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Eva Tesaříková

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SPECIAL DISPERSIONS OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS OF A FINITE TYPE SPECIAL AND THEIR RELATION TO THE KUMMER'S TRANSFORMATION PROBLEM

EVA TESAŘÍKOVÁ

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Abstract: This article is a contribution to the theory of dispersions for the second order linear differential equations of a finite type, special. There is investigated the relation of the special dispersions to the Kummer's transformation problem.

Key words: The second order linear differential equation of a finite type-special, the theory of dispersions, the Kummer's transformation problem.

MS Classification: 34C20, 34A30, 34C10

Introduction

This article is a contribution to the transformation theory of the second order linear differential equations in Jacobian form

$$y'' = q(t)y \qquad (q^{(1)})$$

$$Y'' = Q(T)Y \qquad (Q^{(1)})$$

which are in their definition intervals $j = (a,b)$ $J = (A,B)$ of the same type, 1-special. This paper immediately follows with the publication [5] in which there are defined the general dispersions relative to some canonical mapping p of the space r of all the solutions of $(q^{(1)})$ onto the space R of all the solutions of $(Q^{(1)})$.

At the same time, by the canonical mapping p we will mean the mapping which maps every 1-fundamental basis of the equation $(q^{(1)})$ onto some 1-fundamental basis of the equation $(Q^{(1)})$. The canonical mapping p of the space r onto the space R is always determined by some forming phase basis $(\alpha, \mathcal{A})$, i.e. basis which is composed of the normal directly or indirectly similar first phases of the equations $(q^{(1)})$, $(Q^{(1)})$. At the same time, the direct or indirect general dispersion of these equations relative to the canonical mapping p has the quality of mapping zeros of an arbitrary solution $y \in r$ onto directly or indirectly associated zeros of his image $py \in R$, respectively. General dispersions represent just all the transformation functions of the equations $(q^{(1)})$, $(Q^{(1)})$ defined in the whole interval j. A set of all the general dispersions of the considered pair of equations forms a two-parametric system.

<u>Special dispersions of the differential equations of type $(q^{(1)})$</u>

Now let us consider the assumptions, concepts and notations introduced in the article [5] and let us focus on the functions that associate zeros of an arbitrary solution $y \in r$ at zeros of its image $py \in R$, where p is some canonical mapping of the space r onto R, but those need not be points directly or indirectly associated. These functions will be called the special dispersions of the considered equations in sence of the following definition 1. On the basis of properties of general dispersions and of special central dispersions treated in the article [5], [2] we can derive analogical properties of thus defined functions.

<u>Definition 1</u>

$(zm + k)$ - th direct or indirect special dispersion of the equations $(q^{(1)})$, $(Q^{(1)})$ relative to the canonical mapping p of the space r onto R with the characteristic $\chi_p > 0$ or $\chi_p < 0$

will be called the function X_{zm+k} defined by relation

$$X_{zm+k}(t) = X[\phi_{zm+k}(t)] \qquad \text{for } t \in j, \quad t \neq a_{m-k} ,$$

respectively, where ϕ is a special central dispersion of the 1-st kind of the equation $(q^{(1)})$, for $k = 0,1,\dots,m-1$, $z \in Z$ and where X is a direct or indirect general dispersion of these equations relative to the mapping p.

Theorem 1

The set of all the special dispersions of the equations $(q^{(1)})$, $(Q^{(1)})$ relative to a concrete canonical mapping p of the space r onto R is finite and contains precisely m of various elements $X = X_0, X_1, \dots, X_{m-1}$.

P r o o f . The assertion follows from the properties of special central dispersions of the 1-st kind of the equation $(q^{(1)})$. Regarding the fact $\phi_0 = t$, $\phi_{-k} = \phi_{m-k}$ for $k = 0,1,\dots,m-1$ and that´s way $\phi_{zm+k} = \phi_k$ for an arbitrary $z \in Z$, it is evident also the validity of $X_{-k} = X_{m-k}$, $X_{zm+k} = X_k$ and X_0 is a general dispersion X in the sence of definition 1.

The basic properties of the special dispersions are stated in the theorem 2. For the partial intervals from j or from J the following notation

$$(a_{i-1}, a_i) = j_i \qquad\qquad \text{for } i = 1,2,\dots,m$$
$$(A_{i-1}, A_i) = J_i$$

will be used, where points a_i or A_i are elements of 1-fundamental sequence $(a^{(1)})$ or $(A^{(1)})$, respectively.

