Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 114 (1989), No. 4, 343--355

Persistent URL: http://dml.cz/dmlcz/118389

114 (1989) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 4, 343—355

NECESSARY AND SUFFICIENT CONDITIONS FOR IMBEDDINGS IN WEIGHTED SOBOLEV SPACES

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(Received January 14, 1987)

Summary. The paper deals with imbeddings of weighted Sobolev spaces $W^{1,p}(\Omega; S)$ (S is a collection of weight functions) into weighted Lebesgue spaces $L^q(\Omega; w)$ (w is a weight function). General necessary and sufficient conditions for such imbeddings are established.

Keywords: Weight function, weighted Sobolev space, weighted Lebesgue space, compact imbedding, continuous imbedding.

AMS Classification: 46E35.

1. INTRODUCTION

Let Ω be a domain in R^N. By $\mathscr{W}(\Omega)$ we denote the set of weight functions on Ω, i.e., the set of all measurable, a.e. in Ω positive and finite functions.

For $w \in \mathscr{W}(\Omega)$, $1 \leqq q < \infty$ the weighted Lebesgue space $L^q(\Omega; w)$ is the set of all measurable functions u defined on Ω with a finite norm

$$\|u\|_{q,\Omega,w} = \left(\int_\Omega |u(x)|^q \, w(x) \, dx\right)^{1/q} . \tag{1.1}$$

Obviously, the space $L^q(\Omega; w)$ with the norm (1.1) is complete.

Let $1 \leqq p < \infty$, $p^* = p/(p-1)$ ($p^* = \infty$ for $p = 1$) and let S be a collection of weight functions

$$S = \{v_i \in \mathscr{W}(\Omega);\ i = 0, 1, \ldots, N\} . \tag{1.2}$$

Throughout the paper we assume that

$$v_i \in L^1_{\mathrm{loc}}(\Omega) , \quad v_i^{-1/p} \in L^{p^*}_{\mathrm{loc}}(\Omega) , \quad i = 0, 1, \ldots, N . \tag{1.3}$$

We define the weighted Sobolev space $W^{1,p}(\Omega; S)$ as the set of all functions $u \in L^p(\Omega; v_0)$ which have distributional derivatives $\partial u/\partial x_i \in L^p(\Omega; v_i)$, $i = 1, \ldots, N$. We can easily verify that the space $W^{1,p}(\Omega; S)$ with the norm

$$\|u\|_{1,p,\Omega,S} = \left(\|u\|^p_{p,\Omega,v_0} + \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|^p_{p,\Omega,v_i}\right)^{1/p} \tag{1.4}$$

is a Banach space. Further, we define the space

$$W_0^{1,p}(\Omega; S)$$

as the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (1.4). The norm in this space is again given by (1.4).

For two Banach spaces X, Y we write $X \hookrightarrow\hookrightarrow Y$ or $X \hookrightarrow Y$ if $X \subset Y$ and the natural injection of X into Y is compact or continuous, respectively.

The symbol $M \subset\subset X$ means that M is a closed subspace of a Banach space X. Throughout this paper we will suppose that

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n ,$$

where Ω_n are domains in $\boldsymbol{R}^N$ such that

$$\Omega_n \subset \Omega_{n+1} \subset \Omega , \quad \Omega_{n+1} \neq \Omega .$$

Further, we set

$$\Omega^n = \Omega \setminus \Omega_n , \quad n \in \boldsymbol{N} .$$

In [6], [9] it was shown that

$$X = W_0^{1,p}(\Omega; S) \hookrightarrow\hookrightarrow L^p(\Omega; w)$$

if there are local imbeddings

$$W^{1,p}(\Omega_n; S) \hookrightarrow\hookrightarrow L^p(\Omega_n; w) , \quad n \in \boldsymbol{N} ,$$

and

$$\lim_{n\to\infty} \sup_{u\in X, \|u\|_X \leqq 1} \|u\|_{p,\Omega^n,w} = 0 .$$

The aim of the paper is to generalize this result. In Section 2 we will prove the following assertion:

Let $p, q \in \langle 1, \infty)$, $X \subset\subset W^{1,p}(\Omega; S)$. *Suppose*

$$W^{1,p}(\Omega_n; S) \hookrightarrow\hookrightarrow L^q(\Omega_n; w) \quad \forall n \in \boldsymbol{N} .$$

Then

$$X \hookrightarrow\hookrightarrow L^q(\Omega; w) \tag{1.5}$$

if and only if

$$\lim_{n\to\infty} \sup_{u\in X, \|u\|_X \leqq 1} \|u\|_{q,\Omega^n,w} = 0 . \; ^{*)}$$

Analogous results concerning the continuous imbedding $X \hookrightarrow L^q(\Omega; w)$ are also included in Section 2. Theorems from this section are applied in [7], [4] and [8].

