j - w_k b_k + w_j w_k + b_j b_k,$$

$$\tilde{F} = F' - a_i a_k - b_j b_k + a_i b_j + a_k b_k .$$

2. Let $|U| = 0$. There exist distinct $i, j, k \in \{1, 2, 3\}$ such that w_j belongs to the directed path from w_i to w_k in $\mathbf{C}$ and $a_i a_j, a_j a_k \notin E(F')$. If $b_i b_k \notin E(F')$, we put

$$\tilde{C} = C' - w_i b_i - w_j a_j - w_k a_k - w_k b_k + w_i w_k + w_j w_k + a_j a_k + b_i b_k,$$

$$\tilde{F} = F'.$$

If $b_i b_k \in E(F')$, then $b_j b_k \notin E(F')$, and we put

$$\tilde{C} = C' - w_i a_i - w_j a_j - w_j b_j - w_k b_k + w_i w_j + w_j w_k + a_i a_j + b_j b_k,$$

$$\tilde{F} = F'.$$

We can see that $\tilde{C}$ and $\tilde{F}$ have the desired properties.

Remark 1. Let S be a tree and let $C \in \mathscr{H}(S^3)$. Then for every vertex $v \in V(S)$ with $\deg_S v \geqq 2$ we can find a pair of vertices $\varphi(v), \psi(v) \in V(S - v)$ such that $\varphi(v)\, \psi(v) \in E(C)$, $v\, \varphi(v) \in E(S)$ and $1 \leqq d_S(v, \psi(v)) \leqq 2$.

Lemma 4. *Let T be a tree of even order $p \geqq 4$. Then there exist $C \in \mathscr{H}(T^3)$ and $F \in \mathscr{F}(T^4)$ such that $E(C) \cap E(F) = \emptyset$.*

Proof. If $p = 4$, then T^3 is the complete graph, and the proposition of Lemma 4 is correct. Let $p \geqq 6$. Assume that for every tree T' of order p', where $4 \leqq p' < p$ and p' is even, it is proved that there exist $C' \in \mathscr{H}((T')^3)$ and $F' \in \mathscr{F}((T')^4)$ such that $E(C') \cap E(F') = \emptyset$.

It follows from Lemma 1 that T has a terminal subtree (T_0, r_0) such that either (T_0, r_0) is isomorphic to D_{33} and $\deg_T r_0 \geqq 4$, or (T_0, r_0) is isomorphic to an element of $\mathscr{D}' \cup \mathscr{J}$. For the sake of simplicity we shall assume that $(T_0, r_0) \in \mathscr{D}' \cup \mathscr{J}$, or $(T_0, r_0) = D_{33}$ and $\deg_T r_0 \geqq 4$. If $(T_0, r_0) \in \mathscr{D}$, then $r_0 = u_0$, and there exist $m \geqq 0, n \geqq 1$ such that $V(T_0) = \{u_m, \ldots, u_0, w_1, \ldots, w_n\}$; if $(T_0, r_0) \in \mathscr{J}$, then $r_0 = x_1$ and there exist integers m and n such that $V(T_0) = \{x_1, \ldots, x_3, y_1, \ldots, y_m, z_1, \ldots, z_n\}$. We now distinguish two cases and several subcases.

1. Let $(T_0, r_0) \in \{D_{*21}, D_{*22}, D_{22}, D_{23*}, D_{23}, D_{*33*}\}$. Denote

$$S = T - u_1 - u_2 \text{ if } (T_0, r_0) \neq D_{*33*}, \text{ and}$$

$$S = T - w_1 - w_2 - w_3 - u_3 \text{ if } (T_0, r_0) = D_{*33*}.$$

It is clear that $|V(S)| \geqq 4$. Since $|V(S)|$ is even, it follows from the induction assumption that there exist $C' \in \mathscr{H}(S^3)$ and $F' \in \mathscr{F}(S^4)$ such that $E(C') \cap E(F') = \emptyset$.

1.1. Let $(T_0, r_0) \in \{D_{*21}, D_{*22}\}$. There exist $x, y \in V(S)$ such that $w_1x \in E(C')$ and $w_1y \in E(F')$. Since $E(C') \cap E(F') = \emptyset$, $x \neq y$.
We define

$$C = C' - w_1x + w_1u_1 + u_1u_2 + u_2x \text{ and}$$

$$F = F' - w_1y + w_1u_2 + yu_1;$$

then $C \in \mathscr{H}(T^3)$, $F \in \mathscr{F}(T^4)$, and $E(C) \cap E(F) = \emptyset$.