Theorem 2

a) The k-th direct special dispersion X_k of the equations $(q^{(1)})$, $(Q^{(1)})$ relative to an arbitrary canonical mapping p of the space r onto R, where $\mathcal{X}_p > 0$, has for $k = 0,1,\dots,m-1$ the following properties:

1) the domain of definition of X_k forms $(a, a_{m-k}) \cup (a_{m-k}, b)$
2) the range of values of X_k forms $(A, A_k) \cup (A_k, B)$
3) X_k is an increasing function from class $C^{(3)}$ with the

derivative $X_k' > 0$ in both the definition intervals (a, a_{m-k}), (a_{m-k}, b)

4) for $k = 1,2,\dots,m-1$ holds

$$\lim_{t\to a_{m-k}^-} X_k(t) = B, \quad \lim_{t\to a_{m-k}^+} X_k(t) = A, \quad \lim_{t\to a^+} X_k(t) = \lim_{t\to b^-} X_k(t) = A_k$$

5) the function X_k uniquely maps intervals

j_i	onto	J_{i+k}	for $i = 1,2,\dots,m-k$
j_{m-k+i}	onto	J_i	for $i = 1,2,\dots,k$

b) The k-th indirect special dispersion of the equations $(q^{(1)})$, $(Q^{(1)})$ relative to an arbitrary mapping p of the space r onto R, where $\mathcal{X}_p < 0$, has for $k = 0,1,\dots,m-1$ the following properties:

1) the domain of definition of X_k forms $(a, a_{m-k}) \cup (a_{m-k}, b)$

2) the range of values of X_k forms $(A, A_{m-k}) \cup (A_{m-k}, B)$

3) X_k is an decreasing function from the class $C^{(3)}$ with derivative $X_k' < 0$ in both the intervals (a, a_{m-k}), (a_{m-k}, b)

4) for $k = 1,2,\dots,m-1$ holds

$$\lim_{t\to a_{m-k}^-} X_k(t) = A, \quad \lim_{t\to a_{m-k}^+} X_k(t) = B, \quad \lim_{t\to a^+} X_k(t) = \lim_{t\to b^-} X_k(t) = A_{m-k}$$

5) the function X_k uniquely maps

j_i	onto	$J_{m-k-i+1}$	for $i = 1,2,\dots,m-k$
j_{m-k+i}	onto	J_{m-i+1}	for $i = 1,2,\dots,k$

P r o o f . The above mentioned properties immediately follow from the definition of the direct or indirect general dispersion, from the properties stated in Theorem 4 of [5] and from the properties of special central dispersions of the 1-st kind derived in article [2].

1) the domain of definition of the direct and indirect dispersions is the same as that one of the function $\Phi_k(t)$

2) the range of values of the function $X_k(t)$ forms the directly or indirectly associted points to the values of the function $\Phi_k(t)$

3) from the fact that $X \in C^{(3)}(j)$, $\Phi_k \in C^{(3)}$ in the intervals

(a,a_{m-k}), (a_{m-k},b) holds $X(\Phi_k)\in C^{(3)}$ in intervals (a,a_{m-k}), (a_{m-k},b); from the fact $X'>0$ or $X'<0$ in the interval j and $\Phi_k'>0$ in (a,a_{m-k}), (a_{m-k},b) holds $(X(\Phi_k))'>0$ or $(X(\Phi_k))'<0$ in (a,a_{m-k}), (a_{m-k},b), respectively

4) the expressed limits are directly or indirectly associated points relative to limits of the function $\Phi_k(t)$

5) the direct or indirect associated points of the images relative to the function $\Phi_k(t)$ form the images of the above mentioned intervals.

The following statement can be formulated on the basis of the functional equation (4) from [5] and modification of Abel's equation for special central dispersions of the 1-st kind.

Theorem 3

The k-th special dispersion $X_k(t)$ of the differential equations $(q^{(1)})$, $(Q^{(1)})$ relative to the canonical mapping p of the space r onto R fulfils in its domain of definition the functional equation

$$\mathcal{A}(X_k(t)) = \begin{cases} \alpha(t) + k\pi \operatorname{sign} \alpha' & \text{for } t\in(a,a_{m-k}) \\ \alpha(t) - (m-k)\pi \operatorname{sign} \alpha' & \text{for } t\in(a_{m-k},b) \end{cases} \tag{1}$$

with regard to an arbitrary forming phase basis $(\alpha,\mathcal{A})$ of mapping p.

P r o o f . From relation (4) from [5] for general dispersion X in the point $\Phi_k(t)$ using the relation (1) from [3] for special central dispersion of equation $(q^{(1)})$ follows that

$$\mathcal{A}(X_k(t)) = \mathcal{A}(X\,\Phi_k(t)) = \alpha(\Phi_k(t)) = \begin{cases} \alpha(t) + k\pi \operatorname{sign} \alpha' & \text{for } t\in(a,a_{m-k}) \\ \alpha(t) - (m-k)\pi \operatorname{sign} \alpha' & \text{for } t\in(a_{m-k},b) \end{cases}$$

which is what we wanted to prove.

On the basis of the Theorem 3 we can derive all the other properties of the special dispersions of the above mentioned differential equations, which are analogical to the properties of general dispersions X of these equations, mentioned in the Theorem 4 from [5].

