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## On a problem of Gulevich on nonexpansive maps in uniformly convex Banach spaces

SEHIE PARK

*Abstract.* Let  $X$  be a uniformly convex Banach space,  $D \subset X$ ,  $f : D \rightarrow X$  a nonexpansive map, and  $K$  a closed bounded subset such that  $\overline{\text{co}}K \subset D$ . If (1)  $f|_K$  is weakly inward and  $K$  is star-shaped or (2)  $f|_K$  satisfies the Leray-Schauder boundary condition, then  $f$  has a fixed point in  $\overline{\text{co}}K$ . This is closely related to a problem of Gulevich [Gu]. Some of our main results are generalizations of theorems due to Kirk and Ray [KR] and others.

*Keywords:* uniformly convex, Banach space, Hilbert space, contraction, nonexpansive map, weakly inward map, demi-closed, Rothe condition, Leray-Schauder condition, (KR)-bounded, Opial's condition

*Classification:* Primary 47H10; Secondary 54H25

The well-known theorem of Browder-Göhde-Kirk assures existence of a fixed point for nonexpansive maps  $f : K \rightarrow K$  where  $K$  is a *bounded* closed convex subset of a uniformly convex Banach space  $X$ . In [KR], Kirk and Ray showed that  $f$  can be replaced by a weakly inward nonexpansive map  $f : K \rightarrow X$  while the boundedness of  $K$  can be replaced by that of the geometric estimator

$$G(x, fx) = \{z \in K : \|z - x\| \geq \|z - fx\|\}$$

for some  $x \in K$  or more general sets. Note that any fixed point of  $f$  is contained in  $G(x, fx)$ .

On the other hand, Gulevich [Gu] considered the situation as follows:  $H$  is a Hilbert space,  $K$  is a nonempty bounded closed (not necessarily convex) subset of  $H$ , and  $f : D \subset H \rightarrow H$  is a nonexpansive map, where  $\overline{\text{co}}K \subset D$ . Gulevich's basic theorem [Gu, Theorem 1] states that  $f$  has a fixed point in  $\overline{\text{co}}K$  if  $f$  satisfies the Rothe condition  $f(\text{Bd } K) \subset K$ . He also raised as a problem whether  $H$  can be replaced by a uniformly convex Banach space.

In the present paper, we obtain some fixed point theorems on nonexpansive maps defined on closed (not necessarily bounded or convex) subsets of a Banach space. Our results are closely related to Gulevich's theorem and extend some known results of Kirk and Ray [KR], Goebel and Kuczumow [Go], and Browder [B1], [B2]. Moreover, we adopt more general boundary conditions on those nonexpansive maps. In fact, the weakly inwardness or the so-called Leray-Schauder condition is used in our results instead of the Rothe condition used in [Gu].

Recall that  $f : K \rightarrow X$  is a *contraction* if there exists a  $k \in [0, 1)$  such that

$$\|fx - fy\| \leq k\|x - y\| \quad \text{for all } x, y \in K;$$

and a *nonexpansive map* if

$$\|fx - fy\| \leq \|x - y\| \quad \text{for all } x, y \in K.$$

We say that  $f$  is *weakly inward* if  $fx \in \overline{I_K(x)}$  for any  $x \in \text{Bd } K$  (equivalently, for any  $x \in K$ ), where  $\overline{\phantom{x}}$ ,  $\text{Bd}$ , and  $\text{Int}$  denote the closure, boundary, and interior, respectively, and

$$I_K(x) = \{x + c(y - x) : y \in K, c \geq 1\}.$$

Note that any map satisfying the Rothe condition is weakly inward.

We begin with the following:

**Theorem 0.** *Let  $K$  be a closed subset of a Banach space  $X$  and  $f : K \rightarrow X$  a contraction satisfying one of the following:*

- (i)  $f(\text{Bd } K) \subset K$ .
- (ii)  $f$  is weakly inward.
- (iii)  $0 \in \text{Int } K$  and  $fx \neq mx$  for all  $x \in \text{Bd } K$  and  $m > 1$ .

