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Algebras Presented by Normal Identities

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Abstract

Let $\mathbf{p}$, $\mathbf{q}$ be terms of the same type τ. The identity $\mathbf{p} = \mathbf{q}$ is called *normal* if neither $\mathbf{p}$ nor $\mathbf{q}$ is a variable or else both $\mathbf{p}$ and $\mathbf{q}$ are the same variable. We describe a structure of algebras presented by normal identities and for algebras which have not this property we find a maximal subalgebra of a normally presented algebra satisfying the same identities. We introduce the notion of a *normal congruence* and we shall describe the lattice of all normal congruences of an algebra. Moreover, we expose the connection between *fully invariant* and *normal congruences* of the algebra of terms of a given type τ and *normal varieties* of algebras.

Key words: Normal identities, normal varieties, zero-algebra, assigned term, unary algebra.

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1 Preliminaries

Throughout the paper we suppose that τ is a given similarity type containing at least one at least unary operation symbol. Let $\mathbf{p}$, $\mathbf{q}$ be terms of type τ. An identity $\mathbf{p} = \mathbf{q}$ is said to be *normal* if neither $\mathbf{p}$ nor $\mathbf{q}$ is a variable, or else both $\mathbf{p}$ and $\mathbf{q}$ are the same variable. A term $\mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ of type τ is called *nontrivial* if it is not a single variable, otherwise it is called *trivial.* $Mod(\Sigma)$ denotes the class of all algebras $\mathbf{A}$ of type τ satisfying a given set Σ of identities of type τ. A variety $\mathcal{V}$ is *normally presented* if $\mathcal{V} = Mod(\Sigma)$, where Σ is a set of identities of a given type τ, containing only *normal identities.* Otherwise $\mathcal{V}$ is *not normally presented.* For an algebra $\mathbf{A}$, $Id(\mathbf{A})$ denotes the set of all identities satisfied in $\mathbf{A}$, $N(\mathbf{A})$ denotes the set of all *normal identities* in $Id(\mathbf{A})$. For a variety $\mathcal{V}$ we denote by $Id(\mathcal{V})$ and $N(\mathcal{V})$ the set of all (normal) identities satisfied in $\mathcal{V}$, respectively. The *nilpotent shift* of a variety $\mathcal{V}$ is the variety $\mathcal{N}(\mathcal{V}) = Mod(N(\mathcal{V}))$ (cf. [7], [14]). Hence, $\mathcal{V}$ is *normally presented* iff $\mathcal{V} = \mathcal{N}(\mathcal{V})$ or, equivalently, $Id(\mathcal{V}) = N(\mathcal{V})$. Such varieties were examined by several authors, see [5]–[15]. Some important constructions of $\mathcal{N}(\mathcal{V})$ can be found in [4], [7], [8], [15] and in the case when $\mathcal{V}$ is *congruence distributive* see also [3]. An algebra $\mathbf{A}$ is called *normal* if $Id(\mathbf{A}) = N(\mathbf{A})$ and *non-normal* otherwise.

The aim of the paper is:

(i) describe non-normal algebras;

(ii) If an algebra $\mathbf{A}$ is notnormal and $\mathbf{B} \in \mathcal{N}(V(\mathbf{A}))$, where $V(\mathbf{A})$ is the variety generated by $\mathbf{A}$, find a maximal subalgebra $\mathbf{B}^*$ of $\mathbf{B}$ satisfying $Id(\mathbf{A})$.

In the sequel, H, S, P shall denote the following operators on classes of algebras : H–taking *homomorphic images* of algebras, S–taking *subalgebras*, P–taking *products.* For an algebra $\mathbf{A}$ of type τ, $\mathcal{V}(\mathbf{A})$ denotes the variety (of type τ) generated by $\mathbf{A}$.

2 Non-normal algebras

Lemma 2.1 *Let* $\mathbf{A} = (A, F)$ *be an algebra of type* τ. *Then* $Id(\mathbf{A}) \neq N(\mathbf{A})$ *if and only if there exists a unary nontrivial term* $\mathbf{v}(\mathbf{x})$ *of type* τ *such that* $\mathbf{A} \models \mathbf{v}(\mathbf{x}) = \mathbf{x}$.

