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ON A NONLINEAR PROBLEM FOR THIRD-ORDER DIFFERENTIAL EQUATIONS

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Introduction

The questions of existence and uniqueness of solutions of the boundary problem

$$\begin{aligned} &u'' = f(t,u) \\ &u(0) = c_0,\ u(t) \geqq 0,\ u'(t) \leqq 0 \text{ for } t \geqq 0 \end{aligned} \tag{0.1}$$

have a long history, going back to Kneser (1896). He was the first to establish the existence and uniqueness of solutions of (0.1) with a continuous and nonnegative function f. (See [7]).

In 1927 the boundary problem

$$\begin{aligned} &u'' = t^{-1/2}\, u^{3/2} \\ &u(0) = 1, \quad \lim_{t\to\infty} u(t) = 0 \end{aligned} \tag{0.2}$$

which is a special case of (0.1) appeared in [1] and [16]. It has been found that (0.2) has applications in the study of the distribution of electrons in the heavy atom.

Then such types of problems were considered in [8, 9, 10, 11, 13 - 15]. It is also necessary to recall the important later works [2] and [3].

For $n > 2$ the problem

$$u^{(n)} = f(t,u,u',\dots,u^{(n-1)}) \tag{0.3}$$

$$u(0) = u_0,\ (-1)^i\, u^{(i)}(t) \geqq 0 \text{ for } t \geqq 0,\quad i = 0,1,\dots,n-1 \tag{0.4}$$

was solved for the first time by Kiguradze in [4]. The existence of solutions of (0.3), (0.4) was proved for the case when f does not change its sign. New sufficient conditions for the existence of solutions of (0.3) and

$$\varphi(u(0),\dots,u^{(n-1)}(0)) = 0,\ (-1)^i\, u^{(i)}(t) \geqq 0 \text{ for } t \geqq 0,\ i = 0,\dots,n-1, \tag{0.5}$$

without the assumption of the sign of f were found in [6].

This paper deals with the problem

$$\begin{aligned} &u''' = f(t,u,u',u'') \\ &\varphi(u(0),u'(0),u''(0)) = 0,\ u(t) \geqq 0, \\ &u'(t) \leqq 0,\ u''(t) \geqq 0 \quad \text{for } t \geqq 0 . \end{aligned} \tag{0.6}$$

Here, for $n = 3$, there are proved some more general conditions of existence of solutions of (0.6) than in [6].

<u>Notations</u>

N is the set of all natural numbers,
$R = (-\infty,\infty)$, $R_+ = \langle 0,\infty)$, $R_- = (-\infty,0\rangle$, $R^3 = R\times R\times R$,
$D^3 = R_+\times R_-\times R_+$, $D^3_r = \langle 0,r\rangle\times R_-\times R_+$, $I\subset R^3$, $J\subset R$, $k\in N$,
$C^k(I)$ is the set of all real functions which are continuous with their k-th order derivatives on I ,

$AC^k(I)$ is the set of all real functions which are absolutely continuous with their k-th derivatives on I ,

$L(I)$ is the set of all real Lebesgue-integrable on I functions,

$AC^k_{loc}(I)$ is the set of all real functions which are absolutely continuous with their k-th derivatives on each segment contained in I ,

$L_{loc}(I)$ is the set of all real functions which are Lebesgue-integrable on each segment contained in I ,

$Car(J\times I)$ is the set of all real functions $f:J\times I \to R$ satisfying the local Carathéodory conditions on J, i.e.

$f(.,x_1,x_2,x_3):J \to R$ is measurable for every $(x_1,x_2,x_3)\in I$,

$f(t,.,.,.):I \to R$ is continuous for almost every $t\in J$,

$\sup\left\{|f(.,x_1,x_2,x_3)| : \sum_{i=1}^{3}|x_i| \leq \varrho\right\}\in L(J)$ for any $\varrho\in R_+$,

$Car_{loc}(J\times I)$ is the set of all real functions $f:J\times I \to R$ satisfying the local Carathéodory conditions on each segment contained in J, i.e.

$f(.,x_1,x_2,x_3):J \to R$ is measurable for every $(x_1,x_2,x_3)\in I$,
$f(t,.,.,.):I \to R$ is continuous for almost every $t\in J$,

$\sup\left\{|f(.,x_1,x_2,x_3)| : \sum_{i=1}^{3}|x_i| \leq \varrho\right\}\in L_{loc}(J)$ for any $\varrho\in R_+$.

Fundamental assumptions

Throughout this paper we assume

$$f\in Car_{loc}(R_+\times D^3),\ f(t,0,0,0)=0,\ f(t,x_1,x_2,0)\leq 0 \text{ on } R_+\times D^3 \tag{0.7}$$

$$\varphi\in C(D^3),\ \varphi(0,0,0)<0,\ \varphi(x_1,x_2,x_3)>0 \text{ for } x_1>r,\ r\in R_+ \tag{0.8}$$

1. Boundary value problem

We shall consider the problem

$$u''' = f(t,u,u',u''), \tag{1.1}$$

$$u(t) \geqq 0,\ u'(t) \leqq 0,\ u''(t) \geqq 0 \text{ for } t \in R_+ , \tag{1.2}$$

$$\varphi(u(0),u'(0),u''(0)) = 0 . \tag{1.3}$$

A function $u \in AC^2_{loc}(R_+)$ which fulfils (1.1) for almost every $t \in R_+$ will be called a solution of the equation (1.1). Each solution of (1.1) which satisfies the conditions (1.2), (1.3) will be called a solution of the problem (1.1), (1.2), (1.3).