*Then  $f$  has a unique fixed point.*

Note that Theorem 0(i) is a particular case of Assad and Kirk [AK, Theorem 1], Theorem 0(ii) is due to Martinez-Yanez [M, Theorem] or, in a more general form, to Zhang [Z, Theorem 3.3], and Theorem 0(iii) to Gatica and Kirk [GK, Theorem 2.1]. There are more general results than Theorem 0. However, Theorem 0 is sufficient for our purpose. Note also that (iii) can be replaced by the following:

(iii)' there exists a  $w \in \text{Int } K$  such that

$$fx - w \neq m(x - w) \quad \text{for all } x \in \text{Bd } K \quad \text{and } m > 1.$$

Moreover, (i)  $\Rightarrow$  (ii) and, whenever  $K$  is convex and  $0 \in \text{Int } K$ , we have (ii)  $\Rightarrow$  (iii).

A subset  $K$  of a vector space is said to be *star-shaped* if there exists a given point  $x_0 \in K$  such that  $tx_0 + (1 - t)x \in K$  for any  $t \in (0, 1)$  and  $x \in K$ , where  $x_0$  is called a *center* of  $K$ .

For the  $K$  and  $f$  in Theorem 0, we say that  $K$  is (KR)-*bounded* or bounded in the sense of Kirk-Ray [KR] if, for some bounded set  $A \subset K$ , the set

$$G(A) = \bigcap_{u \in A} G(u, fu)$$

is either empty or bounded.

The following is a generalization of the almost fixed point property of bounded closed subsets of a Banach space for nonexpansive maps.

**Theorem 1.** *Let  $X$  be a Banach space,  $K$  a closed subset of  $X$ , and  $f : K \rightarrow X$  a nonexpansive map such that  $K$  is (KR)-bounded and one of the following holds:*

- (i)  $K$  is star-shaped and  $f(\text{Bd } K) \subset K$ .
- (ii)  $K$  is star-shaped and  $f$  is weakly inward.
- (iii)  $0 \in \text{Int } K$  and  $fx \neq mx$  for all  $x \in \text{Bd } K$  and  $m > 1$ .

*Then there exists a bounded sequence  $\{x_n\}$  in  $K$  such that  $\|x_n - fx_n\| \rightarrow 0$  as  $n \rightarrow \infty$ .*

PROOF: For cases (i) and (ii) we may without loss of generality assume that  $0$  is the center. For  $\alpha \in (0, 1)$ , define  $f_\alpha : K \rightarrow X$  by  $f_\alpha x = \alpha fx$  for  $x \in K$ . Then clearly  $f_\alpha$  is a contraction. We show that one of (i)–(iii) in Theorem 0 holds for  $f_\alpha$ :

(i) Since  $K$  is star-shaped at center  $0$ , we have  $\alpha K \subset K$ . Since  $f(\text{Bd } K) \subset K$ , for  $x \in \text{Bd } K$ , we have  $f_\alpha x = \alpha fx \in \alpha K \subset K$ . Therefore,  $f_\alpha(\text{Bd } K) \subset K$ .

(ii) From  $fx \in \bar{I}_K(x)$ , we have  $f_\alpha x = \alpha fx \in \alpha \bar{I}_K(x) \subset \bar{I}_K(x)$  since  $I_K(x)$  is a star-shaped set with center  $0$ . See Zhang [Z, Theorem 1.2]. Note that (i)  $\Rightarrow$  (ii).

(iii) Suppose that  $f_\alpha x = mx$  for some  $x \in \text{Bd } K$  and  $m > 1$ , then  $fx = \alpha^{-1} f_\alpha x = (\alpha^{-1} m)x$  and  $\alpha^{-1} m > 1$ , which contradicts our assumption.