Theorem 4

The k-th special dispersion of equations $(q^{(1)})$, $(Q^{(1)})$ corresponding to the canonical mapping p of the space r onto R prossesses in its domain of definition in the relation with an arbitrary forming phase basis $(\alpha, \mathcal{A})$ of this mapping the following properties:

1) For every $t \in j$, $t \neq a_{m-k}$ the function X_k is uniquely determined by the relation

$$X_k(t) = \begin{cases} \mathcal{A}^{-1}[\alpha(t) + k\pi \operatorname{sign} \alpha'] & \text{for } t \in (a, a_{m-k}) \\ \mathcal{A}^{-1}[\alpha(t) - (m-k)\pi \operatorname{sign} \alpha'] & \text{for } t \in (a_{m-k}, b) \end{cases} \qquad (2)$$

where $\mathcal{A}^{-1}$ is the function inverse to the phase $\mathcal{A}$.

P r o o f . The relation (2) follows immediately from the relation (1).

2) The function X_k^{-1} inverse to the function $X_k(t)$ is for $X_k' > 0$ the (m-k)-th direct special dispersion $x_{m-k}(T)$ or for $X_k' < 0$ the k-th indirect special dispersion $x_k(T)$ of the equations $(Q^{(1)})$, $(q^{(1)})$ corresponding to the inverse linear canonical mapping p^{-1} of the space R onto r with appropriate forming phase basis $(\mathcal{A}, \alpha)$, respectively.

P r o o f . a) At first, let us consider $X_k' > 0$. From the expression of the derivative $X_k'(t)$ from (1) in form

$$X_k'(t) = \frac{\alpha'(t)}{\mathcal{A}^{\cdot}(X_k)} \qquad \text{for } t \neq a_{m-k}, \quad t \in j$$

it is evident, that $X_k' > 0$ holds only if $\operatorname{sign} \alpha' = \operatorname{sign} \mathcal{A}^{\cdot}$. From the relation (1) at the same time follows that

$$\alpha(t) = \begin{cases} \mathcal{A}[X_k(t)] - k\pi \operatorname{sign} \alpha' = \mathcal{A}[X_k(t)] - k\pi \operatorname{sign} \mathcal{A}^{\cdot} & \text{for } t \in (a, a_{m-k}) \\ \mathcal{A}[X_k(t)] + (m-k)\pi \operatorname{sign} \alpha' = \mathcal{A}[X_k(t)] + \\ \qquad + (m-k)\pi \operatorname{sign} \mathcal{A}^{\cdot} & \text{for } t \in (a_{m-k}, b) \end{cases}$$

Hence, it is evident that the function $X_k^{-1}(T)$ fulfils the functional equation

$$X_k^{-1}(T) = \begin{cases} \alpha^{-1}[\mathcal{A}(T) - k\pi \operatorname{sign} \mathcal{A}^{\cdot}] & \text{for } T \in (A_k, B) \\ \alpha^{-1}[\mathcal{A}(T) + (m-k)\pi \operatorname{sign} \mathcal{A}^{\cdot}] & \text{for } T \in (A, A_k) \end{cases}$$

from which the assertion $X_k^{-1}(T) = x_{m-k}(T)$, $\operatorname{sign} x^{\cdot}_{m-k} =$ $= \operatorname{sign} \alpha' \operatorname{sign} \mathcal{A}^{\cdot} = \operatorname{sign} X_k'$ follows compared with the relation (2) and from the fact $X_{-k} = X_{m-k}$.

b) Now let us consider $X_k' < 0$, which holds only if $\operatorname{sign} \alpha' = -\operatorname{sign} \mathcal{A}^{\cdot}$. From relation (1) the following fact

$$\alpha(t) = \begin{cases} \mathcal{A}[X_k(t)] + k\pi \operatorname{sign} \mathcal{A}^{\cdot} & \text{for } t \in (a, a_{m-k}) \\ \mathcal{A}[X_k(t)] - (m-k)\pi \operatorname{sign} \mathcal{A}^{\cdot} & \text{for } t \in (a_{m-k}, b) \end{cases}$$

analogously follows. Hence, it is evident, that the function $X_k^{-1}(T)$ fulfils the functional equation

$$X_k^{-1}(T) = \begin{cases} \alpha^{-1}[\mathcal{A}(T) + k\pi \operatorname{sign} \mathcal{A}^{\cdot}] & \text{for } T \in (A, A_{m-k}) \\ \alpha^{-1}[\mathcal{A}(T) - (m-k)\pi \operatorname{sign} \mathcal{A}^{\cdot}] & \text{for } T \in (A_{m-k}, B) \end{cases}$$

Comparing that with the relation (2) we receive also the proclaimed validity of follows

$$X_k^{-1}(T) = x_k(T), \ \operatorname{sign} x^{\cdot}_k(T) = \operatorname{sign} \alpha' \operatorname{sign} \mathcal{A}^{\cdot} = \operatorname{sign} X_k'(t)$$

