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Karel Beneš

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MODELLING TRANSFER FUNCTIONS WITH NONZERO INITIAL CONDITIONS

KAREL BENEŠ

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A transfer function is defined as a proportion of Laplace transforms of input and output quantities under zero initial conditions. In practice we often meet with problems where the system under consideration is described by a transfer function, but besides there are placed some initial conditions and, the influence of these initial conditions on the behaviour of the system is investigated.

Introduction:

The transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{Y(s)}{Z(s)} \tag{1}$$

can be brought into a transform of the differential equation

$$\begin{aligned} &a_n s^n Y(s) + a_{n-2} s^{n-1} Y(s) + \dots + a_1 s Y(s) + a_0 Y(s) = \\ &= b_m s^m Z(s) + b_{m-1} s^{m-1} Z(s) + \dots + b_2 s Z(s) + b_0 Z(s) \end{aligned} \tag{2}$$

where $Y(s)$ and $Z(s)$ stand for the transforms of the output and input functions, respectively.

The expression in (2) is a Laplace transform of the differential equation

$$\begin{aligned} a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = b_m z^{(m)} + b_{m-1} z^{(m-1)} + \\ + \dots + b_1 z' + b_0 z, \end{aligned} \tag{3}$$

(a_k; b_k = constants), where the modelling of the transform function (1) can be carried over to the solution of (3). In solutions of the differential equations of the type (3) there are most frequently used:

a) the metod of successive integration

b) a decomposition of the equation into a system of nonhomogeneouns differential equations of the 1st order

c) the metod of reducing the order of the derivative under the introduction of a new variable.

Our task is to determine some equivalent initial conditions on the basis of the given initial conditions $y(0), y'(0), \ldots, y^{(n-1)}(0), z(0), z'(0), \ldots, z(0^{(m-1)})$.

1. The metod of successive integration

We start from equation (3) modified ($n = m$) to the form

$$y^{(n)} = b_n z^{(n)} + b_{n-1} z^{(n-1)} - a_{n-1} y^{(n-1)} + \ldots + b_1 z' - a_1 y' + b_0 z - a_0 y. \quad (4)$$

Integrating the equation term by term we get

$$y^{(n-1)} = b_n z^{(n-1)} + b_{n-1} z^{(n-2)} - a_{n-1} y^{(n-2)} + \ldots + b_1 z - a_1 y + y_1, \quad (5)$$

where

$$y_1 = \int (a_0 y - b_0 z)\, dt = \int_0^t (a_0 y - b_0 t)\, dt + y_1(0), \quad (6)$$

Integrating the equation (5) again and putting

$$y_2 = \int (a_1 y - b_1 z + y_1)\, dt = \int_0^t (a_1 y - b_1 z + y_1)\, dt + y_2(0) \quad (7)$$

gives

$$y^{(n-2)} = b_n z^{(n-2)} + b_{n-1} z^{(n-3)} - a_{n-1} y^{(n-3)} + \ldots + b_2 z - a_2 y - y_2. \quad (8)$$

Continuing in this way, we finally remove all derivatives y till we come to the expression

$$y = b_n z - y_n, \quad (9)$$

where

$$y_n = \int_0^t (a_{n-1} y - b_{n-1} z + y_{n-1})\, dt + y_n(0). \quad (10)$$

Then we construct the expressions (10), (9), ..., (7), (6) in the computing network. This procedure can equaly well be applied even in case of $m \leqq n$ and the corresponding coefficients $b_{m+k} = 0$.

Equivalent initial conditions on integrals can be determined from the equations (5), (8), ..., (9).

In figure 1 we show a programming chart for solving the equation

$$y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = b_4 z^{(4)} + b_3 z''' + b_2 z'' + b_1 z' + b_0 z \quad (11)$$

with initial conditions $y(0)$, $y'(0)$, $y''(0)$, $y'''(0)$.

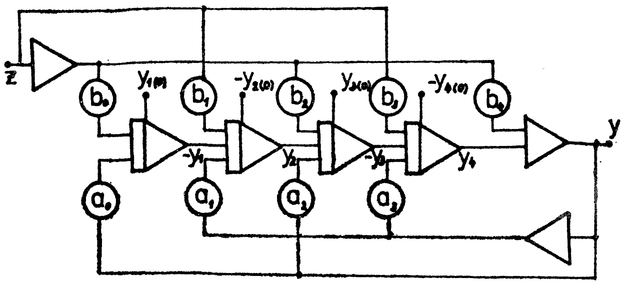

Fig. 1

Equivalent initial conditions are given on the basis of (5), (8), ..., (9) by the relations