In Section 3 we establish some other conditions which are necessary and sufficient for the compactness of the imbedding under investigation. Let us remark that the theorems in Section 3 imply the result by A. Avantaggiati (see [1], Theorem 2.1).

2. NECESSARY AND SUFFICIENT CONDITIONS FOR IMBEDDINGS

The main results of this section are Theorems 2.4, 2.5 and 2.7. The proofs of the first two of them are based on the following two lemmas.

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*) The proof will be quite different from those in [6], [9].

2.1. Lemma. *Suppose* $p, q \in \langle 1, \infty)$ *and*

$$W^{1,p}(\Omega_n; S) \hookrightarrow\hookrightarrow L^q(\Omega_n; w) \quad \forall n \in N. \tag{2.1}$$

Further, for every $\varepsilon > 0$ *let there exist* $\bar{n} \in N$ *such that*

$$\|u\|^q_{q,\Omega,w} \leqq \varepsilon \|u\|^q_{1,p,\Omega,S} + \|u\|^q_{q,\Omega_{\bar{n}},w} \quad \forall u \in W^{1,p}(\Omega; S). \tag{2.2}$$

Then

$$W^{1,p}(\Omega; S) \hookrightarrow\hookrightarrow L^q(\Omega; w). \tag{2.3}$$

Proof. Let $\{u_n\} \subset W^{1,p}(\Omega; S)$, $\|u_n\|_{1,p,\Omega,S} \leqq C$ for all $n \in N$, where $C \in (0, \infty)$. For a given $\varepsilon > 0$ choose $\varepsilon_1 \in (0, \varepsilon^q/(2C^q + 1)\rangle$. By assumption there exists $\bar{n} \in N$ such that

$$\|u\|^q_{q,\Omega,w} \leqq \varepsilon_1 \|u\|^q_{1,p,\Omega,S} + \|u\|^q_{q,\Omega_{\bar{n}},w} \quad \forall u \in W^{1,p}(\Omega; S). \tag{2.4}$$

The imbedding (2.1) implies the existence of a subsequence $\{u_{n_k}\}$ which is a Cauchy sequence in the space $L^q(\Omega_{\bar{n}}, w)$. Hence there exists $k_0 \in N$ such that

$$\|u_{n_k} - u_{n_l}\|^q_{q,\Omega_{\bar{n}},w} \leqq \varepsilon_1 \quad \text{for all} \quad k, l \geqq k_0.$$

This and (2.4) yield

$$\|u_{n_k} - u_{n_l}\|^q_{q,\Omega,w} \leqq \varepsilon_1 \|u_{n_k} - u_{n_l}\|^q_{1,p,\Omega,S} + \|u_{n_k} - u_{n_l}\|^q_{q,\Omega_{\bar{n}},w} \leqq$$
$$\leqq \varepsilon_1 \, 2C^q + \varepsilon_1 \leqq \varepsilon^q.$$

Thus $\{u_{n_k}\}$ is a Cauchy sequence in the space $L^q(\Omega; w)$ and therefore (2.3) holds.

2.2. Lemma. *Suppose* $p, q \in \langle 1, \infty)$ *and*

$$W^{1,p}(\Omega; S) \hookrightarrow\hookrightarrow L^q(\Omega; w). \tag{2.5}$$

Then for every $\varepsilon > 0$ *there exists* $\bar{n} \in N$ *such that*

$$\|u\|^q_{q,\Omega,w} \leqq \varepsilon \|u\|^q_{1,p,\Omega,S} + \|u\|^q_{q,\Omega_{\bar{n}},w} \quad \forall u \in W^{1,p}(\Omega; S). \tag{2.6}$$

Proof. Let us assume, on the contrary, that the statement of Lemma 2.2 is false. Then there exists $\varepsilon > 0$ and a sequence $\{u_n\} \subset W^{1,p}(\Omega; S)$ such that

$$\|u_n\|^q_{q,\Omega,w} > \varepsilon \|u_n\|^q_{1,p,\Omega,S} + \|u_n\|^q_{q,\Omega_n,w} \quad \forall n \in N. \tag{2.7}$$

Taking $v_n = u_n/\|u_n\|_{1,p,\Omega,S}$ (the inequality (2.7) implies $\|u_n\|_{1,p,\Omega,S} \neq 0$), we obtain

$$\|v_n\|^q_{q,\Omega,w} > \varepsilon + \|v_n\|^q_{q,\Omega_n,w} \quad \forall n \in N. \tag{2.8}$$