Remark.

a) From (1.3) and the fundamental assumption (0.8) it follows that the solution u of the problem (1.1), (1.3) satisfies the inequality $u(0) \leqq r$.

b) In the special case $\varphi(x_1,x_2,x_3) = x_1 - r$, the condition (1.3) goes over to

$$u(0) = r. \tag{1.4}$$

Theorem 1. Let there exist $a,b \in R$, $0 < a < b$, $\alpha_i, \beta_i \in R_+$, $i = 1,2,3$, a function $h \in L_{loc}(R_+)$ and a positive function $\omega \in C(R_+)$ such that

$$\int_0^\infty \frac{ds}{\omega(s)} = +\infty , \tag{1.5}$$

on the set $\langle 0,b\rangle \times D^3_r$ the inequality

$$f(t,x_1,x_2,x_3) \geqq -(h(t)+\sum_{i=1}^{2}\alpha_i |x_i| + \alpha_3\sqrt{x_3})(1+\sum_{i=1}^{2} x_i^2 + x_3) \tag{1.6}$$

and on the set $\langle a,\infty)\times D^3_r$ the inequality

$$f(t,x_1,x_2,x_3) \leqq \left[h(t) + \sum_{i=1}^{3} \beta_i |x_i|\right] \omega(x_3) \qquad (1.7)$$

are satisfied.

Then the problem (1.1), (1.2), (1.3) has at least one solution.

Theorem 2. Let all conditions of Theorem 1 be satisfied with the exception of (1.6) and let on the set $\langle 0,b\rangle \times D_r^3$ the inequality

$$f(t,x_1,x_2,x_3) \geqq -\left[h(t) + \sum_{i=1}^{2} \alpha_i |x_i| + \alpha_3 t x_3\right] \omega(x_3) \qquad (1.8)$$

take place.

Then the problem (1.1), (1.2), (1.3) has at least one solution.

Theorem 3. Let there exist $a \in R$, $0 < a$, $\beta \in R_+$ and a positive function $h \in L_{loc}(R_+)$ such that

$$\int_0^a \frac{t\,dt}{H(t)} = +\infty, \quad \text{where } H(t) = \int_0^t h(\tau)d\tau , \qquad (1.9)$$

on the set $\langle 0,a\rangle \times D_r^3$ the inequality

$$f(t,x_1,x_2,x_3) \geqq -h(t)(1 + x_3)^2 \qquad (1.10)$$

and on the set $(0,\infty) \times D_r^3$ the inequality

$$f(t,x_1,x_2,x_3) \leqq h(t)(1 + x_3) + \beta x_3^3 \qquad (1.11)$$

are satisfied.

Then the problem (1.1), (1.2), (1.3) has a least one solution.

Corollary. Let there exist functions $h_i \in L_{loc}(R_+)$, $i = 1,2,3$, such that $h_1(t) \leqq 0$ and $h_2(t) \geqq 0$ for $t \in R_+$.

Then the equation

$$u''' = \sum_{i=1}^{3} h_i(t)u^{(i-1)}(t) \tag{1.12}$$

has at least one solution $u \in AC^2_{loc}(R_+)$ satisfying (1.2) and (1.4).

Remark. Comparing (1.6) and (1.10) we can see that the extension of the exponent of x_3 from 3/2 to 2 forces the additional condition (1.9).

2. Lemmas

Lemma 1 (Chaplying). Let $J \subset R$, $g \in Car_{loc}(R_+ \times J)$ and let the Cauchy problem

$$\frac{dx}{dt} = g(t,x), \; x(t_0) = x_0, \; t_0 \in R_+, \; x_0 \in J,$$

have a unique solution defined on R_+.

Then for any function $y \in AC_{loc}(R_+)$ satisfying the inequalities $y(t_0) \leqq x_0$, $[y'(t) - g(t,y(t))]\,\mathrm{sign}(t - t_0) \leqq 0$ for $t \in R_+$

or

$$y(t_0) \geqq x_0, \; [y'(t) - g(t,y(t))]\,\mathrm{sign}(t-t_0) \geqq 0 \text{ for } t \in R_+,$$

respectively, it holds

$$y(t) \leqq x(t) \quad \text{or} \quad y(t) \geqq x(t), \text{ respectively, for } t \in R_+.$$

Proof. [5], p.42.

Lemma 2 (Gronwall). Let $g(t) \in L(a,b)$, $t_0 \in \langle a,b \rangle$, $x_0 \in R$ and $g(t)\,\mathrm{sign}(t-t_0) \geqq 0$ for $a \leqq t \leqq b$.

Then for any function $y \in C(a,b)$ satisfying on $\langle a,b \rangle$ the inequalities

$$y(t) \leqq x_0 + \int_{t_0}^{t} g(\tau)y(\tau)d\tau \quad \text{or} \quad y(t) \geqq x_0 + \int_{t_0}^{t} g(\tau)y(\tau)d\tau ,$$

respectively, it holds

$$y(t) \leqq x_0 \exp(\int_{t_0}^{t} g(\tau)d\tau) \quad \text{or} \quad y(t) \geqq x_0 \exp(\int_{t_0}^{t} g(\tau)d\tau),$$

respectively, for $a \leqq t \leqq b$.

Proof. [5], p.48-49.

Lemma 3. Let $c > 0$ and $v \in C^2(\langle 0,c \rangle)$ be such that $v(t) \geqq 0$, $v'(t) \leqq 0$, $v''(t) \geqq 0$ for $0 \leqq t \leqq c$. Then the inequality

$$|v'(t)| \leqq v(0)/c + \sqrt{2v(t)w(t)} \quad \text{for } 0 \leqq t \leqq c \tag{2.1}$$

where $w(t) = \max\{|v''(s)| : t \leqq s \leqq c\}$ takes place.