Therefore, by Theorem 0,  $f_\alpha$  has a fixed point  $x_\alpha \in K$ . Suppose that the set  $\{x_\alpha : \alpha \in (0, 1)\}$  is not bounded. Then it is possible to choose  $\alpha \in (0, 1)$  so that

$$\sup_{u \in A} \|fu\| \leq \inf_{u \in A} \|x_\alpha - u\|$$

and in addition, if  $G(A) \neq \emptyset$ , then  $\alpha$  may also be chosen so that

$$\|x_\alpha\| > \sup\{\|x\| : x \in G(A)\}.$$

Therefore, for each  $u \in A$ ,

$$\begin{aligned} \|x_\alpha - fu\| &= \|\alpha fx_\alpha - fu\| \leq \alpha \|fx_\alpha - fu\| + (1 - \alpha)\|fu\| \\ &\leq \alpha \|x_\alpha - u\| + (1 - \alpha)\|x_\alpha - u\| = \|x_\alpha - u\|. \end{aligned}$$

This implies  $x_\alpha \in G(A)$ , which is a contradiction. Thus  $M = \sup\{\|x_\alpha\| : \alpha \in (0, 1)\} < \infty$  and we have

$$\|x_\alpha - fx_\alpha\| = (\alpha^{-1} - 1)\|x_\alpha\| \leq (\alpha^{-1} - 1)M,$$

yielding  $\|x_\alpha - fx_\alpha\| \rightarrow 0$  as  $\alpha \rightarrow 1$ . This completes our proof. □

Note that Kirk and Ray [KR, Theorem 2.3] obtained Theorem 1(ii) for the case  $K$  is convex. In the second half of the proof of Theorem 1, we followed that of [KR, Theorem 2.3]. Note that Theorem 1(i) generalizes Dotson [D, Theorem 1].

A Banach space  $X$  is said to satisfy *Opial's condition* if, whenever a sequence  $\{x_n\}$  converges weakly to  $x_0 \in X$ , then

$$\liminf_{n \rightarrow \infty} \|x_n - x\| > \liminf_{n \rightarrow \infty} \|x_n - x_0\|$$

for all  $x \in X, x \neq x_0$ . Opial [O] showed that if  $C$  is a weakly compact subset of a Banach space  $X$  satisfying this condition and  $f : C \rightarrow X$  is nonexpansive, then  $I - f$  is demi-closed ([B2], [Gö]); that is, if  $\{x_n\} \subset C$  satisfies  $x_n \rightarrow x$  weakly while  $(I - f)x_n \rightarrow y$  strongly, then  $(I - f)x = y$ , where  $I$  is the identity map on  $C$ .

Examples of spaces satisfying Opial's condition are Hilbert spaces,  $l^p$  ( $1 \leq p < \infty$ ), and uniformly convex Banach spaces with weakly continuous duality maps.

From Theorem 1, we have the following:

**Theorem 2.** *Let  $X$  be a Banach space,  $K$  a weakly compact subset of  $X$ , and  $f : K \rightarrow X$  a nonexpansive map satisfying one of (i)–(iii) in Theorem 1.*

- (a) *If  $I - f$  is demi-closed on  $K$ , then  $f$  has a fixed point.*
- (b) *If  $X$  satisfies Opial's condition, then  $f$  has a fixed point.*

PROOF: Since  $K$  is closed and bounded,  $f$  satisfies all the requirements of Theorem 1. Hence, there exists a sequence  $\{x_n\}$  in  $K$  such that  $\|x_n - fx_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $K$  is weakly compact, we may assume that  $x_n \rightarrow x$  weakly to some  $x \in K$ . Since  $x_n - fx_n \rightarrow 0$  strongly,  $x_n - fx_n = (I - f)x_n$ , and  $I - f$  is demi-closed, we conclude that  $(I - f)x = 0$  and hence  $x = fx$ . This completes our proof.  $\square$

Note that Zhang [Z, Theorem 3.8 and Corollaries 3.10, 3.11] obtained the multi-valued version of Theorem 2(ii), with different proof, and that Theorem 2(i) generalizes Dotson [D, Theorem 2]. Note also that if  $K$  is compact, then  $f$  has a fixed point in Theorem 2 without assuming the demi-closedness of  $I - f$ .