3) Each special dispersion X_k, for $k = 0,1,\dots,m-1$ is three times continuously differentiable in its domain of definition and it fulfils in every two homological points $t \in (a, a_{m-k}) \cup (a_{m-k}, b)$, $X_k \in (A, A_k) \cup (A_k, B)$ for $X_k' > 0$ or $X_k \in (A, A_{m-k}) \cup (A_{m-k}, B)$ for $X_k' > 0$ the following formulae

$$X_k'(t) = \frac{\alpha'(t)}{\mathcal{A}^{\cdot}(X_k)}, \quad X_k''(t) = \frac{1}{\mathcal{A}^{\cdot 3}(X_k)} [\alpha''(t)\mathcal{A}^{\cdot 2}(X_k) - \alpha'^{2}(t)\ddot{\mathcal{A}}(X_k)] \tag{3}$$

$$\mathcal{A}^{\cdot}(X_k) = \frac{\alpha'(t)}{X_k'(t)}, \quad \ddot{\mathcal{A}}(X_k) = \frac{1}{X_k'^{3}(t)} [\alpha''(t) . X_k'(t) - X_k''(t)\alpha'(t)] \tag{4}$$

P r o o f . The existence of continuous derivatives in the domain of definition already has been discussed in Theorem 2. The formulae (3) and (4) follow from double derivation of the functional equation (1).

4) For all $t \in j$, $t \neq a_{m-k}$ holds

$$X_k(t) = F_{\varepsilon k}(X(t)) \tag{5}$$

where ε = sign X_k', F(T) is the special central dispersion of the 1-st kind of the equation $(Q^{(1)})$.

P r o o f . The assertion directly follows from the definition of the function $X_k(t)$ and from the property 5) mentioned in the Theorem 4 from [5].

5) The special dispersion $X_k(t)$ satisfies for all $t \in j$, $t \neq a_{m-k}$ the nonlinear differential equation of the second order

$$-\{X_k, t\} + Q(X_k)\, X_k'^2 = q(t) \tag{$Q^{(1)}q^{(1)}$}$$

P r o o f . Let us start from the relation (1)

$$\mathcal{A}[X_k(t)] = \begin{cases} \alpha(t) + k\pi \operatorname{sign} \alpha' & \text{for } t \in (a, a_{m-k}) \\ \alpha(t) - (m-k)\pi \operatorname{sign} \alpha' & \text{for } t \in (a_{m-k}, b) \end{cases},$$

where $(\alpha, \mathcal{A})$ is an arbitrary forming phase basis relative to the canonical mapping p of the space r onto R, where α is the first phase of the certain basis (u,v) of the space r and $\mathcal{A}$ is the first phase of the basis (pu,pv) of the space R. According the relation (1), there alwais exist phases $\overline{\alpha}$, $\overline{\overline{\alpha}}$ in the phase system of the basis (u,v), which satisfy the equations

$$\mathcal{A}[X_k(t)] = \overline{\alpha}(t) \qquad \text{for } t \in (a, a_{m-k})$$

$$\mathcal{A}[X_k(t)] = \overline{\overline{\alpha}}(t) \qquad \text{for } t \in (a_{m-k}, b)$$

The relationship

$$\{\overline{\alpha}, t\} = \{X_k, t\} + \{\mathcal{A}, X_k\}\, X_k'^2 \tag{6}$$

follows for the function $\overline{\alpha}(t)$ in the interval (a, a_{m-k}) from the expression of the Schwarz's derivative of a composed function (17) chapter 8 § 1 from [1]. The same relation is fulfiled also for $\bar{\bar{\alpha}}$ in the interval (a_{m-k},b). Regarding the fact, that the derivative α', $\bar{\alpha}'$, $\bar{\bar{\alpha}}'$ and the Schwarz's derivatives of the functions α, $\bar{\alpha}$, $\bar{\bar{\alpha}}$ are identical for all $t \in j$, we get after the addition of equation $\alpha'^2(t) = a'^2(X_k)X'^2(t)$ and the equation (6) for all $t \in j$, $t \neq a_{m-k}$ to the validity of relation

$$\{X_k, t\} + [\{a, X_k\} + a'^2(X_k)]X_k'^2 = \{\alpha, t\} + \alpha'^2(t)$$

Using (16) of § 5 from [1] for the expression of carriers $(q^{(1)})$, $(Q^{(1)})$ we come also to the validity of $(Q^{(1)}q^{(1)})$.

6) The function $X_k^{-1} = x_{m-k}$ or $X_k^{-1} = x_k$ inverse to the special dispersion X_k for $X_k' > 0$ or $X_k' < 0$ satisfies in its domain of definition, i.e. for $T \in J$, $T \neq A_k$ or $T \neq A_{m-k}$, respectively, the nonlinear diferential equation of the third order

$$-\{X_k^{-1}, T\} + q(X_k^{-1})(X_k^{-1})'^2 = Q(T) \qquad (q^{(1)}Q^{(1)})$$

P r o o f . The statement follows immediately from the properties 2) and 5) mentioned in this theorem.

Now let us consider more equations of the same type 1-special in their definition intervals. We want to proove, that the special dispersions relative to the linear mapping composed of the canonical mappings of individual pairs of spaces of solutions corresponding to the above mentioned equations can be obtained by the composition of the special dispersions relative to these mappings on certain domain of definition. The regulations of this composition are contained in the following theorem.