1.2. Let $(T_0, r_0) \in \{D_{22}, D_{23*}, D_{23}\}$. There exists $x \in V(S)$ such that $w_1x \in E(C')$, and if $n = 3$, then $x \neq w_3$. It is clear that there exists $y \in V(S)$ such that $w_2y \in E(F')$. We define

$$C = C' - w_1x + w_1u_2 + u_2u_1 + u_1x;$$

then $C \in \mathscr{H}(T^3)$. Let $y \neq x$; we define

$$F = F' - w_2y + w_2u_2 + yu_1;$$

then $F \in \mathscr{F}(T^4)$ and $E(C) \cap E(F) = \emptyset$. Let $y = x$; we define

$$F = F' - w_2y + w_2u_1 + yu_2;$$

then $F \in \mathscr{F}(T^4)$ and $E(C) \cap E(F) = \emptyset$.

1.3. Let $(T_0, r_0) = D_{*33*}$. There exist $x, y \in V(S) - \{u_1, u_2\}$ such that $xu_1, yu_2 \in E(C')$. We define

$$C = C' - xu_1 - yu_2 + xw_1 + w_1w_2 + w_2w_3 + w_3u_1 + yu_3 + u_3u_2,$$

$$F = F' + w_1w_3 + w_2u_3;$$

then $C \in \mathscr{H}(T^3)$, $F \in \mathscr{F}(T^4)$, and $E(C) \cap E(F) = \emptyset$.

2. Let $(T_0, r_0) \notin \{D_{*21}, D_{*22}, D_{22}, D_{23*}, D_{23}, D_{*33*}\}$. Then $(T_0, r_0) \in \{D_{*31}, D_{31}, D_{*33}, D_{33}, D_{04**}, D_{04*}, D_{04}\} \cup \mathscr{J}$ and if $(T_0, r_0) = D_{33}$ then $\deg_T r_0 \geqq 4$.

2.1. Let $\deg_T r_0 - \deg_{T_0} r_0 \geqq 2$. If $(T_0, r_0) \notin \mathscr{J}$, then we denote

$$S = T - w_1 - \ldots - w_n - u_1 - \ldots - u_m \text{ if } m = 3 \text{ and}$$

$$S = T - w_1 - \ldots - w_n \text{ if } m = 0.$$

If $(T_0, r_0) \in \mathscr{J}$, then we denote

$$S = T - x_2 - x_3 - y_1 - \ldots - y_m - z_1 - \ldots - z_n.$$

Since $|V(S)| \geqq 4$ and $|V(S)|$ is even, it follows from the induction assumption that there exist $C' \in \mathscr{H}(S^3)$ and $F' \in \mathscr{F}(S^4)$ such that $E(C') \cap E(F') = \emptyset$.

Let $\varphi(r_0)$, $\psi(r_0)$ be vertices selected in accordance with Remark 1. We define

$C = C' - \varphi(u_0)\,\psi(u_0) + \psi(u_0)\,w_1 + w_1u_2 + u_2u_3 + u_3u_1 + u_1\,\varphi(u_0)$ and

$F = F' + w_1u_3 + u_1u_2$ if $(T_0, r_0) \in \{D_{*31}, D_{31}\}$;

$C = C' - \varphi(u_0)\,\psi(u_0) + \psi(u_0)\,w_1 + w_1w_3 + w_3w_2 + w_2u_1 + u_1u_3 + u_3u_2 + u_2\,\varphi(u_0)$ and

$F = F' + u_1w_3 + u_2w_2 + u_3w_1$ if $(T_0, r_0) \in \{D_{*33}, D_{33}\}$;

$C = C' - \varphi(u_0)\,\psi(u_0) + \psi(u_0)\,w_1 + w_1w_3 + w_3w_4 + w_4w_2 + w_2\,\varphi(u_0)$ and

$F = F' + w_1w_4 + w_2w_3$ if $(T_0, r_0) \in \{D_{04**}, D_{04*}, D_{04}\}$;

$C = C' - \varphi(x_1)\,\psi(x_1) + \psi(x_1)\,x_2 + x_2y_1 + y_1y_3 + y_3y_2 + y_2z_1 + z_1z_3 + z_3z_2 + z_2z_3 + x_3\,\varphi(x_1)$ and

$F = F' + x_2x_3 + y_1z_3 + y_3z_1 + y_2z_2$ if $(T_0, r_0) = J_1$;