Proof. The equality $v(0) = v(t) - v'(c)c + \int_0^c \tau v''(\tau)d\tau$

implies $|v'(c)| \leqq v(0)/c.$ (2.2)

Let us put $\tilde{v}(t) = v(t) - v'(c)(t-c)$. Then $v(t) = \tilde{v}(t) + v'(c)(t-c)$ and so

$$\tilde{v}(c) = v(c) \quad \text{and} \quad v(t) \geqq \tilde{v}(t) \quad \text{for } 0 \leqq t \leqq c. \tag{2.3}$$

Further $v'(t) = \tilde{v}'(t) + v'(c)$ and thus

$$\tilde{v}'(c) = 0, \tag{2.4}$$

$$|v'(t)| \leqq |\tilde{v}'(t)| + |v'(c)| . \tag{2.5}$$

From $v''(t) = \tilde{v}''(t)$ it follows that $w(t)=\max\{|\tilde{v}''(t)| : t \leqq s \leqq c\}$ and by (2.3) and (2.4) also

$$\tilde{v}''(t) \geqq 0, \quad \tilde{v}'(t) \leqq 0, \quad \tilde{v}(t) \geqq 0 . \tag{2.6}$$

Multiplying the inequality $|\tilde{v}''(t)| \leq w(t)$ by $-\tilde{v}'(t)$ and integrating from t to c we get by (2.4) and (2.6)

$$- \int_t^c |\tilde{v}''(\tau)| \tilde{v}'(\tau) d\tau \leq - \int_t^c w(\tau)\tilde{v}'(\tau) d\tau \leq w(t)\tilde{v}(t).$$

From this it follows $\tilde{v}^2(t)/2 \leq w(t)v(t)$ and thus

$$|\tilde{v}'(t)| \leq \sqrt{2v(t)w(t)} \quad \text{for } 0 \leq t \leq c \ . \tag{2.7}$$

From (2.2), (2.5) and (2.7) we obtain (2.1).

Lemma 4 (An auxiliary boundary value problem). Suppose that

$$|f(t,x_1,x_2,x_3)| \leq f^*(t) \tag{2.8}$$

takes place on the set $R_+ \times D^3$, where $f^* \in L_{loc}(R_+)$.

Then for any $c \in (0, \infty)$ the boundary value problem

$$u''' = f(t,u,u',u'') \tag{2.9}$$

$$\varphi(u(0),u'(0),u''(0)) = 0, \quad u(c) = u'(c) = 0 \tag{2.10}$$

has at least one solution $u \in AC^2(\langle 0,c \rangle)$ satisfying on $\langle 0,c \rangle$ the inequalities

$$u(t) \geq 0,\ u'(t) \leq 0,\ u''(t) \geq (0) \ . \tag{2.11}$$

Proof. [6].

Lemma 5. Suppose $f_p \in Car_{loc}(R_+ \times D^3)$, $p \in N$, and the following relations are satisfied on the set $R_+ \times D^3$:

$$|f_p(t,x_1,x_2,x_3)| \leq f_o(t,x_1,x_2,x_3) \quad \text{for any } p \in N, \tag{2.12}$$

where $f_o \in Car_{loc}(R_+ \times D^3)$ and

$$\lim_{p\to\infty} f_p(t,x_1,x_2,x_3) = f(t,x_1,x_2,x_3) \,. \tag{2.13}$$

Let for each natural p the differential equation

$$u''' = f_p(t,u,u',u") \tag{2.14}$$

have a solution u_p satisfying the conditions

$$\varphi(u_p(0),u_p'(0),u_p''(0)) = 0, u_p(t) \geqq 0,\ u_p'(t) \leqq 0,$$
$$u_p''(t) \geqq 0 \text{ for } t\in R_+ \tag{2.15}$$

and let $\psi(t)\in C(R_+)$ be such that the inequality

$$\sup\left\{\sum_{i=1}^{3}\left|u_p^{(i-1)}(t)\right| :\ p\in N\right\} \leqq \psi(t) \quad \text{for } t\in R_+ \tag{2.16}$$

takes place. Then the sequance $\{u_p\}_{p=1}^{\infty}$ contains a subsequence $\{u_{p_j}\}_{j=1}^{\infty}$ which is local-uniformly converging together with $\{u'_{p_j}\}_{j=1}^{\infty}$ and $\{u"_{p_j}\}_{j=1}^{\infty}$ on R_+ and $u = \lim_{j\to\infty} u_{p_j}$ is a solution of the problem

$$u''' = f(t,u,u',u"), \tag{2.17}$$

$$\varphi(u(0),u'(0),u"(0)) = 0,\ u(t) \geqq 0,\ u'(t) \leqq 0,$$
$$u"(t) \geqq 0 \quad \text{for} \quad t\in R_+. \tag{2.18}$$

Proof. From (2.12) and (2.16) it follows

$$\left|u_p'''(t)\right| \leqq g_0(t) \quad \text{for} \quad p\in N \quad \text{and} \quad t\in R_+ \,, \tag{2.19}$$

where $g_0(t) = \sup\left\{f_0(t,u_p,u_p',u_p'') : \sum_{i=1}^{3}\left|u_p^{(i-1)}(t)\right| \leqq \psi(t)\right\}$.

Clearly $g_0 \in L_{loc}(R_+)$. By (2.19) we conclude that the sequences $\{u_p(t)\}_{p=1}^{\infty}$, $\{u_p'(t)\}_{p=1}^{\infty}$, $\{u_p''(t)\}_{p=1}^{\infty}$ are equal uniformly continuous on each segment $\langle 0,b\rangle \subset R_+$. Moreover, by (2.16), they are equal bounded on each segment $\langle 0,b\rangle \subset R_+$. So, by Arzelo-Ascoli lemma local uniformly converging subsequences on R_+ can be choosen from each of the three sequences. Integrating (2.14) we get

$$u_{p_j}''(t) = u_{p_j}''(0) + \int_0^t f_{p_j}(\tau, u_{p_j}(\tau), u_{p_j}'(\tau), u_{p_j}''(\tau))d\tau$$

and for $j \longrightarrow \infty$ by Lebesgue theorem, we obtain

$u''(t) = u''(0) + \int_0^t f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau$, thus u is a solution of (2.17), (2.18) on R_+.

Now we shall prove some lemmas on a priori estimates.