Theorem 5

Consider three difeerential equations $(q^{(1)})$, $(Q^{(1)})$, $(\bar{Q}^{(1)})$ of the same finite type m, which are 1-special in their intervals of definition $j = (a,b)$, $J = (A,B)$, $\bar{J} = (\bar{A},\bar{B})$. Let $\Phi(t)$, $F(T)$, $\bar{F}(\bar{T})$ be the special central dispersions of these equations, respectively. Let p be a canonical linear mapping of the space r of all the solutions of $(q^{(1)})$ onto space R of all the solutions of $(Q^{(1)})$, let $X_k(t)$ be the k-th special dispersion relative to this mapping. Let P be a canonical linear mapping of the space R onto space $\overline{R}$ of all the solutions of the equation $(\overline{Q}^{(1)})$, let $\overline{X}_k(t)$ be the k-th special dispersion corresponding to this mapping. Then the composed mapping Pp is canonical again and the composed function $\overline{X}_i(X_k(t))$ represents for all $t \in j$ different from the elements of the 1-fundamental sequence $(a^{(1)})$ some special dispersion $\overline{\overline{X}}_n$ corresponding to the composed mapping Pp. At the same time, the following relations are valid:

a) If $X' > 0$, $\overline{X}' > 0$ or $\overline{X}' < 0$ then $\overline{\overline{X}}' > 0$ or $\overline{\overline{X}}' < 0$, respectively, and the relations

1) $\overline{X}_i\, X_k = \overline{\overline{X}}_{i+k}$

2) $X_i^{-1}\, X_k = \Phi_{k-i}$

hold for $t \in j$, $t \neq a_h$, $i,k,h = 0,1,\dots,m-1$.

b) If $X' < 0$, $\overline{X}' < 0$ or $\overline{X}' > 0$ then $\overline{\overline{X}}' > 0$ or $\overline{\overline{X}}' < 0$, respectively, and the relations

1) $\overline{X}_i\, X_k = \overline{\overline{X}}_{k-i}$

2) $X_i^{-1}\, X_k = \Phi_{k-i}$

hold for $t \in j$, $t \neq a_h$, $i,k,h = 0,1,\dots,m-1$.

P r o o f . Proving the Theorem 5 from [5] we came to the conclusion, that the mapping Pp composed of two canonical mappins is canonical too, and the general dispersion $\overline{\overline{X}}$ relative to it equals the function $\overline{X}(X(t))$. Now we have to prove the validity of above mentioned relations on the interval j except of the points of the sequence $(a^{(1)})$. We start from the validity of (9) of [5] from the Theorem 5 of [5], from the formula (5) and from the property 2) mentioned in the Theorem 4. At the same time we use the introduced notation $X^{-1} = x$, $\overline{X}^{-1} = \overline{x}$, $\overline{\overline{X}}^{-1} = \overline{\overline{x}}$.

a) If $X' > 0$, $\overline{X}' > 0$ then $\overline{\overline{X}}' > 0$ and holds

1) $\overline{X}_i\, X_k = \overline{X}_i\, F_k\, X = \overline{X}\, F_i\, F_k\, X = \overline{X}\, F_{i+k}\, X = \overline{F}_{i+k}\, \overline{X}\, X =$

$= \overline{F}_{i+k}\, \overline{\overline{X}} = X\, \Phi_{i+k} = X_{i+k}$

2) $X_i^{-1} X_k = x_{m-i} X_k = x F_{m-i} X_k = \Phi_{m-i} x X \Phi_k =$

$$= \Phi_{m-i+k} = \Phi_{k-i} \quad ;$$

if $X' > 0$, $\bar{X}' < 0$ then $\bar{\bar{X}}' < 0$ and holds

1) $\bar{X}_i X_k = \bar{X} F_i F_k X = \bar{X} F_{i+k} X = \bar{F}_{-(i+k)} \bar{X} X =$

$$= \bar{F}_{-(i+k)} \bar{\bar{X}} = X \Phi_{i+k} = X_{i+k} \quad .$$

b) If $X' < 0$, $\bar{X}' < 0$ then $\bar{\bar{X}}' > 0$ and holds

1) $\bar{X}_i X_k = \bar{X} F_i F_{-k} X = \bar{X} F_{i-k} X = \bar{F}_{k-i} \bar{X} X =$

$$= \bar{F}_{k-i} \bar{\bar{X}} = X \Phi_{k-i} = X_{k-i} \quad ;$$

2) $X_i^{-1} X_k = x_i X_k = x F_i F_{-k} X = x F_{i-k} X =$

$$= x X \Phi_{k-i} = \Phi_{k-i} \quad ;$$

if $X' < 0$, $\bar{X}' > 0$ then $\bar{\bar{X}}' < 0$ and holds

1) $\bar{X}_i X_k = \bar{X} F_i F_{-k} X = \bar{X} F_{i-k} X = \bar{F}_{i-k} \bar{X} X =$

$$= \bar{F}_{i-k} \bar{\bar{X}} = X \Phi_{k-i} = X_{k-i} \quad .$$

Regarding the property 5) of Theorem 4, it is evident, that also the special dispersions $X_k(t)$ are immediately related to the Kummer's transformation problem. This fact will be proved in the following theorem.