As the imbedding (2.5) holds and the sequence $\{v_n\}$ is bounded in $W^{1,p}(\Omega; S)$, there exists a subsequence $\{v_{n_k}\}$ and a function $v \in L^q(\Omega; w)$ such that $v_{n_k} \to v$ in $L^q(\Omega; w)$. Now, (2.8) yields

$$\|v\|^q_{q,\Omega,w} \geqq \varepsilon + \|v\|^q_{q,\Omega,w}.$$

This is a contradiction because $\varepsilon > 0$.

2.3. Remark. Inequality (2.6) can be rewritten in the form

$$\|u\|^q_{q,\Omega^{\bar{n}},w} \leqq \varepsilon \|u\|^q_{1,p,\Omega,S} \quad \forall u \in W^{1,p}(\Omega; S) . \tag{2.9}$$

Since

$$\|u\|^q_{q,\Omega^n,w} \leqq \|u\|^q_{q,\Omega^{\bar{n}},w} \quad \text{for} \quad n \geqq \bar{n} ,$$

we have by (2.9)

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,\Omega,S}\leqq 1} \|u\|_{q,\Omega^n,w} = 0 . \tag{2.10}$$

Conversely, let us suppose that (2.10) is true. Let $\varepsilon > 0$ and denote $\varepsilon_1 = \varepsilon^{1/q}$. Then by (2.10) there exists $\bar{n} \in N$ such that

$$\sup_{\|u\|_{1,p,\Omega,S}\leqq 1} \|u\|_{q,\Omega^n,w} \leqq \varepsilon_1 \quad \forall n \geqq \bar{n}$$

and so for every $u \in W^{1,p}(\Omega; S)$, $\|u\|_{1,p,\Omega,S} \leqq 1$, we have

$$\|u\|_{q,\Omega^n,w} \leqq \varepsilon_1 \quad \forall n \geqq \bar{n} . \tag{2.11}$$

The last inequality immediately yields

$$\|u\|_{q,\Omega^n,w} \leqq \varepsilon_1 \|u\|_{1,p,\Omega,S} \quad \forall u \in W^{1,p}(\Omega; S) , \quad \forall n \geqq \bar{n} .$$

This implies

$$\|u\|^q_{q,\Omega,w} \leqq \varepsilon_1^q \|u\|^q_{1,p,\Omega,S} + \|u\|^q_{q,\Omega_n,w} \quad \forall n \geqq \bar{n}$$

and therefore the inequality (2.6) is satisfied.

Summarizing the above lemmas and Remark 2.3 we obtain

2.4. Theorem. *Suppose* $p, q \in \langle 1, \infty)$. *If*

$$W^{1,p}(\Omega_n; S) \hookrightarrow\hookrightarrow L^q(\Omega_n; w) \quad \forall n \in N \tag{2.12}$$

and

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,\Omega,S}\leqq 1} \|u\|_{q,\Omega^n,w} = 0 , \tag{2.13}$$

then

$$W^{1,p}(\Omega; S) \hookrightarrow\hookrightarrow L^q(\Omega; w) . \tag{2.14}$$

Conversely, if (2.14) *holds, then the condition* (2.13) *is satisfied.*

Let $X \subset\subset W^{1,p}(\Omega; S)$ (e.g. $X = W^{1,p}_0(\Omega; S)$). For $n \in N$ we set

$$X_n = \{u;\ u = v|_{\Omega_n},\ v \in X\}$$

and in this space we consider the norm $\|\cdot\|_{X_n} = \|\cdot\|_{1,p,\Omega_n,S}$. For simplicity we denote $\|\cdot\|_X = \|\cdot\|_{1,p,\Omega,S}$.

Analogously as we have proved Theorem 2.4 we can verify the following theorem.

2.5. Theorem. *Suppose* $p, q \in \langle 1, \infty)$. *If*

$$X_n \hookrightarrow\hookrightarrow L^q(\Omega_n; w) \quad \forall n \in N \tag{2.15}$$

and

C1. $$\lim_{n \to +\infty} \sup_{u \in X, \|u\|_X \leqq 1} \|u\|_{q,\Omega^n,w} = 0\,,$$

then

$$X \hookrightarrow\hookrightarrow L^q(\Omega; w)\,. \tag{1.5}$$

Conversely, if (1.5) *holds, then the condition* **C1** *is satisfied.*

2.6. Remark. (i) The condition (2.15) will certainly be fulfilled if

$$W^{1,p}(\Omega_n; S) \hookrightarrow\hookrightarrow L^q(\Omega_n; w) \quad \forall n \in N\,.$$

(ii) Theorem 2.4 implies that under the assumption (2.12) the condition (2.13) is necessary and sufficient for the imbedding (2.14) to be compact. Similarly, under the assumption (2.15) the imbedding (1.5) takes place if and only if the condition **C1** is satisfied.