Lemma 6. Suppose $a,b,r \in R$, $0 < a < b$, $0 < r$, α_i, $\beta_i \in R_+$ $(i=1,2,3)$, $h \in L_{loc}(R_+)$ is a nonnegative function and $\omega \in C(R_+)$ is a positive function satisfying

$$\int_0^{\infty} \frac{d\tau}{\omega(\tau)} = +\infty. \tag{2.19}$$

Then there exists $r^* \in \langle r, \infty)$ such that for an arbitrary $c \in (b, \infty)$ and for an arbitrary function $v \in AC^2(\langle 0,c\rangle)$ the inequalities

$$\begin{aligned} v'''(t) \geq -\Big[h(t) + \sum_{i=1}^{2} \alpha_i |v^{(i-1)}(t)| + \\ + \alpha_3 \sqrt{v''(t)}\Big]\Big[1 + \sum_{i=1}^{2} v^{(i-1)}(t)^2 + v''(t)\Big] \end{aligned} \tag{2.20}$$

for $0 \leq t \leq b$,

$$v'''(t) \leqq \Big[h(t) + \sum_{i=1}^{3} \beta_i |v^{(i-1)}(t)|\Big]\, \omega(v''(t)) \qquad (2.21)$$
$$\text{for } a \leqq t \leqq c,$$

$$v(0) = r,\ v(t) \geqq 0,\ v'(t) \leqq 0,\ v''(t) \geqq 0 \qquad (2.22)$$
$$\text{for } 0 \leqq t \leqq c$$

imply the estimates

$$v(t) \leqq r^*,\ v'(t) \geqq -r^*,\ v''(t) \leqq \Omega^{-1}\Big(r^* + tr^* + \int_0^t h(\tau)\,d\tau\Big)$$
$$\text{for } 0 \leqq t \leqq c \quad (2.23)$$

where $\Omega(x) = \int_0^x \frac{ds}{\omega(s)}$.

Remark. Let us remind that r^* depends neither on c nor on v.

Proof. Since (2.22), $v(0) = v(t) + t|v'(t)| + \int_0^t \tau v''(\tau)d\tau$

and thus

$$|v(t)| \leqq r,\ |v'(t)| \leqq r/t,\ \int_0^t \tau v''(\tau)d\tau \leqq r \qquad (2.24)$$
$$\text{for } 0 \leqq t \leqq c.$$

According to Lagrange's theorem there exists $t_0 \in (a,b)$ such that $v''(t_0) = (v'(b)-v'(a))/(b-a) \leqq -v'(a)/(b-a)$ and through (2.24), (2.22) we obtain

$$|v''(t_0)| = r/(a(b-a)). \qquad (2.25)$$

Using Lemma 3 for v on $\langle 0,t_0\rangle$ we have

$$|v'(t)| \leqq r/t_0 + \sqrt{2rw(t)} \leqq r/a + \sqrt{2rw(t)}, \qquad (2.26)$$

where $w(t) = \max\{|v''(s)| : t \leqq s \leqq t_0\}$. If $r_1 = r/a + \sqrt{2r} + 1 + r$, then by (2.25) and (2.26)

$$w(t_o) = v''(t_o) \leqq r/(a(b-a)) = r_1/(b-a),$$
$$|v'(t)| \leqq r_1(1+w(t))^{1/2}, \quad |v''(t)| \leqq r_1(1+w(t)) \; . \tag{2.27}$$

Integrating (2.20) from t to t_o we get according to (2.27)

$$v''(t) \leqq v''(t_o) + \int_t^{t_o} \Big[h(\tau) + \sum_{i=1}^{2} \alpha_i |v^{(i-1)}(\tau)| + $$
$$+ \alpha_3 \sqrt{v''(\tau)}\Big]\Big[1+r^2+(r_1^2 + r_1)(1+w(\tau))\Big] d\tau ,$$

$$1 + w(t) \leqq 1+r_1/(b-a) + 3r_1^2 \int_t^{t_o} \Big[h(\tau) +$$
$$+ \sum_{i=1}^{2} \alpha_i |v^{(i-1)}(\tau)| + \alpha_3 \sqrt{v''(\tau)}\Big]\Big[1 + w(\tau)\Big] d\tau .$$

Now we obtain from the last inequality by Lemma 2

$$1+w(t) \leqq (1+r_1/(b-a))\exp\Big\{3r_1^2 \int_t^{t_o} \Big[h(\tau) +$$
$$+ \sum_{i=1}^{2} \alpha_i |v^{(i-1)}(\tau)| + \alpha_3 \sqrt{v''(\tau)} \Big] d\tau \Big\} \tag{2.28}$$
$$\text{for } 0 \leqq t \leqq t_o \; .$$

The equality $v(0) = v(t) + \int_0^t |v'(\tau)| d\tau$ together with (2.22) imply

$$\int_0^{t_o} |v'(\tau)| d\tau \leqq r. \tag{2.29}$$

Further for any $\varepsilon > 0$ considering (2.24) we have

$$\int_t^{t_o} \sqrt{v''(\tau)}\,d\tau = \int_t^{t_o} \sqrt{\varepsilon/\tau}\,\sqrt{(\tau/\varepsilon)v''(\tau)}\,d\tau \leqq \frac{1}{2}\int_t^{t_o} \frac{\varepsilon}{\tau}\,d\tau +$$

$$+ \frac{1}{2\varepsilon}\int_t^{t_o} \tau\, v''(\tau)\,d\tau \leqq (\varepsilon/2)\ln(t_o/t) + r/(2\varepsilon) \text{ for } 0< t \leqq t_o.$$

Thus

$$\int_t^{t_o} \sqrt{v''(\tau)}\,d\tau \leqq (\varepsilon/2)\ln(t_o/t) + r/(2\varepsilon) \qquad (2.30)$$
$$\text{for } 0< t \leqq t_o.$$

Substituting estimates (2.24), (2.29) and (2.30) into (2.28) we get

$$1+w(t) \leqq (1+r_1/(b-a))\exp\Big\{3r_1^2\Big(\int_0^b h(\tau)\,d\tau + \alpha\, rb + \alpha_2 r +$$

$$+ \alpha_3 \frac{\varepsilon}{2}\ln(b/t) + \alpha_3 \frac{r}{2\varepsilon}\Big) .$$

