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The criteria of strongly exposed points in Orlicz spaces

TINGFU WANG, DONGHAI JI, ZHONGRUI SHI

Abstract. In Orlicz spaces, the necessary and sufficient conditions of strongly exposed points are given.

Keywords: Orlicz spaces, strongly exposed points

Classification: 46E30

Let $X$ be a Banach space, $S(X)$, $B(X)$ denote the unit sphere and unit ball of $X$, respectively. $X^*$ denotes the dual of $X$. $x \in S(X)$ is called an exposed point of $B(X)$ provided there exists $f \in S(X^*)$, such that for all $y \in B(X) \setminus \{x\}$ $f(y) < f(x) = 1$. $x \in S(X)$ is called a strongly exposed point of $B(X)$ provided there exists $f \in S(X^*)$ such that for $x_n \in B(X)$, $f(x_n) \to f(x) = 1$ implies $\|x_n - x\| \to 0$. Obviously, a strongly exposed point is an exposed point.

The exposed points in Orlicz spaces have been discussed (see [1]). In this paper, we will give the criteria of strongly exposed points in Orlicz spaces.

For the sake of convenience, we still present the full proofs. The symbols used in this paper have the same meanings as [2]. $M(u)$, $N(v)$, denote a pair of complemented $N$-functions. $p(u)$, $q(v)$, denote their right-hand derivatives respectively. “$M \in \Delta_2$” (“$M \in \nabla_2$”) means that $M(u)$ satisfies the $\Delta_2$-condition ($\nabla_2$-condition) for large $u$. For the set of $\sum$-measurable functions over a finite nonatomic measure space $(G, \sum, \mu)$,

$$\left\{x(t) : \exists c > 0 \text{ such that } R_M \left(\frac{x}{c}\right) = \int_G M \left(\frac{x(t)}{c}\right) dt < \infty\right\}$$

endowed with Luxemburg norm $\|x\|_M = \inf\{c > 0 : R_M (\frac{x}{c}) \leq 1\}$ and Orlicz norm $\|x\|_M = \sup\{\int_G x(t)y(t) dt : R_N(y(t)) \leq 1\} = \inf\{\frac{1}{k}(1 + R_M(kx)) : k > 0\}$, we denote them as $L_M(G)$, $L_M$, respectively, and call them Orlicz spaces. In addition, for an element $x \in L_M (or L(M))$, we denote

$$K_1(x) = \left\{k > 0 : \|x\|_M = \frac{1}{k}(1 + R_M(kx))\right\},$$

$$\xi_1(x) = \lim_{n \to \infty} \|x - x_n\|_M = \inf \left\{c > 0 : R_M \left(\frac{x}{c}\right) < \infty\right\} = \lim_{n \to \infty} \|x - x_n\|_M,$$

where $x_n(t) = x(t)$ if $|x(t)| \leq n$ and $x_n(t) = 0$ if $|x(t)| > n$. The project supported by NSF of H.L.J. of China
Theorem 1. $x \in S(L(M))$ is a strongly exposed point of $B(L(M))$ if and only if

(i) $M \in \Delta_2$;
(ii) $\mu\{t \in G : x(t) \in R \setminus S_M\} = 0$;
(iii) denote $\{a_i\}, \{b_j\}$ as the sets of all those left extreme points and right extreme points of affine segments of $M(u)$, respectively, satisfying $p_-(a_i) = p(a_i), p_-(b_j) = p(b_j)$, then

$\mu\{t \in G : |x(t)| \in \{a_i\}\} = 0, \text{or}$

$\mu\{t \in G : |x(t)| \in \{b_j\}\} = 0, \text{and there exists } \tau > 0 \text{ such that}$

$\int_G N((1 + \tau)p_-(x(t))) dt < \infty.$

Proof: Necessity. Since a strongly exposed point is a strongly extreme point, by Theorem in [2], (i) and (ii) are trivial. If (iii) were not true, we only need to discuss the following two cases:

(I) there exist affine segments of $M(u), [a, b]$ and $[d, b]$, such that $p_-(a) = p(a), p_-(b) = p(b)$ and $\mu G_a = \mu\{t \in G : |x(t)| = a\} > 0, \mu G_b = \mu\{t \in G : |x(t)| = b\} > 0$, without loss of generality, we assume $x(t) \geq 0$ for all $t \in G$. Take $E \subset G_a, F \subset G_b$ such that

$$M(a)\mu E + M(b)\mu F = M\left(\frac{1}{2}(a + c)\right)\mu E + M\left(\frac{1}{2}(d + b)\right)\mu F.$$ 

Put

$$x'(t) = (x(t) - ky(t))\chi_{G \setminus E} + \frac{1}{2}(a + c)\chi_E + \frac{1}{2}(d + b)\chi_F,$$

then $x \neq x'$, $R_M(x') = R_M(x) = 1$, so $\|x'(t)\|_{(M)} = 1$. Take a support functional $y(t)$ of $x(t)$, and $k \in K_N(y)$, then $p_-(x(t)) \leq ky(t) \leq p(x(t))$. Noticing

$$p_-(x(t)) = p_-(x'(t)) \leq ky(t) \leq p(x(t)) = p(x'(t)) \text{ whenever } t \in G \setminus E \setminus F;$$

$$ky(t) = p(a) = p\left(\frac{1}{2}(a + c)\right) = p(x'(t)) \text{ whenever } t \in E;$$

$$ky(t) = p(b) = p\left(\frac{1}{2}(d + b)\right) = p(x'(t)) \text{ whenever } t \in F;$$

we have $\int_G x'(t)y(t) dt = \int_G x(t)y(t) dt = 1$, hence $x(t)$ is not a strongly exposed point of $B(L(M))$.

