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SIGN CONDITIONS IN NONLINEAR BOUNDARY VALUE PROBLEMS [1]

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Abstract

We consider the second order differential equation $x'' = f(t, x, x')$ with a Carathéodory nonlinearity f and nonlinear boundary conditions $g_1(x(a), x'(a)) = 0$, $g_2(x(b), x'(b)) = 0$. Using the topological degree method we prove the existence of solutions provided f, g_1, g_2 satisfy appropriate sign conditions.

Key words: Nonlinear boundary conditions, topological degree method, Continuation Theorem.

MS Classification: 34B15

1 Introduction

In the paper we study the nonlinear BVP

$$x'' = f(t, x, x') \tag{1.1}$$

$$g_1(x(a), x'(a)) = 0, \qquad g_2(x(b), x'(b)) = 0, \tag{1.2}$$

where $J = [a, b] \subset \mathbb{R}$, $f \in \mathrm{Car}(J \times \mathbb{R}^2)$, $g_1, g_2 \in C(\mathbb{R}^2)$. We show sufficient conditions for the existence of at least one solution of (1.1), (1.2). By a solution we mean a function $u \in AC^1(J)$ (having an absolutely continuous first derivative on J) and satisfying conditions (1.2) and equation (1.1) for a.e. $t \in J$.

Such questions were studied for example in [1], [2], [3]. But in [2] the appropriate linear part of (1.2) was required and in [3] the upper and lower solutions

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method was used and the monotonicity of g_1, g_2 was supposed. Our approach is close to [1], where problem (1.1), (1.2) is studied for a continuous right hand side f satisfying the Bernstein-Nagumo growth conditions and g_1, g_2 monotonous in the second variable.

Here, f, g_1, g_2 satisfy only sign conditions and neither monotonicity of g_1, g_2, nor growth conditions for f are required.

Our proofs are based on the following theorems:

Continuation Theorem [1, p. 40] *Let X, Y be Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ a Fredholm map of index 0 and $\Omega \subset X$ an open bounded set. Let $N : X \to Y$ be L-compact on $\overline{\Omega}$, $Q : Y \to Y$ a continuous projector with $\operatorname{Ker} Q = \operatorname{Im} L$ and $J : \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism. Suppose*

a) for each $\lambda \in (0,1)$ every solution x of $Lx = \lambda N x$ is such that $x \notin \partial\Omega$;

b) $QNx \neq 0$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$ and

c) the Brouwer degree $d[N_0, \Omega \cap \operatorname{Ker} L, 0] \neq 0$, where $N_0 = JQN : \operatorname{Ker} L \to \operatorname{Ker} L$.

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

Generalized Mean Value Theorem [5, p. 178] *Let $D = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$, $a_i < b_i$ and $A_i = \{x \in D : x_i = a_i\}$, $B_i = \{x \in D : x_i = b_i\}$, $i = 1, 2$, $x = (x_1, x_2)$. Further let $f : D \to \mathbb{R}^2$, $x \to (f_1(x), f_2(x))$ be continuous with $f_i(x) f_i(x') < 0$ for any $x \in A_i$, $x' \in B_i$, $i = 1, 2$. Then*

$$d[f, \operatorname{int} D, 0] = \operatorname*{sign}_{x \in B_1} f_1(x) \cdot \operatorname*{sign}_{x \in B_2} f_2(x) = \pm 1.$$

2 The existence results for bounded nonlinearity

First we will prove the existence of solutions to (1.1), (1.2) provided f is bounded by a Lebesgue integrable function φ.

Theorem 2.1 *Let $r \in (0, \infty)$ and $\varphi \in L(J)$ be such that for a.e. $t \in J$ and each $x \in [-r, r]$*

$$g_1(-r, 0)\, g_1(r, 0) < 0, \tag{2.1}$$

$$g_2(-r, 0)\, g_2(r, 0) < 0, \tag{2.2}$$

$$f(t, -r, 0) < 0, \quad f(t, r, 0) > 0, \tag{2.3}$$

$$|f(t, x, y)| \leq \varphi(t) \quad \text{for each } y \in \mathbb{R}. \tag{2.4}$$

