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Vedoucí laboratoře: Antonín Novák, Doc., Ing., CSc.

DYNAMICAL SYSTEMS MODELLED BY THIRD ORDER DIFFERENTIAL EQUATIONS WITH SPECIAL RESPECT TO THE INFLUENCE OF THE RESTORING TERM ON THE PROPERTIES OF SOLUTIONS *)

JAN ANDRES

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Dedicated to Professor M.Laitoch on his 65th birthday

1. The equations considered in this paper are of the form

$$x''' + f(x')x'' + g(x)x' + h(x) = 0, \qquad (L)$$

$$x''' + f(x'') + g(x') + h(x) = 0, \qquad (R)$$

where $h(x)$, $g(y)$, $f(z) \in C^1(-\infty, \infty)$. If we furthermore assume the existence of all their solutions and derivatives $x'(t)$, $x''(t)$ on the interval $(-\infty, \infty)$, then these Liénard (L) and Rayleigh (R) - type differential equations generate dynamical systems in the sense of Barbashin [1], significant for applications.

Since many earlier results [2], obtained for the second order nonautonomous equations (originated from (L), (R) by the substitution $y:=x'$), namely

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$$y'' + f(y)y' + g(t)y = h(t), \qquad (L')$$

$$y'' + f(y') + g(y) = h(t), \qquad (R')$$

may be used to satisfying the relation

$$\limsup_{t\to\infty} (|x'(t)| + |x''(t)|) \leq D' \quad (D'\text{-const.}) \qquad (D')$$

we shall be here mainly concerned with the behaviour of solutions x(t) of (L), (R) under the influence of the restoring force, represented by the term h(x). Hence, in what follows, let us assume that (D') holds.

If the functions g(y), f(z) from (R) are of the form

$$g(y) = by + g_0(y) \quad \text{with } |g_0(y)| \leq G_0 \text{ for all } y, \qquad (G_0)$$

$$f(z) = az + f_0(z) \quad \text{with } |f_0(z)| \leq F_0 \text{ for all } z, \qquad (F_0)$$

where a, b, F_0, G_0 are suitable constants, then the ultimate boundedness of solutions can be easily deduced from the identity, obtained on integrating (R) under the hypothesis

$$h(x)\operatorname{sgn} x \geq F_0 + G_0 \quad \text{for } |x| > R, \qquad (H)$$

while for (L) the condition (H) reduces even to

$$h(x)\operatorname{sgn} x \geq 0 \quad \text{for } |x| > R,$$

where R is a suitable constant.

However, for

$$\liminf_{|x|\to\infty} h(x)\operatorname{sgn} x < 0$$

either some additional restrictions are necessary for the same goal, or there exists such an unbounded solution x(t) of (L) or (R) that

$$\lim_{t\to\infty} |x(t)| = \infty. \qquad (U)$$

2. Let h(x) be such an oscillatory function with isolated

roots $\overline{x}$ that

$$\limsup_{|x|\to\infty} h(x)\operatorname{sgn} x > 0, \qquad \liminf_{|x|\to\infty} h(x)\operatorname{sgn} x < 0$$

throughout this section.

Theorem 1. *If there exist such positive constants* a,R, ε *that the conditions*

$$f(y) \geq a \qquad \textit{for all } y, \tag{1}$$

$$ag(x) - h'(x) \geq \varepsilon \qquad \textit{for } |x| > R, \tag{2}$$

$$g'(\overline{x}) = 0$$

are satisfied, then all solutions of (L) *are ultimately bounded.*

P r o o f. First of all we prove that the roots $\overline{x}=\overline{x}_+$ with

$$h'(x) > 0 \quad \text{for } 0 < |x - \overline{x}_+| < \delta \quad (|x| > R) \tag{4}$$

are stable with the attractivity area determined by $|x| \leq \delta_2$ and $|y| \leq D'$, where $\delta_2 \leq \delta$ are suitable constants. Showing this, the ultimate boundedness of solutions x(t) will be already ensured with respect to (D'), because each suspicious x(t) will be then attracted by some $\overline{x}_+$ with (4) (see bellow).

It is clear that the same is true for a trivial solution instead of $\overline{x}_+$, in considering

$$x''' + f(x')x'' + g^*(x)x' + h^*(x) = 0$$

with $g^*(x) := g(x + \overline{x}_+)$, $h^*(x) := h(x + \overline{x}_+)$. Hence, replacing this equation by

$$x' = y, \quad y' = z, \quad z' = -h^*(x) - g^*(x)y - f(y)z, \tag{L*}$$

we can prove the stability of $(\overline{x}_+,0,0)$ under the following well-known criterium [3], guaranteed by the existence of such a function V(x,y,z) with everywhere continuous first order partial derivatives and such a positive constant δ that in the cylinder $|x - \bar{x}_+| < \delta$, $|y| + |z| < D'$ conditions 1) - 3) are satisfied:

1) $V(x,y,z) > 0$ for $(x,y,z) \neq (\bar{x}_+,0,0)$,

2) $\frac{\partial V}{\partial x} y + \frac{\partial V}{\partial y} z - \frac{\partial V}{\partial z}\left[h^*(x) + g^*(x)y + f(z)z\right] := V' \leq 0$,

3) the set $\{(x,y,z) := V' = 0\}$ does not contain besides the points $(\bar{x}_+,0,0)$ any whole trajectory.