(II) there exists an affine segment $[a, b]$ satisfying $p_-(b) = p(b), \mu G_b = \mu\{t \in G : x(t) = b\} > 0$, and for any $\varepsilon > 0$, $\int_G N((1 + \varepsilon)p_-(x(t))) dt = \infty.$

Take $y \in S(L_N)$ satisfying $\int_G x(t)y(t) dt = 1$ and take $k \in K_N(y)$, then $p_-(x(t)) \leq ky(t) \leq p(x(t))$. Clearly, for any $\varepsilon > 0$, $\int_G N((1 + \varepsilon)ky(t)) dt = \infty$, so $\xi_N(ky) = 1$, hence $\|(ky(t))\chi_{G \setminus G_n}\|_N \to 1$ $(n \to \infty)$, where $G_n = \{t \in G : ky(t) \leq n\}$. By Hahn-Banach theorem, there exist $\{u_n\}_{n=1}^\infty \subset S(L(M))$ such that $u_n(t) = u_n(t)\chi_{G_n \setminus G_n}$, and $\int_{G_n} \int G_n u_n(t)ky(t) dt \to 1$ $(n \to \infty)$. Obviously, for $n$ large enough, $G_b \subset G_n, c = (M(b) - M(a))\mu G_b < 1$. Put

$$x_n(t) = x(t)\chi_{G_n \setminus G_b} + a\chi_{G_b} + cu_n(t),$$
then $R_M(x_n) \leq R_M(x_{\chi G_n \setminus G_b}) + M(a) \mu G_b + cR_M(u_n) = R_M(x_{\chi G_n}) \leq R_M(x) = 1$, hence we have $\|x_n\|(M) \leq 1$, and $\|x_n - x\|(M) \geq (b - a)\|x_{\chi G_b}\|(M) > 0$. On the other hand,

$$
\int_G x_n(t)ky(t) dt = \int_{G_n \setminus G_b} x(t)ky(t) dt + ap(b) \mu G_b + c \int_{G \setminus G_n} u_n(t)ky(t) dt
$$

$$
= \int_{G_n \setminus G_b} (M(x(t)) + N(ky(t))) dt
$$

$$
+ (M(a) + N(p(b))) \mu G_b + (M(b) - M(a)) \mu G_b (1 + o(\frac{1}{n}))
$$

$$
= \int_{G_n} M(x(t)) dt + \int_{G_n} (ky(t)) dt + o(\frac{1}{n}) \rightarrow R_M(x) + R_N(ky)
$$

$$
= 1 + R_N(ky) = k
$$

i.e. $\int_G x_n(t)ky(t) dt \rightarrow 1$. So $x(t)$ is not a strongly exposed point of $B(L(M))$.

Combining (I), (II), we obtain that (iii) is also necessary.

Sufficiency. Still assume $x(t) \geq 0$ for all $t \in G$, only discuss the following two cases.

(I) $\mu\{t \in G : x(t) \in \{b_j\}\} = 0$.

Denote $E_i = \{t \in G : x(t) = a_i\}$, denote the set of all discontinuous points of $p(u)$ as $\{r_n\}$ (either $r_n$ is an extreme point of affine segments or not), denote $e_n = \{t \in G : x(t) = r_n\}$. For every $n$, take $\varepsilon_n > 0$ such that $p_-(r_n) + \varepsilon_n < p(r_n)$, and put $G_0 = G \setminus (\bigcup E_i) \setminus \bigcup e_n$,

$$
W(t) = p_-(x(t))\chi_{G \setminus \bigcup e_n} + \sum_n (p_-(r_n) + \varepsilon_n)\chi_{e_n}, \quad y(t) = \frac{W(t)}{\|W(t)\|_N}.
$$

From

$$
1 \geq \int_G x(t)ky(t) dt = \frac{1}{\|W(t)\|_N} \left( \int_{G \setminus \bigcup e_n} x(t)p_-(x(t)) dt + \sum_n r_n(p_-(r_n) + \varepsilon_n)\mu e_n \right)
$$

$$
= \frac{1}{\|W(t)\|_N} \left( \int_{G \setminus \bigcup e_n} (M(x(t)) + N(p_-(x(t)))) dt 
$$

$$
+ \sum_n (M(r_n) + N(p_-(r_n) + \varepsilon_n))\mu e_n \right)
$$

$$
= \frac{1}{\|W(t)\|_N} \left( \int_G M(x(t)) dt + \int_G N(W(t)) dt \right) \geq \frac{1 + R_N(W(t))}{\|W(t)\|_N} \geq \|y(t)\|_N = 1
$$

we get $\int_G x(t)ky(t) dt = 1$, $k = \|W(t)\|_N \in K_N(y(t))$. For $\{x_n(t)\}_{n=1}^\infty \subset S(L(M))$, $\int_G x_n(t)ky(t) dt \rightarrow 1$, in order to prove $\|x_n - x\|(M) \rightarrow 0$, by $M \in \Delta_2$ (see [2]), we only need to prove $x_n - x \overset{\mu}{\rightarrow} 0$. Noticing

$$
0 \leq 1 + R_N(ky) - \int_G x_n(t)ky(t) dt = \int_G (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t)) dt,
$$
for any $F \subset G$, we have
\[
\int_F (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t)) \, dt \rightarrow 0 \quad (n \rightarrow \infty).
\]

Now we prove $x_n - x \xrightarrow{\mu} 0$ in three steps:

1. $x_n(t) - x(t) \xrightarrow{\mu} 0$ on $G_0$.

Otherwise, there exist $\varepsilon > 0$, $\sigma > 0$, such that $\mu \{ t \in G_0 : |x_n(t) - x(t)| > \varepsilon \} > \sigma$.