Then problem (1.1), (1.2) has a solution u with

$$-r \leq u(t) \leq r \qquad \text{for each } t \in J. \tag{2.5}$$

To prove Theorem 1 we will study a system of auxiliary problems. Choose $n \in \mathbb{N}$ and put

$$f_n(t,x,y) = \begin{cases} f(t,r,0) & \text{for } x \geq r + 1/n \\ f(t,r,y) + [f(t,r,0) - f(t,r,y)]n(x-r) & \text{for } r < x < r + 1/n \\ f(t,x,y) & \text{for } -r \leq x \leq r \\ f(t,-r,y) - [f(t,-r,0) - f(t,-r,y)]n(x+r) & \\ & \text{for } -r - 1/n < x < -r \\ f(t,-r,0) & \text{for } x \leq -r - 1/n, \end{cases}$$

$$g_{in}(x,y) = \begin{cases} g_i(r,0) & \text{for } x \geq r + 1/n \\ g_i(r,y) + [g_i(r,0) - g_i(r,y)]n(x-r) & \text{for } r < x < r + 1/n \\ g_i(x,y) & \text{for } -r \leq x \leq r \\ g_i(-r,y) - [g_i(-r,0) - g_i(-r,y)]n(x+r) & \\ & \text{for } -r - 1/n < x < -r \\ g_i(-r,0) & \text{for } x \leq -r - 1/n, \end{cases}$$
$$i = 1,2.$$

Now, suppose that the conditions of Theorem 2.1 are fulfilled and consider the parameter system of equations

$$x'' = \lambda f_n(t,x,x'), \quad \lambda \in [0,1] \tag{2.6λ}$$

with boundary conditions

$$g_{1n}(x(a), x'(a)) = 0, \qquad g_{2n}(x(b), x'(b)) = 0. \tag{2.7}$$

To apply the Continuation Theorem for problem (2.6λ), (2.7), let us use the notation:

$$X = C^1([a,b]), \quad Y = L(a,b) \times \mathbb{R}^2, \quad \operatorname{dom} L = AC^1([a,b]) \subset X,$$

$$L : \operatorname{dom} L \to Y, \quad x \to (x'', 0, 0), \quad N : X \to Y,$$

$$x \to (f_n(\cdot, x(\cdot), x'(\cdot)), g_{1n}(x(a), x'(a)), g_{2n}(x(b), x'(b))).$$

Problem (2.6λ), (2.7) can be written in the form

$$Lx = \lambda Nx.$$

Lemma 2.2 *L is a Fredholm map of index 0.*

Proof $\operatorname{Ker} L = \{x \in X : x(t) = c(t-a) + d, \ c, d \in \mathbb{R}\}$, $\operatorname{Im} L = L(a,b) \times \{(0,0)\}$ is closed in Y, $\dim \operatorname{Ker} L = \dim \mathbb{R}^2 = \operatorname{codim} \operatorname{Im} L = 2$. □

Lemma 2.3 *For any open bounded set* $\Omega \subset X$, N *is* L*-compact on* $\overline{\Omega}$.

Proof Consider the continuous projectors

$$P : X \to X, x \to x'(a)(t-a) + x(a), \qquad Q : Y \to Y, (y, \alpha, \beta) \to (0, \alpha, \beta).$$

Then the generalized inverse (to L) operator $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ has the form

$$K_p : (y, 0, 0) \to \int_a^t \int_a^\tau y(s)\, ds d\tau.$$

Thus

$$QN : X \to Y, x \to (0, g_{1n}(x(a), x'(a)), g_{2n}(x(b), x'(b))),$$

$$K_p(I-Q)N : X \to X, x \to \int_a^t \int_a^\tau f_n(s, x(s), x'(s))\, ds d\tau.$$

The relative compactness of $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ can be shown similarly as e.g. in [6]. □

Lemma 2.4 *Let problem* (2.6λ), (2.7) *have a solution* u *for some* $\lambda \in (0, 1]$. *Then*

$$-r - 1/n \le u(t) \le r + 1/n, \quad |u'(t)| < \rho \quad \textit{for each } t \in J, \tag{2.8}$$

where

$$\rho = 2(r+2)/(b-a) + \int_a^b \varphi(t)\, dt. \tag{2.9}$$

Proof Suppose that $\max\{u(t) : t \in J\} = u(\bar{t}) > r + 1/n$. Let $\bar{t} \in (a, b)$. Then we can find $\delta > 0$ and $t_0 \ge \bar{t}$ such that

$$u'(t_0) = 0, \quad u'(t) \le 0 \quad \text{and} \quad u(t) \ge r + 1/n$$

for each $t \in (t_0, t_0 + \delta] \subset J$. Thus $\int_{t_0}^{t_0+\delta} u''(\tau)\, d\tau \le 0$. On the other hand

$$\int_{t_0}^{t_0+\delta} u''(\tau)\, d\tau = \lambda \int_{t_0}^{t_0+\delta} f(t, r, 0)\, dt > 0,$$

a contradiction. Now, for $\bar{t} = a$ we have $g_{1n}(u(a), u'(a)) = g_1(r, 0) \ne 0$ and $\bar{t} = b$ implies $g_{2n}(u(b), u'(b)) = g_2(r, 0) \ne 0$. Similar arguments lead to a contradiction provided $\min\{u(t) : t \in J\} < -r - 1/n$.