It follows from (4) that also

$$g(x) \geq \varepsilon/a \quad \text{for } 0 < |x - \bar{x}_+| < \delta \qquad (|x| > R) \qquad (5)$$

with respect to (2). Therefore defining

$V(x,y,z) :=$

$$= a\int_0^x h^*(s)ds + h^*(x)y + \frac{1}{2}\left[g^*(x)y^2 + (ay + z)^2\right] + a\int_0^y \left[f(s) - a\right] sds,$$

we can transform $V(x,y,z)$ for $0 < |x| < \delta$ into the form

$$V(x,y,z) := W(x) + \frac{1}{2}\left\{g^*(x)\left[\frac{h^*(x)}{g^*(x)} + y\right]^2 + (ay+z)^2\right\} + a\int_0^y \left[f(s) - a\right] sds,$$

where $W(x) := \int_0^x \frac{h^*(s)}{g^*(s)}\left[\frac{1}{2}\frac{h^*(s)}{g^*(s)} g^{*\prime}(s) + ag^*(s) - h^{*\prime}(s)\right] ds \geq$

$$\geq \int_0^x \frac{h^*(s)}{g^*(s)}\left[\frac{1}{2}\frac{h^*(s)}{g^*(s)} g^{*\prime}(s) + \varepsilon\right] ds.$$

We can find such a positive number $\delta_1 \leq \delta$ that the relation

$$\frac{1}{2}\left[\frac{h^*(x)}{g^*(x)} g^{*\prime}(x)\right] + \varepsilon > 0$$

will be satisfied with respect to

$$\lim_{x \to 0}\left[\frac{h^*(x)}{g^*(x)} g^{*\prime}(x)\right] = 0,$$

implied by (3), (5). Consequently the first condition of 1) -

3) is satisfied according to the above and (1), (4), (5).

Since we have furthermore

$$V' = -\left[ag^*(x)-h^{*\prime}(x)-\frac{1}{2}g^{*\prime}(x)y\right]y^2 -\left[f(y)-a\right]z^2 \leq -\frac{1}{2}\varepsilon y^2$$

with respect to (L^*) for $|g^{*\prime}(x)y| \leq \varepsilon$ together with (1), (2), the remainder of the proof is completed, when $|x| \leq \delta_2 \leq \delta$, $|y| \leq D'$ for a suitable constant δ_2.

Theorem 2. *If there exist such positive constants* a, F, H', R *that the conditions*

$$F \geq f(z)/z \geq a \quad \textit{for all } z \textit{ and } f(0)=0, \tag{6}$$

$$h'(x) \leq H' \quad \textit{for } |x| > R, \tag{7}$$

$$g(y)/y \geq H' + (F-a)a/4 \quad \textit{for all } y \textit{ and } g(0)=0 \tag{8}$$

are satisfied, then all solutions of (R) *are ultimately bounded.*

P r o o f. The point of the proof is the same; consequently we restrict ourselves only on showing the fact that the roots $\bar{x}=\bar{x}_+$ with (4) are stable.

Hence, replacing (R) by

$$x' = y, \quad y' = z, \quad z' = -h^*(y) - g(y) - f(z)\,, \tag{R^*}$$

where $h^*(x) := h(x + \bar{x}_+)$ again, we can examine the stability of the trivial solution by means of

$$V(x,y,z) := a\int_0^x h^*(s)ds + h^*(x)y + \int_0^y g(s)ds + \frac{1}{2}(ay + z)^2\,.$$

It can be verified just in the same way as in [3, p. 120] that $V(x,y,z)$ is positively defined under our assumptions. Therefore, since we have furthermore

$$V' = -\left[\frac{f(z)}{z} - a\right](z+\frac{a}{2}y)^2 - \left\{-h^{*\prime}(x) + \left[a - \frac{f(z)}{z}\right]\left(\frac{a}{2}\right)^2 + \frac{ag(y)}{y}\right\}y^2 \leq 0$$

with respect to (6) - (8), the proof is done.