Since
\[
1 \geq R_M(x_n) \geq \int_{G(|x| \geq D)} M(x_n(t)) \, dt \geq M(D)\mu G(|x_n(t)| \geq D),
\]
we can take $D$ large enough such that $\mu \{ t \in G : |x_n(t)| \geq D \} < \frac{\sigma}{4}$, $\mu \{ t \in G : |x(t)| \geq \frac{D}{2} \} < \frac{\sigma}{4}$. Since for $t \in G_0$, $x(t) \neq r_n$, $x(t) \neq a_i$, there exist open segments $\delta_n, \delta'_n$ such that $r_n \in \delta_n, a_i \in \delta'_i$, $\mu \{ t \in G_0 : x(t) \in (\bigcup \delta_n) \cup (\bigcup \delta'_i) \} < \frac{\sigma}{4}$. Denote
\[
G_n = \{ t \in G_0 : |x_n(t) - x(t)| \geq \varepsilon, 0 \leq x(t), x_n(t) \leq D, x(t) \notin (\bigcup \delta_n) \cup (\bigcup \delta'_i) \}
\]
then $\mu G_n \geq \frac{\sigma}{4}$. Notice that the function $M(u) + N(v) - uv$ is continuous and positive on the closed bounded set \{ $(u, v) : |u - v| \geq \varepsilon, 0 \leq u, v \leq D, v \in S_M \setminus \bigcup \delta_n \setminus \bigcup \delta'_i$ \}, hence, there exists $\delta > 0$ such that for all $(u, v)$ belonging to this set, we have
\[
M(u) + N(v) - uv \geq \delta.
\]

So, for all $t \in G_n$ we have
\[
M(x_n(t)) + N(ky(t)) - x_n(t)ky(t) = M(x_n(t)) + N(p(x(t))) - x_n(t)p(x(t)) \geq \delta;
\]
we arrive at a contradiction
\[
0 \leftarrow \int_{G_n} (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t)) \, dt \geq \delta \mu G_n \geq \frac{\delta \sigma}{4}.
\]

2. $x_k - x(t) \xrightarrow{\mu} 0$ on $e_n$.

If there exist $\varepsilon > 0$, $\sigma > 0$, such that $e_{nk} = \mu \{ t \in e_n : x_k(t) \geq r_n + \varepsilon \} \geq \sigma$, since $p(x_k(t)) \geq p(r_n) = p_-(r_n) + \varepsilon_n + \tau_n$ whenever $t \in e_{nk}$, we have
\[
M(x_k(t)) + N(p_-(r_n) + \varepsilon_n) - x_k(t)(p_-(r_n) + \varepsilon_n) > \tau_n \varepsilon \quad \text{whenever} \quad t \in e_{nk}.
\]

So we get a contradiction
\[
0 \leftarrow \int_{e_n} (M(x_k(t)) + N(ky(t)) - x_k(t)ky(t)) \, dt
\]
\[
\geq \int_{e_{nk}} (M(x_k(t)) + N(p_-(r_n) + \varepsilon_n) - x_k(t)(p_-(r_n) + \varepsilon_n)) \, dt \geq \tau_n \varepsilon \sigma \quad (k \rightarrow \infty)
\]
hence $\mu\{t \in e_n : x_k(t) \geq r_n + \varepsilon\} \to 0 \ (k \to \infty)$. By the same argument, we can prove $\mu\{t \in e_n : x_k(t) \leq r_n - \varepsilon\} \to 0 \ (k \to \infty)$. So

$$x_k(t) - x(t) \xrightarrow{\mu} 0 \ (on \ e_n).$$

(3) $x_n(t) - x(t) \xrightarrow{\mu} 0 \ (on \ E_i)$.

From the result of (1) and (2), it is easy to know $x_n(t) - x(t) \xrightarrow{\mu} 0$ on $G \setminus \cup_i E_i$. So by Fatou theorem, it follows

$$\lim_{n \to \infty} R_M(x_n(t)\chi_{G \setminus \cup_i E_i}) \geq R_M(x(t)\chi_{G \setminus \cup_i E_i});$$

in view of $R_M(x_n(t)) \leq 1$, we deduce

$$\lim_{n \to \infty} R_M(x_n(t)\chi_{\cup_i E_i}) \leq R_M(x(t)\chi_{\cup_i E_i}). \tag{*}$$

Notice that for all $t \in E_i$, $x(t) = a_i$, and $a_i$ is a left extreme point of affine segments of $M(u)$, analogously to the proof of (2), we can get that for any $\varepsilon > 0$

$$\mu\{t \in E_i : x_n(t) \leq x(t) - \varepsilon\} \to 0 \ (n \to 0).$$

So

$$\lim_{n \to \infty} R_M(x_n(t)\chi_{E_i}) \geq R_M(x(t)\chi_{E_i}) = M(a_i)\mu E_i.$$ 

If there exist $i_0$, $\varepsilon_0 > 0$, $\sigma_0 > 0$, such that $\mu\{t \in E_{i_0} : x_n(t) \geq x(t) + \varepsilon_0\} \geq \sigma_0$, noticing $M(u)$ is increasing monotonously, we deduce

$$\lim_{n \to \infty} R_M(x_n(t)\chi_{E_{i_0}}) > R_M(x(t)\chi_{E_{i_0}}),$$

hence

$$\lim_{n \to \infty} R_M(x_n(t)\chi_{\cup_i E_i}) > R_M(x(t)\chi_{\cup_i E_i})$$

which contradicts (*), i.e. $x_n(t) - x(t) \xrightarrow{\mu} 0$ on $E_i$.