$C = C' - \varphi(x_1)\,\psi(x_1) + \psi(x_1)\,x_2 + x_2z_1 + z_1x_3 + x_3y_2 + y_2y_4 + y_4y_3 + y_3y_5 + y_5y_7 + y_7y_6 + y_6y_1 + y_1\,\varphi(x_1)$ and

$F = F' + x_2x_3 + z_1y_1 + y_2y_7 + y_3y_6 + y_4y_5$ if $(T_0, r_0) = J_2$;

$C = C' - \varphi(x_1)\,\psi(x_1) + \psi(x_1)\,x_2 + x_2z_1 + z_1x_3 + x_3y_1 + y_1y_3 + y_3y_4 + y_4y_2 + y_2y_6 + y_6y_7 + y_7y_5 + y_5\,\varphi(x_1)$ and

$F = F' + x_2x_3 + z_1y_1 + y_2y_7 + y_3y_6 + y_4y_5$ if $(T_0, r_0) \in \{J_3, J_4\}$;

$C = C' - \varphi(x_1)\,\psi(x_1) + \psi(x_1)\,x_3 + x_3y_1 + y_1y_3 + y_3y_2 + y_2x_2 + x_2z_2 + z_2z_3 + z_3z_1 + z_1\,\varphi(x_1)$ and

$F = F' + x_2x_3 + y_1z_3 + y_2z_2 + y_3z_1$ if $(T_0, r_0) = J_5$;

$C = C' - \varphi(x_1)\,\psi(x_1) + \psi(x_1)\,x_2 + x_2y_2 + y_2y_3 + y_3y_1 + y_1y_5 + y_5y_6 + y_6y_4 + y_4x_3 + x_3z_2 + z_2z_3 + z_3z_1 + z_1z_5 + z_5z_6 + z_6z_4 + z_4\,\varphi(x_1)$ and

$F = F' + x_2x_3 + y_1y_6 + y_2y_5 + y_3y_4 + z_1z_6 + z_2z_5 + z_3z_4$ if $(T_0, r_0) = J_6$.

Obviously, $C \in \mathscr{H}(T^3)$, $F \in \mathscr{F}(T^4)$ and $E(C) \cap E(F) = \emptyset$.

2.2 Let $\deg_T r_0 - \deg_{T_0} r_0 < 2$. Then $(T_0, r_0) \neq D_{33}$. Since $|V(T_0)|$ is odd, we have $\deg_T r_0 - \deg_{T_0} r_0 = 1$. We denote

$S = T - w_1 - u_3$ if $(T_0, r_0) \in \{D_{*31}, D_{31}\}$;

$S = T - w_1 - w_2 - w_3 - u_3$ if $(T_0, r_0) = D_{*33}$;

$S = T - w_3 - w_4$ if $(T_0, r_0) \in \{D_{04**}, D_{04*}, D_{04}\}$ and

$S = T - y_1 - \ldots - y_m - z_1 - \ldots - z_n$ if $(T_0, r_0) \in \mathscr{J}$.

Since $|V(S)| \geqq 4$ and $|V(S)|$ is even, it follows from the induction assumption that there exist $C' \in \mathscr{H}(S^3)$ and $F' \in \mathscr{F}(S^4)$ such that $E(C') \cap E(F') = \emptyset$. Since $\deg_T r_0 - \deg_{T_0} r_0 = 1$, S contains a terminal subtree isomorphic to D_{03} or D_{03*}.

2.2.1. Let $(T_0, r_0) \in \{D_{*31}, D_{31}, D_{*33}, D_{04**}, D_{04*}, D_{04}, J_1, J_2, J_3, J_4\}$. Then S contains a terminal subtree isomorphic to D_{03}. Lemma 2 implies that:

if $(T_0, r_0) \in \{D_{*31}, D_{31}, D_{*33}\}$, there exist $i \in \{1, 2\}$, $\tilde{C} \in \mathscr{H}(S^3)$ and $\tilde{F} \in \mathscr{F}(S^4)$ such that $E(\tilde{C}) \cap E(\tilde{F}) \subset \{u_0u_i, u_1u_2\} \subset E(\tilde{C})$;

if $(T_0, r_0) \in \{D_{04**}, D_{04*}, D_{04}\}$, there exist $i \in \{1, 2\}$, $\tilde{C} \in \mathscr{H}(S^3)$ and $\tilde{F} \in \mathscr{F}(S^4)$ such that $E(\tilde{C}) \cap E(\tilde{F}) \subset \{u_0w_i, w_1w_2\} \subset E(\tilde{C})$;

if $(T_0, r_0) \in \{J_1, J_2, J_3, J_4\}$, there exist $i \in \{2, 3\}$, $\tilde{C} \in \mathscr{H}(S^3)$ and $\tilde{F} \in \mathscr{F}(S^4)$ such that $E(\tilde{C}) \cap E(\tilde{F}) \subset \{x_1x_i, x_2x_3\} \subset E(\tilde{C})$.