$$\begin{aligned} y_1(0) &= -y'''(0) + b_4 z'''(0) + b_3 z''(0) - a_3 y''(0) + b_2 z'(0) - \\ &\quad - a_2 y'(0) + b_1 z(0) - a_1 y(0), \\ y_2(0) &= -y''(0) + b_4 z''(0) + b_3 z'(0) - a_3 y'(0) + b_2 z(0) - a_2 y(0), \\ y_3(0) &= -y'(0) + b_4 z'(0) + b_3 z(0) - a_3 y(0), \\ y_4(0) &= -y(0) + b_4 z(0), \end{aligned}$$

generally

$$y_k(0) = \sum_{j=0}^{n-k} b_{n-j} z(0)^{(n-k-j)} - a_{n-j} y(0)^{(n-k-j)}. \quad (12)$$

It becomes apparent from (5), (8), ..., (9) that equivalent initial conditions are always determinable and thus the problem is solvable.

2. A decomposition of the equation into a system of nonhomogeneons differential equations of the 1-st order

In [1] equation is solved in the form of a system of equations

$$y_k = y'_{k-1} - \alpha_{n-k+1}, \qquad 2 \leqq k \leqq n, \quad (13)$$

$$y_1 = y - \alpha_n z, \quad (14)$$

where the coefficients α_i are dependent on the coefficients a_i, b_i of the transfer function and they are computed from a system of linear algebraic equations of such a form giving one and only one solution.

The initial conditions $y_k(0)$ are uniquely given by the relation

$$y_k(0) = y(0)^{(k-1)} - \sum_{j=0}^{k-1} \alpha_{n-j} z(0)^{(k-j-1)}. \tag{15}$$

For the programming chart showing this modelling see figure 2.

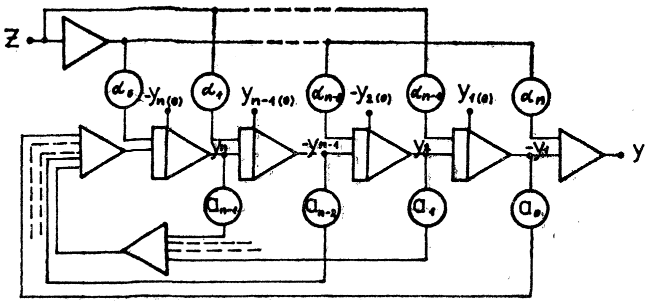

Fig. 2

3. The metod of reducing the order of the derivative under the introduction of a new variable

Equation (3) is programmed in the form of a system of equations

$$\sum_{j=0}^{n} a_j u^{(j)} = z, \tag{16}$$

$$\sum_{r=0}^{m} b_k u^{(k)} = y.$$

The programming chart for solving the system (16), where $n = m$, is given in figure 3. The initial conditions $u(0), u'(0), \ldots, u^{(n-1)}(0)$ will be determined from the initial values $y(0), y'(0), \ldots, y^{(n-1)}(0)$, $z(0), z'(0), \ldots, z^{(m-1)}(0)$ on the basis of (16) from a system of

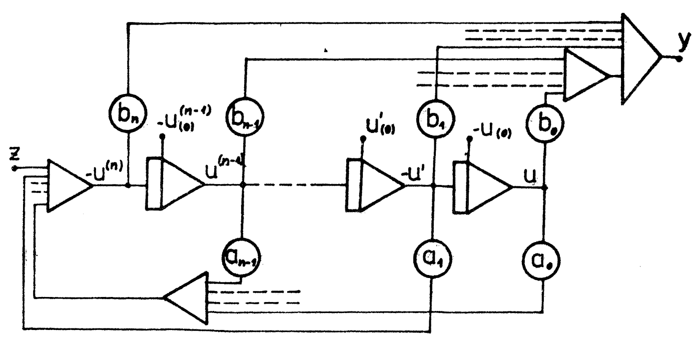

Fig. 3

linear algebraic equations:

$$
\begin{aligned}
y(0) &= \sum_{r=0}^{m} b_k u(0)^{(k)}, \\
y'(0) &= \sum_{r=0}^{m} b_k u(0)^{(k+1)}, \\
&\vdots \\
y(0)^{(n-1)} &= \sum_{r=0}^{m} b_k u(0)^{(k+n-1)}, \\
z(0) &= \sum_{j=0}^{n} a_j u(0)^{(j)}, \\
z'(0) &= \sum_{j=0}^{n} a_j u(0)^{(j+1)}, \\
&\vdots \\
z(0)^{(m-1)} &= \sum_{j=0}^{n} a_j u(0)^{(j+m-1)}.
\end{aligned}
\tag{17}
$$