Theorem 6

Let $X_k(t)$ be the k-th special dispersion of the equations $(q^{(1)})$, $(Q^{(1)})$ corresponding to the canonical mapping p of the space r onto R, for $k = 0,1,\dots,m-1$

a) Let y be an arbitrary element of the space r, let $Y = py$ be its image in the space R. Then the function $Y(X_k) : \sqrt{|X_k'|}$ represents for all $t \in j$, $t \neq a_{m-k}$ a solution of equation $(q^{(1)})$. At the same time, there is fulfilled the relation

$$\frac{Y\,(X_k(t))}{\sqrt{|X_k'(t)|}} = \pm \frac{1}{\sqrt{|\mathfrak{X}_p|}}\, y(t) \quad , \tag{7}$$

where sign + or - is independent on the choice of Y.

b) There exists such a variation $cp = p^*$ of mapping p, where

$$\frac{Y(X_k(t))}{\sqrt{|X_k'(t)|}} = y(t) \tag{8}$$

holds for an arbitrary element $y \in r$ and its image $Y^* = \overset{*}{p} y \in R$ for $t \in j$, $t \neq a_{m-k}$. At the same time, the characteristic of the mapping p^* is determined by relation $\mathfrak{X}_{p^*} = \operatorname{sign} X_k'$.

c) If (U_1, V) is an arbitrary 1-fundamental basis of the equation $(Q^{(1)})$ and W is the Wronskian of this basis, then $(U_1(X_k) : \sqrt{|X_k'|}\,,\ V(X_k) : \sqrt{|X_k'|})$ is a 1-fundamental basis of $(q^{(1)})$ and for its Wronskian w the relationship $w = W \operatorname{sign} X_k'$ holds.

Proof. a) If $(\alpha, \mathcal{A})$ is a forming phase basis of mapping p, then the functional equation (1) is fulfilled for all $t \in j$, $t \neq a_{m-k}$. After expressing of the solutions $y(t)$, $Y(X_k(t))$ in form (3) of [5] and using the relationship $\alpha'(t) = \mathcal{A}^{\cdot}(X_k)X_k'(t)$ we come to the desired validity of (7).

b) At choice of mapping $p^* = cp = p\,\varepsilon E \sqrt{|\mathfrak{X}_p|}$, where ε, $E = \pm 1$ according to the phases α, $\mathcal{A}$ are proper or unproper, $Y^* = Y \varepsilon E \sqrt{|\mathfrak{X}_p|}$ holds. General dispersions corresponding to the mapping p and p^* are coinciding identically and the relationship (8) follows immediately from the relation (7). Besides, from $\mathfrak{X}_{p^*} = \mathfrak{X}_{cp} = c^{-2}\,\mathfrak{X}_p$ follows that $|\mathfrak{X}_{p^*}| = 1$ and thus $\mathfrak{X}_{p^*} =$ $= \operatorname{sign} \mathfrak{X}_p = \operatorname{sign} X_k'$.

c) If (U_1, V) is a 1-fundamental basis of equation $(Q^{(1)})$ and W is its Wronskian, then there exists a canonical mapping p which maps some 1-fundamental basis (u_1, v) of $(q^{(1)})$ onto (U_1, V). With respect to the validity of (7) the solutions u_1, $U_1(X_k) : \sqrt{|X_k'|}$ or v, $V(X_k) : \sqrt{|X_k'|}$ are dependent. Consequently, $(U_1(X_k) : \sqrt{|X_k'|}\,,\ V(X_k) : \sqrt{|X_k'|})$ is also 1-fundamental basis of the equation $(q^{(1)})$. The relationship $w = W \operatorname{sign} X_k'$ follows immediately from the direct computation of w.

With respect to the statement a) of the Theorem 6 and to the properties of the special dispersions X_k it is evident, that these functions represent transformations functions of the equations $(q^{(1)})$, $(Q^{(1)})$ on the both partial intervals (a, a_{m-k}), (a_{m-k}, b). Thus, they are solutions of the nonlinear differential equations $(Q^{(1)} q^{(1)})$ satisfying some initial condition

$$X_k(a_i) = A_{i+k} \quad \text{or} \quad X_k(a_i) = A_{m-k-i} \tag{9}$$

for $i = 1,2,\dots,m-k-1$, $X_k' > 0$ or $X_k' < 0$ on the interval (a, a_{m-k}) and some initial conditions

$$X_k(a_{m-k+i}) = A_i \quad \text{or} \quad X_k(a_{m-k+i}) = A_{m-i} \tag{10}$$

for $i = 1,2,\dots,k-1$, $X_k' > 0$ or $X_k' < 0$ on the interval (a_{m-k}, b), respectively. The converse statement is valid, too.