We define

$$C = \tilde{C} - u_0u_i - u_1u_2 + u_0w_1 + w_1u_i + u_2u_3 + u_3u_1 \text{ and}$$
$$F = \tilde{F} + w_1u_3 \text{ if } (T_0, r_0) \in \{D_{*31}, D_{31}\};$$

$$C = \tilde{C} - u_0u_i - u_1u_2 + u_0w_3 + w_3w_2 + w_2w_1 + w_1u_i + u_1u_3 + u_3u_2 \text{ and}$$
$$F = \tilde{F} + u_3w_2 + w_1w_3 \text{ if } (T_0, r_0) = D_{*33};$$

$$C = \tilde{C} - u_0w_i - w_1w_2 + u_0w_3 + w_3w_i + w_1w_4 + w_4w_2 \text{ and}$$
$$F = \tilde{F} + w_3w_4 \text{ if } (T_0, r_0) \in \{D_{04**}, D_{04*}, D_{04}\};$$

$$C = \tilde{C} - x_1x_i - x_2x_3 + x_1y_1 + y_1y_3 + y_3y_2 + y_2x_i + x_2z_1 + z_1z_2 + z_2z_3 + z_3x_3 \text{ and}$$
$$F = \tilde{F} + y_1z_3 + y_2z_2 + y_3z_1 \text{ if } (T_0, r_0) = J_1;$$

$$C = \tilde{C} - x_1x_i - x_2x_3 + x_1y_2 + y_2y_4 + y_4y_3 + y_3y_5 + y_5y_7 + y_7y_6 + y_6y_1 + y_1x_i + x_2z_1 + z_1x_3 \text{ and}$$
$$F = \tilde{F} + y_1z_1 + y_2y_7 + y_3y_6 + y_4y_5 \text{ if } (T_0, r_0) \in \{J_2, J_3, J_4\}.$$

Obviously, $C \in \mathscr{H}(T^3)$, $F \in \mathscr{F}(T^4)$ and $E(C) \cap E(F) = \emptyset$.

2.2.2. Let $(T_0, r_0) \in \{J_5, J_6\}$. Then S contains a terminal subtree isomorphic to D_{03*}. It follows from Lemma 3 that there exist $\tilde{C} \in \mathscr{H}(S^3)$ and $\tilde{F} \in \mathscr{F}(S^4)$ such that

$$E(\tilde{C}) \cap E(\tilde{F}) \subset \{x_1x_2, x_1x_3, x_2x_3\} \text{ and } |E(\tilde{C}) \cap \{x_1x_2, x_1x_3, x_2x_3\}| = 2.$$

Hence there exists $i \in \{1, 2\}$ such that $x_ix_3 \in E(\tilde{C})$. We put $j = 1$ if $i = 2$ and $j = 2$ if $i = 1$. There exists $k \in \{1, 2, 3\} - \{j\}$ such that $x_jx_k \in E(\tilde{C})$.

We define

$$C = \tilde{C} - x_ix_3 - x_jx_k + x_3y_1 + y_1y_3 + y_3y_2 + y_2x_i + x_kz_1 + z_1z_3 + z_3z_2 + z_2x_j \text{ and}$$
$$F = \tilde{F} + y_1z_3 + y_2z_2 + y_3z_1 \text{ if } (T_0, r_0) = J_5;$$

$$C = \tilde{C} - x_ix_3 - x_jx_k + x_3z_2 + z_2z_3 + z_3z_1 + z_1z_5 + z_5z_6 + z_6z_4 + z_4x_i + x_ky_1 + y_1y_3 + y_3y_2 + y_2y_4 + y_4y_6 + y_6y_5 + y_5x_j \text{ and}$$
$$F = \tilde{F} + y_1y_6 + y_2y_5 + y_3y_4 + z_1z_6 + z_2z_5 + z_3z_4 \text{ if } (T_0, r_0) = J_6.$$

Obviously, $C \in \mathscr{H}(T^3)$, $F \in \mathscr{F}(T^4)$ and $E(C) \cap E(F) = \emptyset$.

Thus the proof of Lemma 4 is complete.

Theorem 1 immediately follows from Lemma 4.

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