3. Now the existence of unbounded solutions with (U) will be studied.

Theorem 3. *If there exist such positive constants g,G,h,H,R that the conditions*

$$g \leq |g(x)| \leq G, \tag{9}$$

$$-H \leq h(x)\operatorname{sgn} x \leq -h \tag{10}$$

are satisfied for $|x| > R$, *then the equation (L) admits an unbounded solution.*

P r o o f. We proceed by the technique developed in [4], consisting of a construction of the Liapunov function V(x,y,z) with everywhere continuous first order partial derivatives, such that the conditions i) - iii) are satisfied:

i) V(x,y,z) is bounded in the cylinder $|x| + |y| + |z| \leq D$,

ii) $\lim\limits_{|x|\to\infty} V(x,y,z) = \infty$ on the set $|y| + |z| \leq D'$,

iii) $V' > \delta$ for $|y| + |z| \leq D'$ and $|x| > R$,

where δ,R are suitable positive constants.

Then the equation studied admits an unbounded solution x(t) with (4) as we wish.

Therefore, denoting

$$F := \max_{|y| \leq D'} \left| \int_0^y f(s)ds \right|, \qquad F_1 := \max_{|y| \leq D'} \left| \int_0^y f(s)ds - y \right| \tag{11}$$

and defining

$$2V(x,y,z) := 2\int_0^x h(s)ds + \left[\int_0^x g(s)ds + \int_0^y f(s)ds + z\right]^2 ,$$

we have

$$\lim_{|x|\to\infty} V(x,y,z) \geq \lim_{|x|\to\infty} \left\{\frac{1}{2}\left[\int_0^x g(s)ds\right]^2 - \int_0^{|x|} |h(s)|\, ds - \right.$$

$$- \int_0^{|x|} |g(s)|\,ds \Big[\int_0^{|y|} |f(s)|\,ds + |z| \Big] - \Big[\int_0^{|y|} |f(s)|\,ds + |z| \Big]^2 \Big\} \geq$$

$$\geq \lim_{|x| \to \infty} \Big\{ \frac{1}{2} (gx)^2 - |x| \big[H + GD'(F+1) \big] \Big\} = \infty$$

for $|y| + |z| \leq D'$ under (9), (10), (11). Since there is furthermore

$$V' = -h(x) \Big\{ z + \int_0^x g(s)ds + \Big[\int_0^y f(s)ds - y \Big] \Big\} \geq \Big| h(x) \int_0^x g(s)ds \Big| -$$

$$- H(D' + F_1) \geq hgR - H(D' + F_1) := \delta > 0$$

with respect to the system equivalent to (L), when $|y| + |z| \leq D'$

and $|x| > R > H(D' + F_1)/hg$, all conditions i) - iii) are satisfied.

Theorem 4. *If there exist such positive constants F_0, G_0, H that the conditions (F_0), (G_0) and*

$$-H < \limsup_{|x| \to \infty} h(x) \operatorname{sgn} x < -F_0 - G_0 \tag{12}$$

are satisfied, then the equation (R) admits an unbounded solution.

P r o o f. Defining

$$2V(x,y,z) := (bx + ay + z)^2,$$

we have (cf. (F_0), (G_0))

$$V' = (z + ay + bx)\big[(az - f(z)) + (by - g(y)) - h(x)\big] \geq |bx| \big[|h(x)| -$$

$$- (F_0 + G_0) \big] - D'(a+1)(F_0 + G_0 + H) \geq |\varepsilon bR| - D'(a+1)(F_0 + G_0 + H) := \delta > 0$$

with respect to the system equivalent to (R), when $|x| > R > D'(a+1)(F_0+G_0+H)/\varepsilon b$, $y + |z| \leq D'$ (ε is a suitable constant, implied by (12)). Thus we can give the same conclusion as in the proof of Theorem 1.

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SOUHRN

Dynamické systémy modelované diferenciálními rovnicemi třetího řádu se zvláštním zřetelem k vlivu členu h(x) na vlastnosti řešení

J a n A n d r e s

Pro rovnice třetího řádu Liénardova a Rayleighova typu ze zobecněné Levinsonovy třídy D´ (generující dynamické systémy ve smyslu Barbašina) je zkoumána otázka vlivu členu h(x) na ohraničenost jejich řešení. Ukazuje se, že zatímco v případě oscilatorické funkce h(x) lze této vlastnosti docílit podobně jako pro (H), existuje za podmínek (10) resp. (12) vždy alespoň jedno neohraničené řešení (U) příslušné rovnice.

РЕЗЮМЕ

Динамические системы моделированные дифференциальными уравнениями третьего порядка с особым отношением к влиянию члена $h(x)$ на свойства решений

Я н   А н д р е с

Для уравнений третьего порядка типа Лиенара и Рэйлея из обобщенного класса D' Левинсона /порождающих динамические системы в смысле Барбашина/ рассматривается вопрос влияния члена $h(x)$ на ограниченность их решений. Между тем как оказывается это свойство в случае осциллирующей функции $h(x)$ подобно как для (H), существует из-за (10) или (12) всегда почти одно неограниченное решение (U) соответствующих уравнений.

Author's address:
RNDr. Jan Andres, CSc.
Laboratoř optiky ČSAV a UP
Gottwaldova 15
771 46 Olomouc ČSSR