Theorem 7

Every in the interval (a, a_{m-k}) defined regular solution X_k of the differential equation $(Q^{(1)} q^{(1)})$, satisfying for some $i = 1,2,\dots,m-k-1$ the initial condition (9), where a_i, A_i are points of 1-fundamental sequences $(a^{(1)})$, $(A^{(1)})$, represents on the interval (a, a_{m-k}) the direct or indirect k-th special dispersion of the equations $(q^{(1)})$, $(Q^{(1)})$ relative to some canonical mapping p of the space r onto R, according $X_k' > 0$ or $X_k' < 0$, respectively.

Every in the interval (a_{m-k}, b) defined regular solution $\overline{X}_k$ of the differential equation $(Q^{(1)} q^{(1)})$ satisfying for some $i = 1,2,\dots,k-1$ the initial condition (10), represents on the interval (a_{m-k}, b) the direct or indirect k-th special dispersion of the equation $(q^{(1)})$, $(Q^{(1)})$ relative to some canonical mapping $\overline{p}$ of the space r onto R, according $\overline{X}_k' > 0$ or $\overline{X}_k' < 0$, respectively.

P r o o f . The proving of validity of this statement is analogous to the proof of Theorem 8 from [5].

1) Let us choose an arbitrary point $a_i \in (a, a_{m-k})$, $a_i \in (a^{(1)})$.

Let X_k be a regular solution of $(Q^{(1)} q^{(1)})$ defined in the interval (a, a_{m-k}) and satisfying for $X_k' > 0$ or $X_k' < 0$ at the point a_i the condition (9). Choosing a phase α of the phase system of some 1-fundamental basis relative $(q^{(1)})$, which is vanishing in the point a_i, for example the phase determined by conditions

$$\alpha(a_i) = 0, \qquad \alpha'(a_i) = 1, \qquad \alpha''(a_i) = 0. \tag{11}$$

With respect to it we will choose the phase $\mathcal{A}$ of $(Q^{(1)})$ fulfilling initial conditions

$$\mathcal{A}(X_0) = k\pi, \quad \dot{\mathcal{A}}(X_0) = \frac{1}{X_0'}, \quad \ddot{\mathcal{A}}(X_0) = -\frac{X_0''}{X_0'^3} \tag{12}$$

where X_0, X_0', X_0'' are values of function X_k and its first and second derivative in the point a_i. Thus, this phase $\mathcal{A}$ is with respect to the phase α directly or indirectly similar phase of the some 1-fundamental basis of equation $(Q^{(1)})$. By relationships (18) of § 5 and 17 of § 1 from [1] from following expressions of carriers of both equations on the interval (a, a_{m-k}) in the forms

$$-\{\operatorname{tg}\alpha, t\} = q(t), \quad -\{\operatorname{tg}\mathcal{A}(X_k), X_k(t)\} = Q(X_k(t)) \tag{13}$$

follows the relationships

$$-\{X_k, t\} - \{\operatorname{tg}\mathcal{A}, X_k\}\, X_k'^2 = -\{\operatorname{tg}\alpha, t\}, \tag{14}$$

$$\{\operatorname{tg}\mathcal{A}(X_k), t\} = \{\operatorname{tg}\alpha, t\}, \tag{15}$$

and thus, from the point of wiew of 8, § 1 of [1], also the relation

$$\operatorname{tg}\mathcal{A}(X_k) = \frac{c_{11}\operatorname{tg}\alpha(t) + c_{12}}{c_{21}\operatorname{tg}\alpha(t) + c_{22}}, \tag{16}$$

where $c_{11}, \ldots, c_{22}$ are constants. By putting the initial conditions of phases $\alpha, \mathcal{A}$ we get: $c_{12} = 0$, $c_{11} = c_{22}$, $C_{21} = 0$. From this fact follows the equality

$$\operatorname{tg}\mathcal{A}(X_k(t)) = \operatorname{tg}\alpha(t)$$

and also the validity of

$$\mathcal{A}(X_k(t)) = \alpha(t) + k\pi \operatorname{sign}\alpha' \quad \text{for } t \in (a, a_{m-k}),$$

with respect to the initial conditions (11), (12) and the fact sign $\alpha' = 1$. Consequently, $X_k(t)$ is on the interval (a, a_{m-k}) the k-th special dispersion of the equations $(q^{(1)})$, $(Q^{(1)})$ corresponding to the canonical mapping p determined by the forming phase basis $(\alpha, \mathcal{A})$.

2) Let us choose again an arbitrary point $a_i \in (a_{m-k}, b)$, $a_i \in (a^{(1)})$. Let $\bar{X}_k$ be in (a_{m-k}, b) defined regular solution of the equation $(Q^{(1)} q^{(1)})$ satisfying for $\bar{X}_k' > 0$ or $\bar{X}_k' < 0$ at the point a_i the condition (10). If we choose again a phase $\bar{\alpha}$ of $(q^{(1)})$ determined by initial conditions

$$\bar{\alpha}(a_i) = 0\,, \quad \bar{\alpha}'(a_i) = 1\,, \quad \bar{\alpha}''(a_i) = 0 \tag{17}$$