(II) $\mu\{t \in G : x(t) \in \{a_n\}\} = 0$, there exists $\tau > 0$ such that $\int_G N((1 + \tau)p_-(x(t))) dt < \infty$. Denote the set of all discontinuous points of $p(u)$ as $\{r_n\}$, denote $e_n = \{t \in G : x(t) = r_n\}$, $E_j = \{t \in G : x(t) = b_j\}$. Take $\varepsilon_n > 0$ such that $p_-(r_n) + \varepsilon_n < p(r_n)$,

$$\int_{G \setminus \cup_n e_n} N((1 + \tau)p_-(x(t))) dt + \sum_n N((1 + \tau)(p_-(r_n) + \varepsilon_n))\mu e_n < \infty.$$ 

Put $G_0 = G \setminus \cup_j E_j \setminus \cup_n e_n$, $W(t) = p_-(x(t))\chi_{G \setminus \cup_n e_n} + \sum_n (p_-(r_n) + \varepsilon_n)\chi_{e_n},$ $y(t) = \frac{W(t)}{\|W(t)\|_N}$. Then $k = \|W(t)\|_N \in K_N(y(t))$ and $\int_G x(t)y(t) dt = 1$. For $x_n \in B(L(M))$, $\int_G x_n(t)y(t) dt \to 1 \ (n \to \infty)$, it is enough to show $x_n(t) - x(t) \xrightarrow{\mu} 0$ on $G$. 

First we prove

(1) \[ \lim_{\mu \delta \to 0} \{ \sup_n R_N(x_n(t) \chi_\delta) \} = 0. \]

Otherwise, there exist \( \varepsilon > 0, \delta_n \subset G, \mu \delta_n \to 0 \), such that \( R_M(x_n \chi_{\delta_n}) \geq \varepsilon > 0 \), we get a contradiction

\[
0 \leftarrow \int_{\delta_n} (M(x_n(t)) + N(ky(t)) - x_n(t)ky(t)) \, dt \\
\geq \int_{\delta_n} (M(x_n(t)) - \frac{1}{1 + \tau} x_n(t)(1 + \tau)ky(t)) \, dt \\
\geq \int_{\delta_n} (M(x_n(t)) - \frac{1}{1 + \tau} (M(x_n(t)) + N((1 + \tau)ky(t)))) \, dt \\
= \int_{\delta_n} \frac{1}{1 + \tau} M(x_n(t)) \, dt - \int_{\delta_n} \frac{1}{1 + \tau} N((1 + \tau)ky(t)) \, dt \\
\geq \frac{\tau \varepsilon}{1 + \tau} - \frac{1}{1 + \tau} R_N((1 + \tau)ky(t) \chi_{\delta_n}) \to \frac{\tau \varepsilon}{1 + \tau}.
\]

Similarly to the proof of (I), we can get \( x_n(t) - x(t) \xrightarrow{\mu} 0 \) on \( G \setminus \bigcup_j E_j = G_0 \cup (\bigcup_n e_n) \). Using (1) we deduce

\[
\lim_{n \to \infty} R_M(x_n(t) \chi_{G \setminus \bigcup_j E_j}) = R_M(x(t) \chi_{G \setminus \bigcup_j E_j});
\]

moreover, by \( \int_G x_n(t)y(t) \, dt \to 1 \), we know \( \|x_n\|_{(M)} \to 1 \), so \( R_M(x_n) \to 1 = R_M(x) \), thus

(2) \[ \lim_{n \to \infty} R_M(x_n \chi_{\bigcup_j E_j}) = R_M(x \chi_{\bigcup_j E_j}). \]

Noticing \( b_j \) is a right extreme point of affine segments of \( M(u) \), using the same method as above, we can get that for any \( \varepsilon > 0 \), \( \mu \{t \in E_j : x_n(t) - x(t) \geq \varepsilon\} \to 0 \) so

\[ \lim_{n \to \infty} R_M(x_n(t) \chi_E_j) \leq R_M(x(t) \chi_{E_j}). \]

If there exist \( j_0, \varepsilon > 0, \sigma > 0 \), such that \( \mu \{t \in E_{j_0} : x_n(t) \leq x(t) - \varepsilon\} > \sigma \), then

\[ \lim_{n \to \infty} R_M(x_n(t) \chi_{E_{j_0}}) < R_M(x(t) \chi_{E_{j_0}}), \]

combining with (1), we get a contradiction

\[ \lim_{n \to \infty} R_M(x_n(t) \chi_{\bigcup_j E_j}) < R_M(x(t) \chi_{\bigcup_j E_j}). \]

\( \square \)
Corollary 1. \( L(M) \) has strongly exposed property, i.e. all points in \( S(L(M)) \) are strongly exposed points of \( B(L(M)) \) if and only if
(i) \( M(u) \in \Delta_2 \);
(ii) \( M(u) \) is strictly convex.

Theorem 2. \( x \in S(L_M) \) is a strongly exposed point of \( B(L_M) \) if and only if
(i) \( M(u) \in \Delta_2 \) and \( K_M(x) = \{k\} \) is a singleton set;
(ii) \( \mu\{t \in G : kx(t) \in (R \setminus S_M) \cup \{a_i\} \cup \{b_j\} = 0\} \), where \( \{a_i\}, \{b_j\} \) denote the sets of all continuous left extreme points and right extreme points of affine segments of \( M(u) \) respectively;
(iii) there exists \( y(t) \in S(L_N) \) and \( \tau > 0 \), such that \( \int_G x(t)y(t) \, dt = 1 \),
\[
R_N((1 + \tau)y(t)) < \infty;
\]
(iv) \( R_N(p- (kx(t))) = 1 \) implies \( \mu\{t \in G : kx(t) \in \{b_j\}\} = 0 \),
\[
R_N(p(kx(t))) = 1 \text{ implies } \mu\{t \in G : kx(t) \in \{a_i\}\} = 0,
\]
where \( \{a_i\}, \{b_j\} \), denote the sets of all discontinuous left extreme points and right extreme points of affine segments of \( M(u) \) respectively.