If $\varepsilon = 1/(3r_1^2\alpha_3)$ and $r_2 = (1+r_1/(b-a)b^{\frac{1}{2}} \exp\Big\{3r_1^2\Big(\int_0^b h(\tau)\,d\tau +$

$$+ r(\alpha_1 b + \alpha_2 + \alpha_3/2\varepsilon)\Big)\Big\},$$

then

$$1 + w(t) \leqq r_2 t^{-1/2} \quad \text{for} \quad 0< t \leqq t_o . \qquad (2.31)$$

Thus we have by (2.31) and (2.27)

$$\int_t^{t_o} \sqrt{v''(\tau)}\,d\tau = \int_t^{t_o} \sqrt{r_1}\,\sqrt{1+w(\tau)}\,d\tau \leqq \int_t^{t_o} \sqrt{r_1 r_2}\,\tau^{-1/4}\,d\tau \leqq \sqrt{r_1 r_2}\,\frac{4}{3}\,b^{3/4}.$$

If $r_3 = (4/3)\sqrt{r_1 r_2}\, b^{3/4}$, then

$$\int_0^{t_o} \sqrt{v''(\tau)}\,d\tau \leqq r_3 . \qquad (2.32)$$

Substituting estimates (2.24), (2.29) and (2.32) into (2.28) we obtain

$$1+w(t) \leqq (1+r_1/(b-a))\exp(3r_1^2(\int_0^b h(\tau)d\tau + \alpha_1 rb + \alpha_2 r + \alpha_3 r_3)).$$

If $r_4 = r_1(1+r_1/(b-a))\exp(3r_1^2(\int_0^b h(\tau)d\tau + \alpha_1 rb + \alpha_2 r + \alpha_3 r_3))$,

then $1+w(t) \leqq r_4/r_1$ for $0 \leqq t \leqq t_o$, and according to (2.22) and (2.27) it follows from the last inequality

$$|v'(t)| \leqq r_4 \quad \text{for } 0 \leqq t \leqq c, \quad |v''(t)| \leqq r_4 \quad \text{for } 0 \leqq t \leqq t_o. \qquad (2.33)$$

It remains to find an estimation for $v''(t)$ on $\langle t_o, c\rangle$. Integrating (2.21) from t_o to t and using (2.33) we have

$$\Omega(v''(t)) \leqq \Omega(v''(t_o)) + \int_{t_o}^{t} h(\tau)d\tau + t\beta_1 r + \beta_2 r + \beta_3 r_4 \leqq$$

$$\leqq \Omega(r_4) + \beta_2 r + \beta_3 r_4 + t\beta_1 r + \int_{t_o}^{t} h(\tau)d\tau .$$

Let us put $r^* = \Omega(r_4) + \beta_2 r + (\beta_3 + 1)r_4 + \beta_1 r$. Then from the latter inequality and from (2.31), (2.24) we obtain estimates (2.23).

Lemma 7. Suppose $a, b, r \in R$, $0 < a < b$, $0 < r$, $\alpha_i, \beta_i \in R_+$ $(i=1,2,3)$, $h \in L_{loc}(R_+)$ is a nonnegative function and $\omega \in C(R_+)$ is a positive function satisfying (2.19).

Then there exists $r^{*} \in \langle r, \infty)$ such that for an arbitrary $c \in (b, \infty)$ and for an arbitrary function $v \in AC^2(\langle 0,c\rangle)$ the inequalities (2.21), (2.22) and

$$v'''(t) \geqq -\Big[h(t) + \sum_{i=1}^{2} \alpha_i |v^{(i-1)}(t)| + \alpha_3 t v''(t)\Big] \omega(v''(t)) \quad \text{for } 0 \leqq t \leqq b \tag{2.34}$$

imply estimates (2.23).

Proof. We obtain inequalities (2.24), (2.25) and (2.29) in the same way as in the proof of Lemma 6. Integrating (2.34) from t to t_o we get

$$\Omega(v''(t)) \leqq \Omega(v''(t_o)) + \int_0^b h(\tau)d\tau + \alpha_1 br + \alpha_2 r + \alpha_3 r \quad \text{and}$$

thus

$$v''(t) \leqq r_1 \quad \text{for } 0 \leqq t \leqq t_o, \tag{2.35}$$

where $r_1 = \Omega^{-1}(\Omega(r/(a(b-a))) + \int_0^b h(\tau)d\tau + r(\alpha_1 b + \alpha_2 + \alpha_3)$.

Using Lemma 3 for v on $\langle 0, t_o\rangle$ we have

$$|v'(t)| \leqq r/t_o + \sqrt{2rr_1} \leqq r/a + \sqrt{2rr_1} = r_2 \quad \text{for } 0 \leqq t \leqq t_o \tag{2.36}$$

If $r_3 = \max\{r_1, r_2, r\}$, then by (2.22), (2.24) and (2.35), (2.36)

$$|v(t)| \leqq r_3, \ |v'(t)| \leqq r_3 \ \text{ for } 0 \leqq t \leqq c, \ |v''(t)| \leqq r_3 \ \text{ for } 0 \leqq t \leqq t_o. \tag{2.37}$$

Integrating (2.21) from t_o to t we obtain in accordance to (2.37)

$$\Omega(v''(t)) \leq \Omega(r_3) + \int_{t_0}^{t} h(\tau)d\tau + t\beta_1 r + (\beta_2 + \beta_3)r_3.$$ Let us put $r^* = \Omega(r_3) + (\beta_2 + \beta_3)r_3 + \beta_1 r$. Then from the last inequality and (2.37) it follows (2.23).

Lemma 8. Suppose $a, r, \beta \in R$, $0 < a$, $0 < r$, $0 < \beta$, and $h \in L_{loc}(R_+)$ is a positive function satisfying (1.9).