So, we have proved $-r - 1/n \le u(t) \le r + 1/n$ for each $t \in J$. And therefore we can find $t_0 \in (a, b)$ such that $|u'(t_0)| \le 2(r + 1/n)/(b-a)$. Integrating (2.6$\lambda$) from t_0 to t we get $|u'(t)| < \rho$ for each $t \in J$. □

Lemma 2.5 *For any* $n \in \mathbb{N}$ *problem (2.61), (2.7) has at least one solution* u *satisfying (2.8).*

Proof Let us put $\Omega = \{x \in X : |x(t)| < r+2, |x'(t)| < \rho$ for each $t \in J\}$. Then Lemma 2.4 implies that the condition a) of the Continuation Theorem is fulfilled. Let Q be the projection of the proof of Lemma 2.3. Then

$$QNx = (0, g_{1n}(x(a), x'(a)), g_{2n}(x(b), x'(b)))$$

for $x \in X$. Since $\operatorname{Ker} L = \{x \in X : x(t) = c(t-a)+d,\ c, d \in \mathbb{R}\}$, $\operatorname{Ker} L \cap \Omega = \{x \in X : x(t) = c(t-a)+d,\ |c(b-a)+d| < r+2,\ |d| < r+2\}$. Naturaly, the conditions $|c(b-a)+d| < r+2$, $|d| < r+2$ and $\rho > 2(r+2)/(b-a)$ imply $|x'(t)| = |c| < \rho$ for each $t \in J$.

Let us suppose $QNx = 0$ for some $x \in \operatorname{Ker} L \cap \partial\Omega$. It is equivalent to $g_{1n}(d,c) = 0$ and $g_{2n}(c\,(b-a)+d, c) = 0$ for one of four possibilities:

$$\begin{array}{lll}
\text{a)} & d = r+2, & |c\,(b-a)+d| \le r+2, \\
\text{b)} & d = -r-2, & |c\,(b-a)+d| \le r+2, \\
\text{c)} & c(b-a)+d = r+2, & |d| \le r+2, \\
\text{d)} & c(b-a)+d = -r-2, & |d| \le r+2.
\end{array}$$

But in these cases we have by (2.1) and (2.2)

$$\begin{array}{ll}
\text{a)} & g_{1n}(r+2, c) = g_1(r, 0) \ne 0, \\
\text{b)} & g_{1n}(-r-2, c) = g_1(-r, 0) \ne 0, \\
\text{c)} & g_{2n}(r+2, c) = g_2(r, 0) \ne 0, \\
\text{d)} & g_{2n}(-r-2, c) = g_2(-r, 0) \ne 0.
\end{array}$$

Thus $QNx \ne 0$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$ and the condition b) is fulfilled.

Now, put $J : \operatorname{Im} Q \to \operatorname{Ker} L, (0, \alpha, \beta) \to \alpha(t-a)+\beta$. Then $N_0 = JQN : \operatorname{Ker} L \to \operatorname{Ker} L$ has the form

$$N_0(c(t-a)+d) = g_{1n}(d,c)(t-a) + g_{2n}(c\,(b-a)+d, c).$$

Therefore, since $\{(t-a), 1\}$ is a basis for $\operatorname{Ker} L$,

$$d[N_0, \operatorname{Ker} L \cap \Omega, 0] = d[(g_{1n}(d,c), g_{2n}(c\,(b-a)+d, c)), \Gamma, 0],$$

where $\Gamma = \{(d, c\,(b-a)+d) : |c\,(b-a)+d| < r+2,\ |d| < r+2\}$. Using (2.1), (2.2) and a), b), c), d) we get by means of The Generalized Mean Value Theorem

$$d[(g_{1n}(d,c), g_{2n}(c\,(b-a)+d, c)), \Gamma, 0] = \operatorname{sign} g_{1n}(r+2, c) \cdot \operatorname{sign} g_{2n}(r+2, c) \ne 0.$$

So, the condition c) of the Continuation Theorem is satisfied and problem (2.61), (2.7) has at least one solution $u \in \operatorname{dom} L \cap \overline{\Omega}$. By Lemma 2.4 u satisfies (2.8). □

Proof of Theorem 2.1 For $n \in \mathbb{N}$ let us consider the sequence of BVPs

$$x'' = f_n(t, x, x'), \tag{2.61}$$

$$g_{1n}(x(a), x'(a)) = 0, \qquad g_{2n}(x(b), x'(b)) = 0. \tag{2.7}$$

In Lemma 2.5 we proved for any $n \in \mathbb{N}$ the existence of a solution u_n satisfying (2.8). By the Arzelà-Ascoli Theorem and the integrated form of the equation, one gets the existence of a converging subsequence of $(u_n)_1^\infty$ whose limit is a solution u of (1.1), (1.2) satisfying (2.5). □

3 The existence results for unbounded nonlinearity

Theorem 3.1 *Let r, $R \in (0, \infty)$ be such that for a.e. $t \in J$ and each $x \in [-r, r]$ the conditions (2.1), (2.2), (2.3) and*

$$f(t, x, R) > 0, \quad f(t, x, -R) < 0, \tag{3.1}$$

$$g_2(x, R) \cdot g_2(x, -R) < 0 \tag{3.2}$$

are fulfilled.