Proof: Necessity. Without loss of generality, we assume \( x(t) \geq 0 \), since a strongly exposed point is a strongly extreme point, we get (i) and \( \mu\{t \in G : kx(t) \in R \setminus S_M\} = 0 \). If there exists \( b'_j \) satisfying \( \mu G_{b'_j} = \mu\{t \in G : kx(t) = b'_j\} > 0 \), take \( a < b'_j, E \subset G_{b'_j} \), such that \( p(a) = p(b'_j), 0 < \mu(G_{b'_j} \setminus E) < \mu G_{b'_j} \). Put
\[
x'(t) = x(t) x_{G \setminus E} + \frac{a}{k} x_E,
\]
noticing \( R_N(p(kx'(t))) = R_N(p(kx(t))) \geq 1 \) and for any \( \varepsilon > 0 \),
\[
R_N(p(1 - \varepsilon)kx(t)) \leq R_N(p(1 - \varepsilon)kx(t))) \leq 1, \text{ we have } k \in K_M(x').
\]
Take \( y(t) \in S(L_N) \) with \( \int_G x(t)y(t) \, dt = 1 \), obviously
\[
p_-(kx'(t)) = p_-(kx(t)) \leq y(t) \leq p(kx(t)) = p(kx'(t)) \text{ whenever } t \in G \setminus E
\]
\[
y(t) = p(kx(t)) = p(b'_j) = p(a) = p(kx'(t)) \text{ whenever } t \in E.
\]
So \( y(t) \) is a supporting functional of \( x'(t) \), in view of \( \|x'|_M \neq x \), which contradicts that \( x(t) \) is an exposed point of \( B(L_M) \). So we have
\[
\mu\{t \in G : kx(t) \in \{b_j\}\} = 0.
\]
If there exists \( a_i' \) such that \( \mu G_{a'_i} = \mu\{t \in G : kx(t) = a'_i\} > 0 \), take \( b > a'_i, E \subset G_{a'_i} \) satisfying \( p_-(a'_i) = p(a_i') = p(b), 0 < \mu(G_{a'_i} \setminus E) < \mu G_{a'_i} \). Put
\[
x'(t) = x(t) x_{G \setminus E} + \frac{b}{k} x_E,
\]
for any $\tau > 0$, we have
\[ R_N(p((1 - \tau)kx')) \leq \int_{G\setminus E} N(p((1 - \tau)kx)) \, dt + N(p(a_i))\mu E \]
\[ \leq \lim_{m \to \infty} \left( \int_{G\setminus E} N(p((1 - \tau/m)kx)) \, dt \right) + \lim_{m \to \infty} N(p((1 - \tau/m)a_i))\mu E \]
\[ = \lim_{m \to \infty} \int\limits_{G} N(p((1 - \tau/m)kx)) \, dt \leq 1. \]

Notice that $R_N(p(kx')) = R_N(p(kx)) \geq 1$, we get $k \in K_M(x')$. For any $y \in S(L(N))$ with $\int_{G} x(t)y(t) \, dt = 1$, similarly to the above we can get $\int_{G}(\frac{x'(t)}{\|x'(t)\|_M})y(t) \, dt = 1$. Noticing $\frac{x'}{\|x'\|_M} \neq x$, and the arbitrariness of $y(t)$, we get a contradiction that $x(t)$ is not an exposed point of $B(L_M)$, so $\mu\{t \in G : kx(t) \in \{a_i\} \} = 0$. Thus we have showed that the condition (ii) is necessary.

If (iii) is not necessary, then for any $y(t) \in S(L(N))$ with $\int_{G} x(t)y(t) \, dt = 1$ and $\varepsilon > 0$, $R_N((1 + \varepsilon)y(t)) = \infty$. Hence $\xi_N(y) = 1$ and $\lim_{n \to \infty} \|y(x'G \setminus G_n)\|_N = 1$, where $G_n = \{t \in G : |y(t)| \leq n\}$. By Hahn-Banach theorem, there exist $u_n(t) = \int_{G} x(t)\chi_{G_n} \, dt = 1, \int_{G\setminus G_n} u_n(t)y(t) \, dt \to 1$. Put
\[ x_n(t) = \frac{1}{2}(x(t)\chi_{G_n} + u_n(t)), \]
then
\[ 1 \geq \frac{1}{2}(\|x\chi_{G_n}\|_M + \|u_n\|_M) \geq \|x_n\|_M \geq \int_{G} x_n(t)y(t) \, dt \]
\[ = \frac{1}{2} \left( \int_{G} x(t)y(t) \, dt + \int_{G\setminus G_n} u_n y(t) \, dt \right) \to 1. \]
So $\|x_n\|_M \to 1, \int_{G} x_n(t)y(t) \, dt \to 1$, noticing $\|x - x_n\|_M \geq \frac{1}{2}\|x\chi_{G_n}\|_M \to \frac{1}{2}\|x\|_M = \frac{1}{2}$, we obtain that $y(t)$ is not a strongly exposed functional of $x(t)$.

Now we prove that (iv) is necessary. Otherwise, we only need to consider the following two cases.