Then there exists $r^* \in \langle r, \infty)$ such that for an arbitrary $c \in (a, \infty)$ and for an arbitrary function $v \in AC^2(\langle 0, c\rangle)$ the inequalities (2.22),

$$v'''(t) \geq -h(t)(1+v''(t))^2 \quad \text{for } 0 \leq t \leq a, \tag{2.38}$$

$$v'''(t) \leq h(t)(1+v''(t)) + \beta v''^2(t) \quad \text{for } 0 < t \leq c \tag{2.39}$$

imply the estimates

$$v(t) \leq r^*, \; v'(t) \geq -r^*, \; v''(t) \leq r^* \exp\left(\int_0^t h(\tau)d\tau\right) \quad \text{for } 0 \leq t \leq c. \tag{2.40}$$

Proof. Since (2.38),

$$\frac{d}{dt}(1+v''(t)) \geq -h(t)(1+v''(t))^2 \quad \text{for } 0 \leq t \leq a. \tag{2.41}$$

Let us consider the differential equation

$$\varphi'(t) = -h(t)\varphi^2(t) \quad \text{for } 0 \leq t \leq a. \tag{2.42}$$

Integrating (2.42) from 0 to t we get $\varphi(t) = \left[1/\varphi(0) + H(t)\right]^{-1}$, where $H(t) = \int_0^t h(\tau)d\tau$. According to (1.9) there exists $\varepsilon \in (0,1)$ and $a_0 \in (0,a)$ such that

$$\int_{a_0}^{a} t(\varphi(t) - 1)dt > r, \quad \text{where } \varphi(0) = 1/\varepsilon. \tag{2.43}$$

Let us suppose that $1+v''(t) \geqq \varphi(t)$ takes place for $a_o \leqq t \leqq a$. Then by (2.43) we obtain

$$\int_{a_o}^{a} tv''(t)dt > r \ ,$$

which contradicts inequality (2.24). Thus it is necessary that there exists $t_o \in (a_o, a)$ such that

$$1 + v''(t_o) \leqq \varphi(t_o) \ . \tag{2.44}$$

Since (2.41), (2.42) and (2.44) by Lemma 1,

$$1 + v''(t) \leqq \varphi(t) \leqq \varphi(0) = 1/\varepsilon \quad \text{for } 0 \leqq t \leqq t_o, \tag{2.45}$$

and on the other hand (2.22) implies

$$|v(t)| \leqq r \quad \text{for } 0 \leqq t \leqq c \ . \tag{2.46}$$

Using Lemma 3 for v on $\langle 0, t_o \rangle$ we obtain according to (2.45), (2.22), $|v'(t)| \leqq r/t_o + \sqrt{2r/\varepsilon}$ for $0 \leqq t \leqq t_o$ and then putting $r_o = r/a_o + \sqrt{2r/\varepsilon} + r + 1$, we obtain

$$|v'(t)| \leqq r_o \quad \text{for } 0 \leqq t \leqq c \ . \tag{2.47}$$

Integrating (2.39) from t_o to t and using (2.45) we get

$$1 + v''(t) \leqq 1 + v''(t_o) + \int_{t_o}^{t} \left[h(\tau)(1+v''(\tau)) + \beta v''^2(\tau) \right] d\tau \leqq$$

$$\leqq 1/\varepsilon + \int_{t_o}^{t} \left[h(\tau) + \beta v''(\tau) \right] \left[1+v''(\tau) \right] d\tau$$

and from (2.47), Lemma 2 and the last inequality we have

$$1+v''(t) = (1/\varepsilon) \exp \int_{t_o}^{t} \left[h(\tau) + \beta v''(\tau) \right] d\tau \leqq$$

$$\leqq (1/\varepsilon) \exp(\beta r_o) \exp \int_{t_o}^{t} h(\tau) d\tau \ . \tag{2.48}$$

Therefore if $r^* = \max\{r_o, (1/\varepsilon)\exp(\beta r_o)\}$, then (2.46), (2.47) and (2.48) imply the estimates (2.40).

Lemma 9. Suppose that $a, r \in R$, $0 < a$, $0 < r$, $h_i \in L_{loc}(R_+)$, $(i=1,2,3)$, and $h_1(t) \geqq 0$, $h_2(t) \leqq 0$ for $t \in R_+$.

Then there exists $r^* \in \langle r, \infty)$ such that for an arbitrary $c \in (a, \infty)$ and for any $v \in AC^2(\langle 0,c\rangle)$ the conditions (2.22) and

$$v'''(t) = \sum_{i=1}^{3} h_i(t)v^{(i-1)}(t) \qquad \text{for } 0 \leqq t \leqq c \tag{2.49}$$

imply the estimates

$$v(t) \leqq r^*, \ v'(t) \geqq -r^*, \ v''(t) \leqq r^* \exp \int_0^t \sum_{i=1}^{3} |h_i(\tau)| \, d\tau . \tag{2.50}$$

Proof. (2.22) yields $|v(t)| \leqq r$ for $0 \leqq t \leqq c$. Further, we can prove in the same way as in the proof of Lemma 6 that there exists $t_o \in (\frac{a}{2}, a)$ such that

$$|v''(t_o)| \leqq 4r/a^2, \ |v'(t)| \leqq |v'(t_o)| \leqq 2r/a \qquad \text{for } t_o \leqq t \leqq c . \tag{2.51}$$

Moreover, by (2.49) we get

$$v'''(t) \leqq \sum_{i=1}^{3} |h_i(t)||v^{(i-1)}(t)|$$

and integrating the latter inequality from t_o to t we obtain according to (2.51)

$$v''(t) \leqq v''(t_o) + \int_{t_o}^{t} \left[r|h_1(\tau)| + (2r/a)|h_2(\tau)| + |h_3(\tau)| v''(\tau) \right] d\tau ,$$

thus $v''(t) + 1 \leqq 1 + v''(t_o) + (r+2r/a)\int_{t_o}^{t} (\sum_{i=1}^{3} |h_i(\tau)|)(1+v''(\tau))d\tau.$

It follows from here by Lemma 2

$$1 + v''(t) \leqq r_1 \exp \int_0^t \sum_{i=1}^{3} |h_i(\tau)|\, d\tau \quad \text{for } t_0 \leqq t \leqq c\ , \qquad (2.52)$$

where $r_1 = (1+4r/a^2)\exp(r+2r/a)$. Now, we estimate $v'(t)$ and $v''(t)$ on $\langle 0, t_0 \rangle$. From (2.49) we get

$$v'''(t) \geqq -\sum_{i=1}^{3} |h_i(t)|\,|v^{(i-1)}(t)|$$

and integrating the latter from t to t_0 we have

$$v''(t) = v''(t_0) + \int_t^{t_0} \sum_{i=1}^{3} |h_i(\tau)|\ |v^{(i-1)}(\tau)|\, d\tau\ . \qquad (2.53)$$