From Theorem 1, we also have the following:

**Theorem 3.** *Let  $X$  be a uniformly convex Banach space,  $D$  a subset of  $X$ , and  $f : D \rightarrow X$  a nonexpansive map. Let  $K$  be a closed  $(KR)$ -bounded subset of  $X$  such that  $\overline{\text{co}} K \subset D$  and one of (i)–(iii) of Theorem 1 holds for  $f|_K$ . Then  $f$  has a fixed point in  $\overline{\text{co}} K$ .*

PROOF: Since  $f|_K$  satisfies all the requirements of Theorem 1, there exists a bounded sequence  $\{x_n\}$  in  $K$  such that  $\|x_n - fx_n\| \rightarrow 0$ . Since  $\{x_n\}$  is contained in a bounded closed convex subset  $L \subset \overline{\text{co}} K$  and  $L$  is weakly compact, we may assume  $x_n \rightarrow x_0$  weakly to some  $x_0 \in L$ . Since  $I - f$  is demi-closed on  $L$  ([B2], [Gö]) and  $(I - f)x_n \rightarrow 0$  strongly, we conclude that  $(I - f)x = 0$ , and hence  $x = fx$ . This completes our proof.  $\square$

If we can eliminate the star-shapedness in (i), then Theorem 3(i) will be the required affirmative answer to Gulevich's problem. Moreover, Gulevich [Gu] noted that, for case (i) of Theorem 3 in a Hilbert space  $H$ ,  $f$  has a fixed point in  $K$ .

In case  $0 \in \text{Int } K$ , Theorem 3(iii) generalizes [Gu, Theorem 1].

Note that the set  $A \subset K$  for the (KR)-boundedness can be chosen so that  $A \subset D$  and  $f(A) \subset K$ . See the proof of Theorem 1. Therefore, Theorem 3 generalizes Ray [R, Lemma 1].

For  $D = K = \overline{\text{co}} K$ , Theorem 3 reduces to the following:

**Theorem 4.** *Let  $X$  be a uniformly convex Banach space,  $K$  a closed convex subset, and  $f : K \rightarrow X$  a nonexpansive map such that  $K$  is (KR)-bounded and one of (i)–(iii) in Theorem 0 holds. Then  $f$  has a fixed point.*

Note that Theorem 4(i) and (ii) are due to Kirk and Ray [KR, Theorem 2.3], which extends Goebel and Kuczumow [Go, Theorem 6]. Also note that Theorem 4(iii) extends Browder [B2, Theorem 1] for nonexpansive maps. For a Hilbert space  $X$  and a closed ball in  $X$ , Theorem 4(i) is due to Browder [B1, Theorem 2], which was used to show existence of periodic solutions for nonlinear equations of evolution.

Recently Canetti, Marino, and Pietramala [CMP] obtained multi-valued versions of Theorem 4(ii) and, under the stronger assumption of convexity, some other results similar to Theorems 1–3 for case (ii).

Finally, we note that the so-called Rothe condition (i) was first adopted by Knaster, Kuratowski, and Mazurkiewicz [KKM]. Also, the origin of the so-called Leray-Schauder condition (iii) seems to be Schaefer [S], and the following are well-known examples of that condition:

$$(A) \quad \|fx - x\|^2 \geq \|fx\|^2 - \|x\|^2 \text{ for } x \in \text{Bd } K.$$

$$(K) \quad \text{Re} \langle fx, x \rangle = \|x\|^2 \text{ for } x \in \text{Bd } K, x \neq 0, \text{ in a Hilbert space } H.$$

Or more generally,

$$(P) \quad (fx, Jx) \leq (x, Jx) \text{ for } x \in \text{Bd } K, 0 \in \text{Int } K, \text{ where } J \text{ is any duality map of } X \text{ into } 2^{X^*}.$$

Condition (A) is due to Altman [A], (K) to Krasnosel'skii [K] and Shinbrot [Sh], and (P) to Petryshyn [P].

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