Then problem (1.1), (1.2) has at least one solution u satisfying (2.5) and

$$-R \le u'(t) \le R \qquad \text{for each } t \in J. \tag{3.3}$$

Proof Let us put

$$\tilde{f}(t, x, y) = \begin{cases} f(t, x, R) & \text{for } y > R \\ f(t, x, y) & \text{for } -R \le y \le R \\ f(t, x, -R) & \text{for } y < -R, \end{cases}$$

$$\tilde{g}_2(x, y) = \begin{cases} g_2(x, R) & \text{for } y > R \\ g_2(x, y) & \text{for } -R \le y \le R \\ g_2(x, -R) & \text{for } y < -R, \end{cases}$$

and consider the problem

$$x'' = \tilde{f}(t, x, x') \tag{3.4}$$

$$g_1(x(a), x'(a)) = 0, \qquad \tilde{g}_2(x(b), x'(b)) = 0. \tag{3.5}$$

The functions $\tilde{f}, g_1, \tilde{g}_2$ fulfil the conditions of Theorem 2.1 with

$$\varphi(t) = \sup\{|f(t, x, y)| : x \in [-r, r], y \in [-R, R]\}.$$

So, problem (3.4), (3.5) has a solution u with $-r \leq u(t) \leq r$ on J. Suppose $\max\{u'(t) : t \in J\} = u'(t_0) > R$. Let $t_0 \in [a, b)$. Then we can find $\delta > 0$ such that $R < u'(t) \leq u'(t_0)$ for each $t \in (t_0, t_0 + \delta)$. On the other hand by (3.1)

$$\int_{t_0}^{t_0+\delta} u''(\tau)\, d\tau = \int_{t_0}^{t_0+\delta} f(\tau, u(\tau), R)\, d\tau > 0,$$

a contradiction. Further $u'(b) > R$ implies $\tilde{g}_2(u(b), u'(b)) = g_2(u(b), R) \neq 0$. So $u'(t) \leq R$ for each $t \in J$. The inequality $-R \leq u'(t)$ for each $t \in J$ can be proved by similar arguments. Thus (3.3) is valid and therefore u is a solution of (1.1), (1.2) as well. □

Theorem 3.2 *Let $r, R \in (0, \infty)$ be such that for a.e. $t \in J$ and each $x \in [-r, r]$ the conditions (2.1), (2.2), (2.3) and*

$$f(t, x, R) < 0, \qquad f(t, x, -R) > 0, \tag{3.6}$$

$$g_1(x, R) \cdot g_1(x, -R) < 0 \tag{3.7}$$

are fulfilled. Then problem (1.1), (1.2) has at least one solution u satisfying (2.5) and (3.3).

Proof Theorem 3.2 can be proved similarly as Theorem 3.1. □

4 Examples

Let us show some possibilities for f satisfying the conditions of Theorem 3.1: Suppose k, $n \in \mathbb{N}$, $f_1, f_2 \in L(J)$, $f_3 \in L^\infty(J)$ and $f_i(t) > 0$ for a.e. $t \in J$, $i = 1, 2$. Then we can choose
a) f superlinear:

$$f(t, x, y) = f_1(t)x^{2k-1}e^x + f_2(t)y^{2n-1}e^y + f_3(t);$$

b) f linear:

$$f(t, x, y) = f_1(t)x + f_2(t)y + f_3(t);$$

c) f sublinear:

$$f(t, x, y) = f_1(t)\sqrt[2k-1]{x} + f_2(t)\sqrt[2n-1]{y} + f_3(t);$$

d) f nonmonotonous:

$$f_1(t)\sin(x + y) + f_2(t)y\cos y.$$

Similarly for g_i, $i = 1, 2$, we have e.g. the following possibilities. Suppose $k, n \in \mathbb{N}$, $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$, $\alpha_1 \neq 0$, $\alpha_2\beta_2 \neq 0$.

a) $g_i(x, y) = \alpha_i x^{2k-1} e^x + \beta_i y^{2n-1} e^y + \gamma_i$;

b) $g_i(x, y) = \alpha_i x + \beta_i y + \gamma_i$;

c) $g_i(x, y) = \alpha_i \sqrt[2k-1]{x} + \beta_i \sqrt[2n-1]{y} + \gamma_i$;

d) $g_i(x, y) = \alpha_i \sin(x + y) + \beta_i y \cos y$.

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