Using Lemma 3 for v on $\langle 0, t_0 \rangle$ we obtain $|v'(t)| \leqq r/t_0 + \sqrt{2rw(t)}$, where $w(t) = \max\{|v''(\tau)| : t \leqq \tau \leqq t_0\}$. If $r_2 = 2r/a + r + 1$, then

$$|v'(t)| \leqq r_2(w(t)+1) \quad \text{for } 0 \leqq t \leqq t_0\ . \qquad (2.54)$$

Let us substitute (2.54) into (2.53). Then

$$w(t)+1 \leqq 1+4r/a^2 + \int_0^a r|h_1(\tau)|\, d\tau + \int_t^{t_0} \Big[|h_2(\tau)|\, r_2 + |h_3(\tau)|\Big]\ \Big[w(\tau) + 1\Big]\, d\tau\ .$$

If $r_3 = 1 + 4r/a^2 + \int_0^a r|h_1(\tau)|\, d\tau$, then from the last inequality it follows by Lemma 2 $w(t) + 1 \leqq r_3 \exp \int_t^{t_0} (|h_2(\tau)|\, r_2 + |h_3(\tau)|)\, d\tau$.

Putting $r_4 = r_3 \exp \int_0^a (|h_2(\tau)|\, r_2 + |h_3(\tau)|)\, d\tau$ we have

$$|v''(t)| \leq r_4 \qquad \text{for } 0 \leq t \leq t_0 \quad \text{and} \tag{2.55}$$

$$|v'(t)| \leq r_4 r_2 \qquad \text{for } 0 \leq t \leq t_0 \ .$$

For $r^* = \max\{r_4 r_2, r_1\}$ from (2.51), (2.52) and (2.55) it follows (2.50).

3. The proofs of the existence theorems

The proof of Theorem 1. Without loss of generality we can assume that h is a nonnegative function.

Let $\Omega(x) = \int_0^x \frac{ds}{\omega(s)}$ and r^* be the constant from Lemma 6.

Put $\varrho(t) = 2r^* + \Omega^{-1}(r^* + tr^* + \int_0^t h(\tau)d\tau)$,

$$\chi(t,s) = \begin{cases} 1 & \text{for } 0 \leq s \leq \varrho(t) \\ 2-s/\varrho(t) & \text{for } \varrho(t) \leq s \leq 2\varrho(t) \\ 0 & \text{for } s \geq 2\varrho(t) \ , \end{cases}$$

$$\tilde{f}(t,x_1,x_2,x_3) = \chi(t,\sum_{i=1}^{3}|x_i|)\ f(t,x_1,x_2,x_3) \ . \tag{3.1}$$

Since $\tilde{f}$ satisfies the assumptions of Lemma 4, the boundary value problem

$$u''' = \tilde{f}(t,u,u',u'')$$
$$u(b+p) = u'(b+p) = 0$$
$$\varphi(u(0),u'(0),u''(0)) = 0 \tag{3.2}$$

has for any $p \in N$ at least one solution $u_p \in AC^2(\langle 0,b+p\rangle)$ satisfying in $\langle 0,b+p\rangle$ the inequalities

$$u_p(t) \geq 0,\ u_p'(t) \geq 0,\ u_p''(t) \leq 0 \ . \tag{3.3}$$

From (3.3) and the inequality $|u_p(0)| \leqq r$ it follows $|u_p(t)| \leqq r$ for $0 \leqq t \leqq b+p$. According to (1.6) and (3.1) we have

$$u_p'''(t) = \tilde{f}(t,u_p,u_p',u_p'') = \chi\Big(t,\sum_{i=1}^{3} |u_p^{(i-1)}(t)|\Big) f(t,u_p,u_p',u_p'') \geqq$$

$$\geqq -\chi\Big(t,\sum_{i=1}^{3}|u_p^{(i-1)}(t)|\Big)\Big[h(t)+\sum_{i=1}^{2}\alpha_i |u_p^{(i-1)}(t)| + \alpha_3 \sqrt{u_p''(t)}\Big]\Big[1 +$$

$$+ u_p'^2(t) + u_p''(t)\Big] \geqq -\Big[h(t) + \sum_{i=1}^{2} \alpha_i |u_p^{(i-1)}(t)| + \alpha_3 \sqrt{u_p''(t)}\Big]\Big[1 +$$

$$+ u_p'^2(t) + u_p''(t)\Big] \quad \text{for } 0 \leqq t \leqq b$$

and similarly from (1.7) we obtain

$$u_p'''(t) \leqq \Big[h(t)+ \sum_{i=1}^{3} \beta_i |u_p^{(i-1)}(t)|\Big] \cdot \omega(u_p''(t))$$
$$\text{for } a \leqq t \leqq b+p \ .$$

Using Lemma 6 for u_p on $\langle 0,b+p\rangle$ we get the estimates

$$u_p(t) \leqq r^*, \ u_p'(t) \geqq -r^*, \ u_p''(t) \leqq \Omega^{-1}\Big(r^* + tr^* + \int_0^t h(\tau)d\tau\Big) , \tag{3.4}$$

where r^* depends nether on p nor on u_p. Considering (3.3) and (3.4) we obtain $\sum_{i=1}^{3} |u_p^{(i-1)}(t)| \leqq \varrho(t)$ for $0 \leqq t \leqq b+p$ and thus u_p is also a solution of equation (1.1) on $\langle 0,b+p\rangle$.
Denote

$$f_p(t,x_1,x_2,x_3) = \begin{cases} f(t,x_1,x_2,x_3) & \text{for } 0 \leqq t \leqq b+p \\ 0 & \text{for } t > b+p \end{cases} .$$