For computing the initial conditions $u(0), u'(0), \ldots, u(0^{(n-1)})$, the system (17) will be modified to a system of n-linear equations for the unknowns $u(0), u'(0), \ldots, u(0^{(n-1)})$. So, for instance, for $n = 3$ the system has the following form:

$$
\begin{aligned}
&u''(0)\,(b_2 - b_3a_2) + u'(0)\,(b_1 - b_3a_1) + u(0)\,(b_0 - b_3a_0) = y(0) - b_3z(0) \\
&u''(0)\,(b_3a_2^2 - b_3a_1 - b_2a_2 + b_1) + u'(0)\,(b_3a_2a_1 - b_3a_0 - b_2a_1 + b_0) + \\
&\quad + u(0)\,(b_3a_2a_0 - b_2a_0) = y'(0) - b_3z'(0) + (b_3a_2 + b_2)\,z(0), \\
&u''(0)\left[b_3(-a_2^3 + 2a_2a_1 - a_0) + b_2(a_2^2 - a_1) - b_1a_2 + b_0\right] + \\
&\quad + u'(0)\left[b_3(-a_2^2a_1 + a_2a_0 + a_1^2) + b_2(a_2a_1 - a_0) - b_1a_1\right] + \\
&\quad + u(0)\left[b_3(-a_2^2a_0 + a_1a_0) + b_2a_2a_0 - b_1a_0\right] = \\
&\quad = -b_3(z''(0) - a_2z'(0) + a_2^2z(0) - a_1z(0)) - b_2(z'(0) + a_2z(0)) - b_1z(0).
\end{aligned}
\tag{18}
$$

In modelling tansfer functions with nonzero initial conditions there may arise difficulties caused by the unsolvability of the system of n-linear algebraic equations for the initial values $u(0), u'(0), \ldots, u(0)$. If $h = h' = n$ (h and h' stand for the ranks of the matrix and of the expanded matrix respectively) then the system has one and only one solution. If $h = h' < n$, then the system has an infinite number of solutions. If $h \neq h'$, the system has no solution, the initial values of the new function and their derivatives are undeterminable, and the modelling by reducing the order of the derivative under the introduction of a new variable becomes impossible.

A very interesting case occurs in $h = h' < n$, where with infinitely many n-tuples of initial conditions $u(0)$, $k = 0, 1, \ldots, n - 1$ the system of equations (16) has still the same solution y. For instance, in solving equation (3) having the form

$$y''' + y'' - 5y' - 5y = z'' - z, \tag{19}$$

with initial conditions $y(0) = 1$, $y'(0) = 5$, $y''(0) = 10$, $z = \sin(t + 20°30')$, i.e. $z(0) = 3$, $z'(0) = 8$ the above equation is carried over to a system of equations

$$u''' + u'' - 5u' - 5u = z, \tag{19a}$$
$$u'' - u = y.$$

The system of equations for determining the initial conditions $u^{(j)}(0)$ has the form

$$\begin{aligned} u''(0) - u(0) &= 1, \\ -u''(0) + 4u'(0) + 5u(0) &= 2, \\ 5u''(0) - 5u(0) &= 5. \end{aligned} \tag{20}$$

The matrix $\boldsymbol{A}$ of the system and the matrix $\boldsymbol{A}'$ of the expanded system have the form

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 4 & 5 \\ 5 & 0 & -5 \end{pmatrix}, \qquad \boldsymbol{A}' = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 4 & 5 & 2 \\ 5 & 0 & -5 & 5 \end{pmatrix}.$$
$$h = 2 \qquad\qquad h' = 2$$

On account of the fact that $h = h' < n$, the system has an infinite number of solutions. If we choose the value of an unknown, say

$$u(0) = \varphi, \qquad \text{then} \qquad u''(0) = 1 + \varphi, \qquad u'(0) = \frac{3 - 4\varphi}{4}. \tag{21}$$

Choosing now a concrete value for φ, say $\varphi = 0$, we obtain

$$u(0) = 0, \qquad u'(0) = 0{,}75, \qquad u''(0) = 1,$$

if we choose $\varphi = 1$ we get

$$u(0) = 1, \qquad u'(0) = -0{,}25, \qquad u''(0) = 2.$$

The programming chart for solving equations from (19) in case of $\varphi = 0$ is shown in figure 4.