(1) $R_N(p_-(kx)) = 1$ and there exist $b_j$ satisfying $p_-(b_j) < p(b_j)$, and $\mu G_{b_j} = \mu\{t \in G : kx(t) = b_j\} > 0$. Take $a < b_j, E \subset G_{b_j}$ such that $p_-(a) = p(a) = p_-(b_j), 0 < \mu(G_{b_j} \setminus E) < \mu G_{b_j}$, put
\[ x'(t) = x(t)\chi_{G\setminus E} + \frac{a}{k}\chi_E. \]
From $R_N(p_-(kx')) = R_N(p_-(kx)) = 1$, we derive that $k \in K_M(x')$, and $y = p_-(kx) = p_-(kx')$ is the unique support functional of $x(t)$. Obviously $\int_{G} \frac{x'(t)}{\|x'\|_M}y(t) \, dt = 1$ and $\frac{x'}{\|x'\|_M} \neq x$, so $x(t)$ is not a strongly exposed point.
(2) \( R_N(p(kx)) = 1 \) and there exist \( a_i \) satisfying \( p_-(a_i) < p(a_i) \), \( \mu G_{a_i} = \mu \{ t \in G : kx(t) = a_i \} > 0 \). Take \( b > a_i \), \( E \subset G_{a_i} \) such that \( p(b) = p_-(b) = p(a_i) \), \( 0 < \mu E < \mu G_{a_i} \). Put \( x'(t) = x(t) \chi_{G \setminus E} + \frac{b}{k} \chi_{E} \). From \( R_N(p(kx')) = R_N(p(kx)) = 1 \), we derive that \( k \in K_M(x') \), and \( y = p(kx') = p(kx) \) is the unique support functional of \( x(t) \). Obviously we have \( \frac{x'}{\|x'\|_M} \neq x \), and \( \int_G \frac{x'(t)}{\|x'\|_M} y(t) \, dt = 1 \), so \( x(t) \) is not a strongly exposed point of \( B(L_M) \).

Sufficiency. First we prove that if \( y \in S(L_M) \) with \( \int_G x(t)y(t) \, dt = 1 \) and for some \( \tau > 0 \), \( R_N((1 + \tau)y) < \infty \), there exist \( \{ x_n \}_{n=1}^{\infty} \subset S(L_M) \) such that \( \int_G x_n(t)y(t) \, dt \to 1 \), then

\[
(3) \quad \lim_{\mu \to 0} \sup_n R_M(k_n x_n \chi_e) = 0,
\]

\[
(4) \quad \lim_{\mu \to 0} \sup_n R_N(p(k_n x_n \chi_e)) = 0,
\]

where \( k_n \in K_M(x_n) \).

Otherwise, there exist \( e_i \), satisfying \( \mu e_i \to 0 \) and \( \varepsilon > 0 \), such that for some \( \{ x_{ni} \}_{i=1}^{\infty} \subset \{ x_n \}_{n=1}^{\infty} \), \( R_M(k_{ni} x_{ni} \chi_{ei}) \geq \varepsilon \), so it follows a contradiction:

\[
0 \leftarrow 1 + R_M(k_{ni} x_{ni}) - k_{ni} \int_G x_{ni}(t)y(t) \, dt
\]

\[
= \int_G (M(k_{ni} x_{ni}(t)) + N(y(t)) - k_{ni} x_{ni} y(t)) \, dt
\]

\[
\geq \int_{e_i} (M(k_{ni} x_{ni}(t)) + N(y(t)) - k_{ni} x_{ni}(t)y(t)) \, dt
\]

\[
\geq \int_{e_i} (M(k_{ni} x_{ni}(t)) - \frac{1}{1 + \tau} k_{ni} x_{ni}(1 + \tau)y(t)) \, dt
\]

\[
\geq \int_{e_i} (M(k_{ni} x_{ni}(t)) - \frac{1}{1 + \tau} (M(k_{ni} x_{ni}(t)) + N((1 + \tau)y(t)))) \, dt
\]

\[
= \frac{\tau}{1 + \tau} R_M(k_{ni} x_{ni} \chi_{ei}) - \frac{1}{1 + \tau} R_N((1 + \tau)y \chi_{ei})
\]

\[
\geq \frac{\tau \varepsilon}{1 + \tau} - \frac{R_N((1 + \tau)y \chi_{ei})}{1 + \tau} \to \frac{\tau \varepsilon}{1 + \tau},
\]

the contradiction shows that (3) is true. Noticing that \( M \in \Delta_2 \), and \( \lim_{\mu \to 0} \sup_n \int_G k_{ni} x_{ni}(t)p(k_{ni} x_{ni}(t)) \, dt = 0 \), we get (4).

In the following, we prove the sufficiency in three cases.

(I) \( R_N(p_-(kx)) = 1 \), \( \mu \{ t \in G : |kx(t)| \in \{ b_j \} \} = 0 \).

In these cases we have that \( y(t) = p_-(kx(t)) \) is the unique support functional of \( x(t) \). For any \( \{ x_n(t) \}_{n=1}^{\infty} \subset S(L_M) \) with \( \int_G x_n(t)y(t) \, dt \to 1 \), take \( k_n \in K_M(x_n) \), and denote \( E_i = \{ t \in G : kx(t) = a_i \} \). Analogously to the proof of the sufficiency of Theorem 1, we can get \( k_n x_n(t) - kx(t) \overset{p_+}{\to} 0 \) on \( G \setminus \bigcup_i E_i \). Since \( p_-(t) \) is not
decreasing and continuous on the left hand, we have
\[
\lim_{n \to \infty} R_N(p-(k_nx_n))_{\chi G \setminus \bigcup_i E_i} \geq R_N(p-(kx(t)))_{\chi G \setminus \bigcup_i E_i},
\]
mwre, by \(R_N(p-(k_nx_n)) \leq R_N(p-(kx)))\, we have
\[
(5) \quad \lim_{n \to \infty} R_N(p-(k_nx_n))_{\chi \bigcup E_i} \leq R_N(p-(kx))_{\chi \bigcup E_i}.
\]
For every \(i\) and any \(\varepsilon > 0\), in view of that \(a_i\) is a left extreme point of affine segment of \(M(u)\), we have
\[
\mu\{t \in E_i : k_nx_n(t) \leq kx(t) - \varepsilon\} \to 0 \quad (n \to \infty),
\]
hence
\[
\lim_{n \to \infty} R_N(p-(k_nx_n))_{\chi E_i} \geq R_N(p-(kx))_{\chi E_i}.
\]
If there exist \(i_0, \varepsilon > 0, \sigma > 0\), such that
\[
\mu\{t \in E_{i_0} : k_nx_n(t) \geq kx(t) + \varepsilon\} \geq \sigma,
\]
then
\[
\lim_{n \to \infty} R_N(p-(k_nx_n))_{\chi E_{i_0}} > R_N(p-(kx))_{\chi E_{i_0}},
\]
and hence
\[
\lim_{n \to \infty} R_N(p-(k_nx_n))_{\chi \bigcup E_i} > R_N(p-(kx))_{\chi \bigcup E_i}.
\]
This is in contradiction with (5). So \(k_nx_n(t) - kx(t) \xrightarrow{\mu} 0\) on \(\bigcup E_i\). Combining (3) we get
\[
k_n = 1 + R_M(k_nx_n) \to 1 + R_M(kx) = k \quad (n \to \infty),
\]
hence \(x_n(t) - x(t) \xrightarrow{\mu} 0\). By \(M(u) \in \Delta_2\), we deduce \(\|x_n - x\|_M \to 0 \quad (n \to \infty)\).