There is an analogue of Theorem 2.5 for continuous imbeddings.

2.7. Theorem. *Suppose* $p, q \in \langle 1, \infty)$. *If*

$$X_n \hookrightarrow L^q(\Omega_n; w) \quad \forall n \in N \tag{2.16}$$

and

$$\lim_{n \to \infty} \sup_{u \in X, \|u\|_X \leqq 1} \|u\|_{q,\Omega^n,w} < \infty\,, \tag{2.17}$$

then

$$X \hookrightarrow L^q(\Omega; w)\,. \tag{2.18}$$

Conversely, if (2.18) *holds, then the condition* (2.17) *is fulfilled.*

Proof. As

$$\|u\|_{q,\Omega,w} \leqq \|u\|_{q,\Omega^n,w} + \|u\|_{q,\Omega_n,w}\,,$$

we have

$$\sup_{u \in X, \|u\|_X \leqq 1} \|u\|_{q,\Omega,w} \leqq \sup_{u \in X, \|u\|_X \leqq 1} \|u\|_{q,\Omega^n,w} + \sup_{u \in X_n, \|u\|_{X_n} \leqq 1} \|u\|_{q,\Omega_n,w}$$

and thus the assumptions (2.16), (2.17) imply (2.18).

The converse assertion follows by a contradiction argument from the inequality

$$\|u\|_{q,\Omega^n,w} \leqq \|u\|_{q,\Omega,w}\,.$$

2.8. Remark. Let us note that under the additional assumptions

$$W_0^{1,p}(\Omega; S) \subset\subset X\,; \tag{i}$$

(ii) there exist an open non-empty set $Q \subset \Omega$ and a constant C, $0 < C < \infty$ such that

$$C^{-1} < w(x) < C\,, \quad C^{-1} < v_i(x) < C$$

for a.e. $x \in Q$ and for $i = 0, \ldots, N$;

the imbedding (1.5) yields

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + 1 > 0$$

(or equivalently $N^{-1} > p^{-1} - q^{-1}$) while (2.18) implies

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + 1 \geqq 0$$

(or equivalently $N^{-1} \geqq p^{-1} - q^{-1}$) – cf. [8], Lemma 8.10.

3. SOME OTHER NECESSARY AND SUFFICIENT CONDITIONS FOR COMPACT IMBEDDINGS

The aim of this section is to establish some other conditions that are necessary and/or sufficient for the imbedding (1.5) to be compact (see Theorems 3.8 and 3.9). The symbols p, q, Ω, X and X_n have the same meaning as in Section 2.

Let us consider a measurable space $(\Omega, \mathscr{A})$ where $\mathscr{A}$ is a σ-algebra of all Lebesgue measurable subsets of the domain Ω, and let $\{\sigma_n\}$ be a sequence of non negative and finite measures on $\mathscr{A}$. By $B(\delta)$ $(\delta > 0)$ we denote the ball $\{x \in \boldsymbol{R}^N;\ |x| < \delta\}$. The Lebesgue measure of a set $E \in \mathscr{A}$ is denoted by $|E|$. If $\{E_k\} \subset \mathscr{A}$ then the notation $E_k \searrow E$ means that $E_{k+1} \subset E_k$ for $k \in \boldsymbol{N}$ and $E = \lim_{k\to\infty} E_k \ (= \bigcap_{k=1}^{\infty} E_k)$.

In order to facilitate a concise formulation of our next result, we first recall the definitions of certain kinds of continuity for set functions.