Then on the set $R_+ \times D^3$ we have

$$|f_p(t,x_1,x_2,x_3)| \leq |f(t,x_1,x_2,x_3)| \quad \text{for any } p \in N$$

and

$$\lim_{p\to\infty} f_p(t,x_1,x_2,x_3) = f(t,x_1,x_2,x_3) \quad .$$

Moreover $\sup\left\{ \sum_{i=1}^{3} |u_p^{(i-1)}(t)| : p \in N \right\} \leq \varphi(t)$ for $t \in R_+$. Thus, by Lemma 5, the sequence $\{u_p\}_{p=1}^{\infty}$ contains a subsequence $\{u_{p_j}\}_{j=1}^{\infty}$ which is local uniformly converging together with $\{u'_{p_j}\}_{j=1}^{\infty}$ and $\{u''_{p_j}\}_{j=1}^{\infty}$ on R_+ and $u(t) = \lim_{j\to\infty} u_{p_j}(t)$ is a solution of equation (1.1) on R_+. Since all functions u_p ($p \in N$) satisfy (3.2) and (3.3), the solution u satisfies these conditions in R_+ .

The proofs of Theorems 2 and 3 can be obtained in a similar way as the proof of Theorem 1. We only use Lemma 7 or 8 instead of Lemma 6.

The proof of Corollary. The functions $f(t,x_1,x_2,x_3) = \sum_{i=1}^{3} h_i(t)x_i$ and $\varphi(x_1,x_2,x_3) = x_1 - r$, satisfy fundamental assumptions (0.7), (0.8). Thus using Lemma 9 we can find a solution of problem (1.12), (1.2), (1.4) in the same way as in the proof of Theorem 1.

REFERENCES

[1] F e r m i, E.: Un metodo statistico per la determinazione di alcune proprietà dell' atomo, Rend.R.Acc.Naz. dei Lincei. 1927, 6, 602-607.

[2] H a r t m a n, P., W i n t n e r, A.: On the non-increasing solutions of y" = f(x,y,y´), Amer.Journ.of Math., 1951, 73, 390-404.

[3] K i g u r a d z e, I.T.: On the non-negative non-increasing solutions of non-linear second order differential equations, Ann.di Mat. pura ed appl., 1969, 81, 169-192.

[4] K i g u r a d z e, I.T.: On monotone solutions of non-linear ordinary differential equations of order n, Izv. Akad.Nauk SSSR, ser.mat., 1969, 33, 1293-1317.

[5] K i g u r a d z e, I.T.: Nekotoryje singuljarnyje krajevyje zadači dlja obyknověnnych differencial´nych uravněnij, ITU, Tbilisi, 1975.

[6] K i g u r a d z e, I.T., R a ch ů n k o v á, I.: O razrešimosti nelinejnoj zadači tipa Knesera, Diff.uravněnija, 1979, XV, 10, 1754-1765.

[7] K n e s e r, A.: Untersuchung und asymptotische Darstellung der Integrale gewisser Differentialgleichungen bei grossen reelen Werten des Arguments, J.reine und angew.Math., 1896, 166, 178-212.

[8] L a m p a r i e l l o, G.: Su una classe notevole di equazioni differenziali del secondo ordine non lineari, Rend.R.Acc.Naz. dei Lincei, 1934, 9, 284-290, 386-393.

[9] M a m b r i a n i, A.: Su una particolare equazione differenziale, Rend.R.Acc.Naz. dei Lincei, 1929, 9, 142-144.

[10] M a m b r i a n i, A.: Su un teorema relativo alle equazioni differenziali ordinarie del 2° ordine, Rend. R.Acc.Naz. dei Lincei, 1929, 9, 620-622.

[11] M i r a n d a, C.: Teoremi e metodi per l´integrazione numerica dell´ equazione differenziale di Fermi, Mem. della R.Acc.d´Italia, 1934, 5, 285-322.

[12] S c o r z a - D r a g o n i, G.: A proposito di un´ equazione differenziale, Rend.R.Acc.Naz. dei Lincei, 1928, 8, 361-362.

[13] S c o r z a - D r a g o n i, G.: Su un´ equazione differenziale particolare, Rend.R.Acc.Naz. dei Lincei, 1929, 9, 623-625.

[14] S c o r z a - D r a g o n i, G.: Il problema dei valori ai limiti studiato in grande per gli integrali di una equazione differenziale del secondo ordine, Giorn. di Mat. di Battaglini, 1931, 69, 77-112.

[15] S o m m e r f e l d, A.: Integrazione asimptotica dell´ equazione di Thomas - Fermi, Rend.R.Acc.Naz. dei Lincei, 1932, 15, 788-792.

[16] T h o m a s, L.H.: The calculation of atomic fields, Proc. of the Cambridge Phil.Soc., 1927, 23, 542-548.

NELINEÁRNÍ PROBLÉM PRO DIFERENCIÁLNÍ ROVNICE 3.ŘÁDU

Souhrn

V tomto článku jsou nalezeny postačující podmínky pro existenci řešení diferenciální rovnice

$$u''' = f(t,u,u',u'')$$

splňujícího podmínky

$$\varphi(u(0),u'(0),u''(0)) = 0,\ u(t) \geqq 0,\ u'(t) \leqq 0,$$
$$u''(t) \geqq 0 \quad \text{pro } t \geqq 0 .$$

Existenční věty jsou dokázány pro případ, že f splňuje Carathéodoryho podmínky a může měnit znaménko.

О НЕЛИНЕЙНОЙ ЗАДАЧЕ ДЛЯ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТРЕТЬЕГО ПОРЯДКА

Резюме

В статье найдены достаточные условия для существования решения дифференциального уравнения

$$u''' = f(t,u,u', u'')$$

удовлетворяющего условиям

$$\varphi(u(0),u'(0),u''(0)) = 0,u(t)\geqq 0,u'(t) \leqq 0,u''(t) \geqq 0$$

для $t \geqq 0$ **. Теоремы существования доказаны в случае, когда f знакопеременная функция и удовлетворяет условиям Каратеодори.**

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