The solution of (19a) with initial conditions (21) is the function

$$u = \left(\varphi - \frac{7}{8}\right)e^{-t} + \frac{11\sqrt{5} + 19}{48\sqrt{5}}\,e^{t\sqrt{5}} + \frac{11\sqrt{5} - 19}{48\sqrt{5}}\,e^{-t\sqrt{5}} + $$
$$+ \frac{8{,}57}{12}[\cos(t + 20^0 30') - \sin(t + 20^0 30')],$$

and the solution of (19) is the function

$$y = \frac{44\sqrt{5} + 19{,}4}{48\sqrt{5}}\,e^{t\sqrt{5}} + \frac{44\sqrt{5} - 19{,}4}{48\sqrt{5}}\,e^{-t\sqrt{5}} -$$
$$- \frac{8{,}57}{6}[\cos(t + 20^0 30') - \sin(t + 20^0 30')].$$

The solution y is independent of φ which in this way is able to assume arbitrary values.

Conversely, in modelling the transfer function

$$H(s) = \frac{4s^2 + 4s + 2}{2s^3 - s - 1},$$

by means of the solution of the differential equation

$$2y''' - y' - y = 4z'' + 4z' + 2z \tag{22}$$

with the above given initial conditions $y(0) = 1$, $y'(0) = 5$, $y''(0) = 10$, $z = \sin(t + 20°30')$, that is $z(0) = 3$, $z'(0) = 8$, the system (16) has the form

$$\begin{aligned} 2u''' - u' - u &= z, \\ 4u'' + 4u' + 2u &= y. \end{aligned} \tag{23}$$

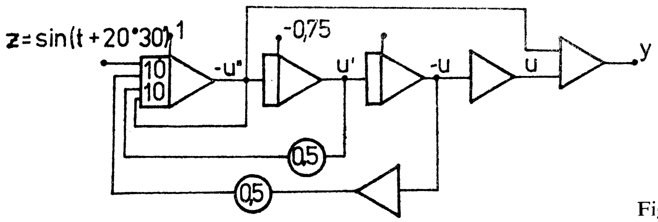

Fig. 4

The system of linear algebraic equations for determining the values $u(0)$, $u'(0)$, $u''(0)$ is of the form

$$\begin{aligned} 4u''(0) + 4u'(0) + 2u(0) &= 1, \\ 4u''(0) + 4u'(0) + 2u(0) &= -7, \\ 4u''(0) + 4u'(0) + 2u(0) &= -34. \end{aligned} \tag{24}$$

The matrice $\mathbf{A}$ of the system and the matrix $\mathbf{A}'$ of the expanded system have the following forms

$$\mathbf{A} = \begin{pmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 4 & 4 & 2 \end{pmatrix}, \quad \mathbf{A}' = \begin{pmatrix} 4 & 4 & 2 & 1 \\ 4 & 4 & 2 & -7 \\ 4 & 4 & 2 & -34 \end{pmatrix}.$$

$$h = 1 \qquad\qquad h' = 3$$

Due to $h \neq h'$, the system (24) has no solution and the equation (22) cannot be solved by the method of reducing the order of the derivative under introducing new variables.

The above discussion leads us to conclude that both the method of successive integration and the method using the decomposition into a system of nonhomogeneons equations of the order are always useable, while the method of reducing the order of the derivative under the introduction of a new variable is generally not always useable.

References

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Souhrn

MODELOVÁNÍ PŘENOSOVÝCH FUNKCÍ S NENULOVÝMI POČÁTEČNÍMI PODMÍNKAMI

KAREL BENEŠ

V práci je popsáno určení ekvivalentních počátečních podmínek při modelování přenosových funkcí s nenulovými počátečními podmínkami. Je ukázáno, že metoda postupné integrace a metoda rozkladu rovnice na soustavu nehomogenních rovnic 1. řádu jsou vždy použitelné. Dále je ukázáno, že metoda snižování řádu derivace se zavedením nové proměnné není obecně vždy použitelná (nedovedeme-li určit počáteční hodnoty v diferenciální rovnici pro novou proměnnou), na druhé straně může existovat n-tice počátečních hodnot nekonečně mnoho, resp. též jen jedna n-tice. V těchto případech lze pro modelování přenosové funkce použít výše uvedené metody.

Резюме

МОДЕЛИРОВАНИЕ ПЕРЕДАТОЧНЫХ ФУНКЦИЙ С НЕНУЛОВЫМИ НАЧАЛЬНЫМИ УСЛОВИЯМИ

КАРЕЛ БЕНЕШ

В статье описано определение начальных условий при моделированию передаточных функций с ненуловыми условиями. Показано, что метод постепенной интеграции и метод разложения уравнения на систему негомогенных уравнений 1. порядка всегда можно применить. Далее показано, что метод понижения порядка производной с применением новой переменной нельзя в общем всегда использовать (когда мы незнаем определить начальные условия в дифференциальном уравнении для новой переменной), и может существовать n-групп начальных условий бесконечне много, или тоже только одна группа. В этих случаях можно для моделирования передаточной функции использовать метод понижения порядка производной с применением новой переменной.