(II) \(R_N(p(kx)) = 1, \mu\{t \in G : kx(t) \in \{a_i\}\} = 0\).
In this case, \(y(t) = p(kx)\) is the unique support functional of \(x(t)\). For any \(\{x_n(t)\}_{n=1}^\infty \subset S(L_M)\) with \(\int_G x_n(t)y(t) dt \to 1\), take \(k_n \in K_M(x_n)\), and denote \(F_j = \{t \in G : kx(t) = b_j\}\). Similarly, we can get \(k_nx_n \xrightarrow{\mu} kx\) on \(G \setminus \bigcup_j F_j\). Since \(p(u)\) is not decreasing and continuous on the right hand, by (4) it follows
\[
\lim_{n \to \infty} R_N(p(k_nx_n))_{\chi G \setminus \bigcup_j F_j} \leq R_N(p(kx))_{\chi G \setminus \bigcup_j F_j}.
\]
Noticing \(R_N(p(k_nx_n)) \geq 1 = R_N(p(kx))\), we have
\[
(6) \quad \lim_{n \to \infty} R_N(p(k_nx_n))_{\chi \bigcup_j F_j} \geq R_N(p(kx))_{\chi \bigcup_j F_j}.
\]
Since $b_j$ is a right extreme point of affine segment of $M(u)$, for any $\varepsilon > 0$ and every $j$, we have
\[ \mu\{t \in F_j : k_n x_n(t) \geq kx(t) + \varepsilon\} = 0, \]
hence
\[ \lim_{n \to \infty} R_N(p(k_n x_n(t)) \chi_{F_j}) \leq R_N(p(kx(t)) \chi_{F_j}). \]
If there exist $j_0$, $\varepsilon > 0$, $\sigma > 0$, such that
\[ \mu\{t \in F_{j_0} : k_n x_n(t) \leq kx(t) - \varepsilon\} \geq \sigma, \]
then
\[ \lim_{n \to \infty} R_N(p(k_n x_n(t)) \chi_{F_{j_0}}) < R_N(p(kx(t)) \chi_{F_{j_0}}), \]
and hence
\[ \lim_{n \to \infty} R_N(p(k_n x_n(t)) \chi_{\bigcup_j F_j}) < R_N(p(kx(t)) \chi_{\bigcup_j F_j}), \]
which contradicts (6). So $k_n x_n - kx \xrightarrow{\mu} 0$. From (4), it follows $k_n \to k$, and hence $x_n - x \xrightarrow{\mu} 0$. Noticing $M \in \Delta_2$, we get $\|x_n - x\|_M \to 0$.

(III) $R_N(p-(kx)) < 1 < R_N(p(kx))$.

By the condition (iii) of this theorem, there exist $y(t) \in S(L(N))$, and $\tau > 0$, such that $\int_{G} x(t) y(t) \, dt = 1$, $R_N((1 + \tau)y) < \infty$. Denote all discontinuous points of $p(u)$ as $\{r_n\}$ (including $\{a_i\}$, $\{b_j\}$), denote $e_n = \{t \in G : kx(t) = r_n\}$. By $R_N(y) = 1$, for all $t \in G$, $p-(kx(t)) \leq y(t) \leq p(kx(t))$, and
\[ y(t) = p-(kx(t)) = p(kx(t)) \quad \text{whenever} \quad t \in G \setminus \bigcup_n e_n, \]
we have
\[ \mu(\bigcup_n \{t \in e_n : y(t) > p-(r_n)\}) > 0; \quad \mu(\bigcup_n \{t \in e_n : y(t) < p(r_n)\}) > 0. \]

Denote $e'_n = \{t \in e_n : y(t) = p-(r_n)\}$. For $r_n$ take $\varepsilon_n > 0$, such that $p-(r_n) + \varepsilon_n < p(r_n)$ and
\[ \int_{G \setminus \bigcup_n e'_n} N((1 + \tau)y) \, dt + \sum_n N(p-(r_n) + \varepsilon_n) \mu e'_n = 1. \]

Constructing a function $z(t)$ satisfying the following conditions
\[ z(t) = y(t) \quad \text{whenever} \quad t \in G \setminus \bigcup_n e_n, \]
\[ z(t) = p-(r_n) + \varepsilon_n \quad \text{whenever} \quad t \in e'_n, \]
\[ p-(r_n) < z(t) \leq y(t) \quad \text{whenever} \quad t \in e_n \setminus e'_n, \]
and such that \( R_N(z) = R_N(y) = 1 \), we can get \( R_N((1 + \tau)z) < \infty \) and

\[
\mu\{t \in e_n : z(t) = p_{-}(r_n)\} = 0.
\]