3.1. Definition. (i) We say that the terms of the sequence $\{\sigma_n\}$ are *equicontinous from above at* $\emptyset$ (and write $\{\sigma_n\} \in \mathrm{ECA}\emptyset$), if the following condition holds:

$$\{E_k\} \subset \mathscr{A}\,, \quad E_k \searrow \emptyset \Rightarrow \lim_{k\to\infty} \sup_{n\in\boldsymbol{N}} \sigma_n(E_k) = 0\,. \tag{3.1}$$

(ii) We say that the terms of the sequence $\{\sigma_n\}$ are *uniformly absolutely continuous* (and write $\{\sigma_n\} \in \mathrm{UAC}$), whenever for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma_n(E) < \varepsilon$ for every positive integer n and for every set $E \in \mathscr{A}$, $|E| < \delta$.
[Obviously, $\{\sigma_n\} \in \mathrm{UAC}$ if and only if the following condition is satisfied:

$$\{E_k\} \subset \mathscr{A}\,, \quad \lim_{k\to\infty} |E_k| = 0 \Rightarrow \lim_{k\to\infty} \sup_{n\in\boldsymbol{N}} \sigma_n(E_k) = 0.] \tag{3.2}$$

(iii) The terms of the sequence $\{\sigma_n\}$ are said to be *uniformly absolutely continuous in the narrower sense* (and we write $\{\sigma_n\} \in \mathrm{UAC}^*$), if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma_n(E) < \varepsilon$ for every positive integer n and for every set $E \in \mathscr{A}$, $|E| < \delta$ or $E \cap B(1/\delta) = \emptyset$.

The relations between these three notions of continuity are established in the following two lemmas proved in Section 4.

3.2. Lemma. $\mathrm{UAC}^* = \mathrm{UAC} \cap \mathrm{ECA}\emptyset$.

3.3. Lemma. *If* $|\Omega| < \infty$, *then* $\mathrm{UAC}^* = \mathrm{UAC}$.

In this section we shall deal with a specific sequence of measures defined by

$$\sigma_n(E) = \int_E |u_n(x)|^q \, w(x) \, dx \,, \quad E \in \mathscr{A} \,, \quad n \in N \,, \tag{3.3}$$

where $w \in \mathscr{W}(\Omega)$, $q \in \langle 1, \infty)$ and $\{u_n\}$ is a sequence of functions from X.

Now we are ready to introduce some conditions on the measures (3.3) and investigate their relations to the compactness of the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega; w) \,. \tag{1.5}$$

C2. $\{\sigma_n\} \in \mathrm{ECA}\emptyset$ for every bounded sequence $\{u_n\} \subset X$.

C3. $\{\sigma_n\} \in \mathrm{UAC}$ for every bounded sequence $\{u_n\} \subset X$.

C4. $\{\sigma_n\{ \in \mathrm{UAC}^*$ for every bounded sequence $\{u_n\} \subset X$.

The following assertion is an immediate consequence of Lemma 3.2.

3.4. Lemma. **C4** $\Leftrightarrow$ **C2** $\wedge$ **C3**.

Further relations between the conditions **C1**—**C4** are given by the next lemma.*)

3.5. Lemma.
(i) **C2** $\Rightarrow$ **C1**;
(ii) **C4** $\Rightarrow$ **C1**;
(iii) *if* $|\Omega| < \infty$, *then* **C3** $\Rightarrow$ **C1**.

Proof. ad (i). Evidently, the condition **C1** is equivalent with the following one:

C1*. $\lim\limits_{k \to \infty} \sup\limits_{n \in N} \|u_n\|_{q,\Omega^k,w} = 0$ for every bounded sequence $\{u_n\} \subset X$.
Therefore we will prove the implication **C2** $\Rightarrow$ **C1***.
Let $\{u_n\} \subset X$ be a bounded sequence. As $\Omega^k \searrow \emptyset$, condition **C2** yields

*) For the condition **C1** see Theorem 2.5.

$$\limsup_{k\to\infty}\sup_{n\in N} \|u_n\|_{q,\Omega^k,w} = \limsup_{k\to\infty}\sup_{n\in N} \sigma_n(\Omega^k)^q = 0$$

and so the condition **C1*** holds.

The statement (ii) follows from (i) and Lemma 3.4. Finally, (iii) is a consequence of (ii) and Lemma 3.3

3.6. Lemma. *Let*

$$X \subset\subset L^q(\Omega; w)\,. \tag{1.5}$$

Then the conditions **C2** *and* **C3** *are satisfied.*

Proof. Let $X \subset\subset L^q(\Omega; w)$. We shall only prove that condition **C2** holds. The proof of **C3** is analogous.