and a phase $\bar{\mathcal{A}}$ of $(Q^{(1)})$ fulfilling the conditions

$$\bar{\mathcal{A}}(\bar{X}_0) = -(m-k)\pi\,, \quad \dot{\bar{\mathcal{A}}}(\bar{X}_0) = \frac{1}{\bar{X}_0'}\,, \quad \ddot{\bar{\mathcal{A}}}(\bar{X}_0) = -\frac{\bar{X}_0''}{\bar{X}_0'^3}\,, \tag{18}$$

where $\bar{X}_0$, $\bar{X}_0'$, $\bar{X}_0''$ are values of the function $\bar{X}_k$ and its first and second derivative at the point a_i, the phase $\bar{\mathcal{A}}$ represents a directly or indirectly similar phase of some 1-fundamental basis of equation $(Q^{(1)})$. The relationships (14), (15), (16) follow also for the values of the functions $\bar{X}_k$, $\bar{\alpha}$, $\bar{\mathcal{A}}$ from the expressing of carriers of the both equations on the interval (a_{m-k}, b) in form (13). By putting the initial conditions of the phases $\bar{\alpha}, \bar{\mathcal{A}}$ we come again to the values of constants: $c_{12} = c_{21} = 0$, $c_{11} = c_{22}$. The equality

$$\operatorname{tg} \bar{\mathcal{A}}(\bar{X}_k(t)) = \operatorname{tg} \bar{\alpha}(t)$$

and also the validity of

$$\bar{\mathcal{A}}(\bar{X}_k(t)) = \bar{\alpha}(t) - (m-k)\pi \operatorname{sign} \bar{\alpha}' \quad \text{for } t \in (a_{m-k}, b)$$

follows immediately from this fact with respect to the initial conditions (17), (18) and the fact sign $\bar{\alpha}' = 1$. Thus, the function $\bar{X}_k(t)$ is on the interval (a_{m-k}, b) the k-th special dispersion of $(q^{(1)})$, $(Q^{(1)})$ corresponding to the canonical mapping p of the space r onto R determined by the forming phase basis $(\bar{\alpha}, \bar{\mathcal{A}})$.

In agreement with the statement of the Theorem 7 we can describe the structure of a set of all the k-th special dispersions of $(q^{(1)})$, $(Q^{(1)})$ corresponding to different canonical mappings of the space r onto R.

Theorem 8

A set M_k of all the k-th special dispersions X_k of equations $(q^{(1)})$, $(Q^{(1)})$ by the choice of $k \in \{1,2,\dots,m-1\}$, composed of two disjunct subsets M_{kp} and M_{kn} of direct or indirect k-th special dispersions, respectively, is a two-parametric system which we will call a bunch. The bunch of the k-th special dispersions is a one-parametric system of one-parametric subsystems M_{σ_k}, where σ is a real number, which are called bundels. Every bundel M is composed of two disjunct subbundels $M_{\sigma_{kp}}$, $M_{\sigma_{kn}}$ which are composed of only direct or only indirect k-th special dispersions of equations $(q^{(1)})$, $(Q^{(1)})$. All curves $[t, X_k(t)]$ for $X_k \in M_{kp}$ or for $X_k \in M_{kn}$ pass through m-2 common points $P(a_i, A_{i+k})$ for $i = 1,2,\dots,m-k-1$ and $P(a_{m-k+i}, A_i)$ for $i = 1,2,\dots,k-1$ or $P(a_i, A_{m-k-i})$ for $i = 1,2,\dots,m-k-1$ and $P(a_{m-k+i}, A_{m-i})$ for $i = 1,2,\dots$ $\dots,k-1$, respectively, where a_i, A_i are points of the 1-fundamental sequences $(a^{(1)})$, $(A^{(1)})$. All the curves $[t, X_k(t)]$ for $X_k \in M_{\sigma_{kp}}$ or for $X_k \in M_{\sigma_{kn}}$ and for the fixed value of the parametr σ pass through also m the common points $P(t_i, T_{i+k})$ for $i = 1,2,\dots,m-k$ and $P(t_{m-k+i}, T_i)$ for $i = 1,2,\dots,k$ or $P(t_i, T_{m-k-i+1})$ for $i = 1,2,\dots,m-k$ and $P(t_{m-k+i}, T_{m-i+1})$ for $i =$ $= 1,2,\dots,k$, respectively, where $t_i \in (a_{i-1}, a_i)$ for $i = 1,2,\dots,m$ are 1-conjugate points of $(q^{(1)})$, $T_i \in (A_{i-1}, A_i)$ for $i = 1,2,\dots,m$ are 1-conjugate points of $(Q^{(1)})$.

Proof. All the general dispersions of the equations $(q^{(1)})$, $(Q^{(1)})$ are determined by the different independent canonical mappings of the space r onto the space R. The k-th special dispersion corresponding to the mapping p is determined uniquely by the general dispersion X(t) corresponding to the mapping p. Consequently, the structure of the set M_k immediately follows from the structure of the set M, described in Theorem 11 of [5] and from the properties of the special central dispersions of the 1-st kind, mentioned in the article [2].

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Department of Biophysic
Palacký University
Dr.S.Allenda 3, 775 15 Olomouc
Czechoslovakia

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