Similarly to the above, we can construct a function \( u(t) \) satisfying \( R_N(u(t)) = R_N(z) = R_N(y) = 1 \), \( R_N((1 + \tau)u(t)) < \infty \), and

\[
p_{-}(r_n) < u(t) < p(r_n) \quad \text{whenever} \quad t \in e_n.
\]

Obviously, \( u(t) \) is a support functional of \( x(t) \). For \( \{x_n(t)\}_{n=1}^{\infty} \subset S(L_M) \) with \( \int_G x_n(t)u(t) \, dt \to 1 \), take \( k_n \in K_M(x_n) \), then \( k_nx_n(t) \) satisfies (3), (4). Analogously to the proof of the sufficiency of Theorem 1, we can get

\[
k_nx_n(t) \to kx(t) \quad \text{in} \quad L^\mu_M.
\]

By (4), we have \( k_n \to k \), hence we have \( x_n(t) \to x(t) \) \( \quad \text{in} \quad L^\mu_M \). In view of \( M(u) \in \Delta_2 \), we deduce \( \|x_n - x\|_{L^\mu_M} \to 0 \).

**Corollary 2.** \( L_M \) has the strongly exposed property if and only if

(i) \( M \in \Delta_2 \);

(ii) \( M(u) \) is strictly convex;

(iii) there exist \( u_0 > 0 \), \( \tau > 0 \), \( D > 0 \), when \( u \geq u_0 \),

\[
N((1 + \tau)p(u)) \leq DN(p(u)).
\]

**Proof:** Sufficiency. For \( x(t) \in S(L_M) \), by (i) and (ii) of the corollary, it immediately follows that (i), (ii) and (iv) of the Theorem 2 hold. Notice that when \( u > u_0 \), \( N((1 + \tau)p(u)) \leq DN(p(u)) \), it follows \( N((1 + \tau)p_{-}(u)) \leq DN(p_{-}(u)) \). If \( R_N(p_{-}(kx)) = 1 \), then \( x(t) \) has the unique support functional \( y(t) = p_{-}(kx(t)) \), so

\[
\int_G N((1 + \tau)y(t)) \, dt = \int_G N((1 + \tau)p_{-}(kx(t))) \, dt \\
\leq N((1 + \tau)p_{-}(u_0)) \mu G + D \int_G N(p_{-}(kx(t))) \, dt \\
= N((1 + \tau)p_{-}(u_0)) \mu G + D < \infty.
\]

If \( R_N(p(kx)) = 1 \), for the same reason, we have

\[
\int_G N((1 + \tau)y(t)) \, dt = \int_G N((1 + \tau)p(kx(t))) \, dt < \infty.
\]

If \( R_N(p_{-}(kx)) < 1 < R_N(p(kx)) \), take \( G_0 < G \) such that

\[
\int_{G \setminus G_0} N(p_{-}(kx(t))) \, dt + \int_{G_0} N(p(kx(t))) \, dt = 1,
\]
put
\[ y(t) = p_-(kx(t))\chi_{G\setminus G_0} + p(kx(t))\chi_{G_0}, \]
then \( y(t) \) is a support functional of \( x(t) \), and
\[
R_N((1 + \tau)y) = \int_{G\setminus G_0} N((1 + \tau)p_-(kx(t))) \, dt + \int_{G_0} N((1 + \tau)p(kx(t))) \, dt
\leq N((1 + \tau)p(u_0))\mu G + D < \infty.
\]
Combining the above, we get that the (iii) of Theorem 2 is true, so \( x(t) \) is a strongly exposed point of \( B(L_M) \).

Necessity. By (i) and (ii) of Theorem 2, it immediately follows that (i) and (ii) of the corollary hold. If (iii) is not true, then there exist \( u_n \not\to \infty \), such that \( N((1 + \frac{1}{n})p(u_n)) > 2^n N(p(u_n)) \). Take a sequence \( \{G_n\}_{n=1}^\infty \) of subsets of \( G \) with \( G_i \cap G_j = \emptyset \) whenever \( i \neq j \), such that \( N(p(u_n))\mu G_n = \frac{1}{2^n} \). Put
\[
y(t) = \sum_{n=1}^\infty p(u_n)\chi_{G_n}.
\]
For any \( \varepsilon > 0 \), take \( n_0 \) so that \( \frac{1}{n_0} < \varepsilon \), then
\[
R_N((1 + \varepsilon)y) = \sum_{n=1}^\infty N((1 + \varepsilon)p(u_n))\mu G_n \geq \sum_{n \geq n_0}^\infty N((1 + \frac{1}{n})p(u_n))\mu G_n
\geq \sum_{n=n_0}^\infty 2^n N(p(u_n))\mu G_n = \infty.
\]
But
\[
R_N(y) = \sum_{n=1}^\infty N(p(u_n))\mu G_n = 1,
\]
put
\[
x(t) = \frac{\sum_{n=1}^\infty u_n\chi_{G_n}}{\|\sum_{n=1}^\infty u_n\chi_{G_n}\|_M}, \quad \text{then } x(t) \in S(L_M).
\]
By \( R_N(p(\|\sum_{n=1}^\infty u_n\chi_{G_n}\|_M x)) = R_N(\sum_{n=1}^\infty p(u_n)\chi_{G_n}) = R_N(y) = 1 \), we know \( k_x = \|\sum_{n=1}^\infty u_n\chi_{G_n}\|_M \), since \( R_N(p(kx x)) = 1 \), \( x(t) \) has the unique support functional \( y(t) \), but \( y(t) \) does not satisfy (iii) of Theorem 2, so \( x(t) \) is not a strongly exposed point of \( B(L_M) \). \( \square \)

**Remark.** Under \( M \in \Delta_2 \), the condition (iii) is equivalent to \( M \in \nabla_2 \).
References


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