Suppose on the contrary that condition **C2** is not satisfied. Then there exist a bounded sequence $\{u_n\} \subset X$, a number $\varepsilon > 0$ and a sequence $\{E_k\} \subset \mathscr{A}$, $E_k \searrow \emptyset$, such that

$$\sigma_n(E_k) \geqq \varepsilon\,, \quad n, k \in N\,. \tag{3.5}$$

The imbedding (1.5) and the boundedness of $\{u_n\}$ in the space X imply that there exist a subsequence $\{u_{n_i}\}$ and a function $v \in L^q(\Omega; w)$ such that

$$u_{n_i} \to v \quad \text{in} \quad L^q(\Omega; w)\,. \tag{3.6}$$

The inequalities (3.5) yield

$$\begin{aligned} \varepsilon \leqq \sigma_{n_i}(E_k) &= \int_{E_k} |u_{n_i}(x)|^q\, w(x)\, dx \leqq \\ &\leqq 2^{q-1}\{\int_\Omega |u_{n_i} - v|^q\, w\, dx + \int_{E_k} |v|^q\, w\, dx\}\,. \end{aligned} \tag{3.7}$$

If $|E_k| < \infty$ for some k, we have $|E_n| \to 0$ and, consequently,

$$\lim_{k\to\infty} \int_{E_k} |v(x)|^q\, w(x)\, dx = 0\,, \tag{3.8}$$

which contradicts (3.6) and (3.7).

Hence, suppose that $|E_k| = \infty$ for every $k \in N$. In view of

$$\int_\Omega |v(x)|^q\, w(x)\, dx < \infty$$

there exists $\delta > 0$ such that

$$\int_E |v(x)|^q\, w(x)\, dx < \varepsilon/2^q$$

for every set $E \in \mathscr{A}$ for which $|E| < \delta$ or $E \cap B(1/\delta) = \emptyset$. For $k \in N$ we set

$$F_k = E_k \cap B\left(\frac{1}{\delta}\right), \quad G_k = E_k \setminus B\left(\frac{1}{\delta}\right).$$

As

$$G_k \cap B\left(\frac{1}{\delta}\right) = \emptyset \quad \text{for} \quad k \in N\,,$$

we have

$$\int_{G_k} |v(x)|^q \, w(x) \, dx < \varepsilon/2^q . \tag{3.9}$$

The inequalities (3.5), (3.9) together with (3.6) imply

$$\begin{aligned} \varepsilon \leqq 2^{q-1}\{\int_\Omega |u_{n_i} - v|^q \, w \, dx + \int_{F_k} |v|^q \, w \, dx + \int_{G_k} |v|^q \, w \, dx\} < \\ < 2^{q-1}\{\int_\Omega |u_{n_i} - v|^q \, w \, dx + \int_{F_k} |v|^q \, w \, dx\} + \varepsilon/2 . \end{aligned} \tag{3.10}$$

Since $|F_k| \leqq |B(1/\delta)| < \infty$, $k \in N$, and $F_k \searrow \emptyset$, we have $|F_k| \to 0$, and consequently

$$\lim_{k \to \infty} \int_{F_k} |v(x)|^q \, w(x) \, dx = 0 . \tag{3.11}$$

Letting $i \to \infty$ in (3.10), we again obtain (in view of (3.6) and (3.11)) a controversial inequality $\varepsilon \leqq \varepsilon/2$. Thus the condition **C2** holds.

3.7. Remark. As an immediate consequence of Lemmas 3.6 and 3.4 we obtain the following assertion:

If

$$X \subset\subset L^q(\Omega; w) ,$$

then the condition **C4** *holds.*

Summarizing the above lemmas and Theorem 2.5 we conclude:

3.8. Theorem. *Suppose* $p, q \in \langle 1, \infty)$. *If*

$$X_n \subset\subset L^q(\Omega_n; w) \quad \forall n \in N \tag{2.15}$$

and if at least one of the conditions **C1, C2, C4** *is fulfilled, then*

$$X \subset\subset L^q(\Omega; w) . \tag{1.5}$$

Conversely, if (1.5) *holds, then all the conditions* **C1, C2, C3, C4** *are satisfied.*

3.9. Theorem. *Suppose* $p, q \in \langle 1, \infty)$, $|\Omega| < \infty$ *and*

$$X_n \subset\subset L^q(\Omega; w) \quad \forall n \in N .$$

If the condition **C3** *is fulfilled, then*

$$X \subset\subset L^q(\Omega; w) .$$

3.10. Remark. Theorem 3.8 implies that under the assumption (2.15) all the conditions **C1, C2, C4** and (1.5) are equivalent.

If we suppose in addition that $|\Omega| < \infty$, then by Theorem 3.8 and 3.9 all the conditions **C1, C2, C3, C4** and (1.5) are equivalent.

Proof of Lemma 3.2. As

$$\mathrm{UAC}^* \subset \mathrm{UAC},$$

it is sufficient to verify the inclusions

$$\mathrm{UAC}^* \subset \mathrm{ECA}\emptyset, \tag{4.1}$$

$$\mathrm{UAC} \cap \mathrm{ECA}\emptyset \subset \mathrm{UAC}^*. \tag{4.2}$$

a) Let $\{\sigma_n\} \in \mathrm{UAC}^*$ and $\{E_k\} \subset \mathscr{A}$, $E_k \searrow \emptyset$. Further, let ε be a positive number. Then there exists $\delta > 0$ such that

$$\sup_{n \in N} \sigma_n(E) < \varepsilon \quad \text{for every} \quad E \in \mathscr{A} \quad \text{with} \tag{4.3}$$

$$|E| < \delta \quad \text{or} \quad E \cap B\left(\frac{1}{\delta}\right) = \emptyset.$$

We have to distinguish the following two cases:
(i) there exists $k_0 \in N$ such that $|E_{k_0}| < \infty$;
(ii) $|E_k| = \infty$ for every $k \in N$.

In the case (i) we have $\lim_{k \to \infty} |E_k| = 0$, and consequently there exists $k_1 \in N$ such that $|E_k| < \delta$ for every $k \geqq k_1$, $k \in N$. Now (4.3) implies

$$\sup_{n \in N} \sigma_n(E_k) < \varepsilon \quad \text{for every} \quad k \geqq k_1, \quad k \in N,$$

i.e.

$$\lim_{k \to \infty} \sup_{n \in N} \sigma_n(E_k) = 0, \tag{4.4}$$

which completes the proof of (4.1) in the case (i).

To prove (4.1) in the case (ii) we denote

$$F_k = E_k \cap B\left(\frac{1}{\delta}\right), \quad G_k = E_k \setminus B\left(\frac{1}{\delta}\right), \quad k \in N.$$

Since $E_k \searrow \emptyset$, we have $F_k \searrow \emptyset$. Further, $|F_k| \leqq |B(1/\delta)| < \infty$ for $k \in N$, hence $\lim_{k \to \infty} |F_k| = 0$. Therefore there exists $k_1 \in N$ such that $|F_k| < \delta$ for every $k \geqq k_1$, $k \in N$. Using (4.3) we obtain

$$\sup_{n \in N} \sigma_n(F_k) < \varepsilon \quad \text{for every} \quad k \geqq k_1, \quad k \in N. \tag{4.5}$$

Since $G_k \cap B(1/\delta) = \emptyset$, (4.3) yields

$$\sup_{n \in N} \sigma_n(G_k) < \varepsilon \quad \text{for every} \quad k \in N.$$

The last estimate and (4.5) imply

$$\sup_{n\in N} \sigma_n(E_k) = \sup_{n\in N} [\sigma(F_k) + \sigma_n(G_k)] \leqq \sup_{n\in N} \sigma_n(F_k) + \sup_{n\in N} \sigma_n(G_k) < 2\varepsilon$$

for every $k \geqq k_1$, $k \in N$, and consequently, (4.4) again holds. The proof of (4.1) is complete.

b) To prove (4.2) we assume that the statement of (4.2) is false, i.e., there exists $\{\sigma_n\}$, $\{\sigma_n\} \in UAC \cap \mathrm{ECA}\emptyset$ but $\{\sigma_n\} \notin \mathrm{UAC}^*$. Then it is possible to find $\varepsilon > 0$ and a sequence $\{E_k\} \subset \mathscr{A}$ such that

$$|E_k| < \frac{1}{k} \quad \text{or} \quad E_k \cap B(k) = \emptyset$$

and

$$\sup_{n\in N} \sigma_n(E_k) \geqq \varepsilon \quad \text{for every} \quad k \in N. \tag{4.6}$$

We have to distinguish the following two cases:

(i) there exists an infinite set $N_1 \subset N$ such that $E_k \cap B(k) = \emptyset$ for every $k \in N_1$;

(ii) there exists an infinite set $N_2 \subset N$ such that $|E_k| < 1/k$ for every $k \in N_2$.

If we denote $G_k = R^N \setminus B(k)$, $k \in N_1$ in the case (i), we have $G_k \searrow \emptyset$ for $k \to \infty$, $k \in N_1$. Then the condition $\{\sigma_n\} \in \mathrm{ECA}\emptyset$ implies

$$\lim_{\substack{k\to\infty \\ k\in N_1}} \sup_{n\in N} \sigma_n(G_k) = 0\,. \tag{4.7}$$

Since $E_k \subset G_k$ for $k \in N_1$, (4.7) implies

$$\lim_{\substack{k\to\infty \\ k\in N_1}} \sup_{n\in N} \sigma_n(E_k) = 0\,.$$

Hence there exists $k_0 \in N$ such that

$$\sup_{n\in N} \sigma_n(E_k) \leqq \frac{\varepsilon}{2} \quad \text{for every} \quad k \geqq k_0\,, \quad k \in N_1\,.$$

However, this contradicts (4.6).

Now, let us consider the case (ii). Since $\{\sigma_n\} \in \mathrm{UAC}$, there is $\delta > 0$ such that $\sup_{n\in N} \sigma_n(E) \leqq \varepsilon/2$ for every set $E \in \mathscr{A}$, $|E| < \delta$. As $|E_k| < 1/k$ for every $k \in N_2$, we can find $k_0 \in N$ such that $|E_k| < \delta$ for $k \geqq k_0$, $k \in N_2$. Consequently,

$$\sup_{n\in N} \sigma_n(E_k) \leqq \frac{\varepsilon}{2} \quad \text{for every} \quad k \geqq k_0\,, \quad k \in N_2\,,$$

which contradicts (4.6). The proof of Lemma 3.2 is complete.

Proof of Lemma 3.3. By Lemma 3.2 it is sufficient to verify

$$\mathrm{UAC} \subset \mathrm{ECA}\emptyset\,. \tag{4.8}$$

If $\{E_k\} \subset \mathscr{A}$, $E_k \searrow \emptyset$, we have $|E_k| \leqq |\Omega| < \infty$ for every $k \in N$, and consequently

$$\lim_{k \to \infty} |E_k| = 0 .$$

This and the condition $\{\sigma_n\} \in \mathrm{UAC}$ imply

$$\lim_{k \to \infty} \sup_{n \in N} \sigma_n(E_k) = 0$$

and therefore $\{\sigma_n\} \in \mathrm{ECA}\emptyset$.

References

[1] *A. Avantaggiati:* On compact embedding theorems in weighted Sobolev spaces, Czechoslovak Math. J., *29* (*104*) (1979), no. 4, 635—648.

[2] *P. R. Halmos:* Measure Theory, Graduate texts in mathematics 18, Springer-Verlag, New York—Heidelberg—Berlin, 1974.

[3] *A. Kufner, B. Opic:* How to define reasonably weighted Sobolev spaces, Comment. Math. Univ. Carolinae, *25* (*3*) (1984), 537—554.

[4] *P. Gurka, B. Opic:* Continuous and compact imbeddings of weighted Sobolev spaces I, Czechoslovak Math. J., *38*(*113*) (1988), 730—744.

[5] *B. Opic:* Some remarks to a compact imbedding of a weighted Sobolev space defined on an unbounded domain, A. Haar. Mem. Conf., Budapest/Hungary 1985, Colloq. Math. Soc. János Bolyai *49* (1987), 667—673.

[6] *B. Opic:* Compact imbedding of weighted Sobolev space defined on an unbounded domain I, Časopis Pěst. Mat., No. 1, *113* (1988), 60—73.

[7] *B. Opic, P. Gurka:* A_r-condition for two weight fnnctions and compact imbeddings of weighted Sobolev spaces, Czechoslovak Math. J., *38* (*113*) (1988), 611—617.

[8] *B. Opic, P. Gurka:* Continuous and compact imbeddings of weighted Sobolev spaces II, Czechoslovak Math. J., *39* (*114*) (1989), 78—94.

[9] *B. Opic, A. Kufner:* Remark on compactness of imbeddings in weighted spaces, Math. Nachr. *133* (1987), 63—70.

Souhrn

NUTNÉ A POSTAČUJÍCÍ PODMÍNKY PRO VNOŘENÍ VÁHOVÝCH SOBOLEVOVÝCH PROSTORŮ

BOHUMÍR OPIC

V článku jsou zkoumána vnoření váhových Sobolevových prostorů $W^{1,p}(\Omega; S)$ (S je systém váhových funkcí) do váhových Lebesgueových prostorů $L^q(\Omega; w)$ (w je váhová funkce). Jsou nalezeny nutné a postačující podmínky pro vyšetřovaná vnoření.

Резюме

НЕОБХОДИМЫЕ И ДОСТАТОЧНЫЕ УСЛОВИЯ ДЛЯ ВЛОЖЕНИЙ ВЕСОВЫХ ПРОСТРАНСТВ СОБОЛЕВА

Bohumír Opic

В работе исследуются вложения весовых прсстранств Соболева $W^{1,p}(\Omega; S)$ (S — система весовых функций) в весовые пространства Лебега $L^q(\Omega; w)$ (w — весовая функция). Установлены необходимые и достаточные условия для существования рассматриваемых вложений.

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