Proof *Sufficiency* is obvious.
Necessity. Let $Id(\mathbf{A}) \neq N(\mathbf{A})$. Then $\mathbf{A} \models \mathbf{p}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}_k$ for some n-ary nontrivial term $\mathbf{p}$. Setting $\mathbf{v}(\mathbf{x}) = \mathbf{p}(\mathbf{x}, \ldots, \mathbf{x})$ we conclude that $\mathbf{A} \models \mathbf{v}(\mathbf{x}) = \mathbf{x}$. □

Remark 2.1 If $\mathbf{A}$ is non-normal, then the term $\mathbf{v}(\mathbf{x})$ of Lemma 2.1 need not be uniquely determined. On the other hand, if $\mathbf{w}(\mathbf{x})$ is a nontrivial term of type τ and $\mathbf{A} \models \mathbf{w}(\mathbf{x}) = \mathbf{x}$ then clearly $\mathbf{A} \models \mathbf{w}(\mathbf{x}) = \mathbf{v}(\mathbf{x})$. Hence, $\mathbf{v}(\mathbf{x})$ is "determined uniquely up to an identity". It justifies to call $\mathbf{v}(\mathbf{x})$ the *assigned term* of $\mathbf{A}$.

Example 2.1 If $\mathbf{L}$ is a lattice, one can set $\mathbf{v}(\mathbf{x}) = \mathbf{x} \vee \mathbf{x}$ or $\mathbf{v}(\mathbf{x}) = \mathbf{x} \wedge \mathbf{x}$. Of course, $\mathbf{x} \vee \mathbf{x} = \mathbf{x} = \mathbf{x} \wedge \mathbf{x}$ in $\mathbf{L}$. If $\mathbf{G}$ is a group, one can set $\mathbf{v}(\mathbf{x}) = \mathbf{x}\mathbf{x}^{-1}\mathbf{x}$ or $\mathbf{v}(\mathbf{x}) = e\mathbf{x}$, where e is the unit of $\mathbf{G}$. Then $\mathbf{G} \models \mathbf{v}(\mathbf{x}) = \mathbf{x}$.

An endomorphism w of an algebra $\mathbf{A}$ is called *idempotent* if $w \circ w = w$ in $\mathbf{A}$, where $\circ$ denotes the superposition.

Lemma 2.2 *Let* $\mathbf{A}$ *be a non-normal algebra of type* τ. *Then the assigned term* $\mathbf{v}(\mathbf{x})$ *corresponds to an idempotent endomorphism of every algebra* $\mathbf{B} \in Mod(N(\mathbf{A}))$.

Proof Of course, the identity $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ implies $\mathbf{v}(\mathbf{v}(\mathbf{x})) = \mathbf{v}(\mathbf{x})$, which is a normal identity, i.e. it is satisfied by each $\mathbf{B} \in Mod(N(\mathbf{A}))$.

Moreover, let $f \in F$ be n-ary. Thus $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ also implies:

$$\mathbf{v}(\mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n)) = \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{f}(\mathbf{v}(\mathbf{x}_1), \ldots, \mathbf{v}(\mathbf{x}_n)). \tag{1}$$

However, such identities are normal, hence satisfied by $\mathbf{B}$. Thus $\mathbf{v}(\mathbf{x})$ is an idempotent endomorphism of $\mathbf{B}$. □

Definition 2.1 An algebra $\mathbf{A} = (A, F)$ of type τ is called zero-algebra (of type τ) if there exists an element $0 \in A$ (so-called zero of A) such that

$$f(a_1, \ldots, a_n) = 0$$

for every $f \in F$ with $\tau(f) = n$ ($n \in \boldsymbol{N}$) and all $a_1, \ldots, a_n \in A$.

For the role of *zero-algebras* in varieties and pseudovarieties (cf. [1]) see [8], [9]. The two-element zero-algebra of type τ is denoted by $\mathbf{1}_\tau$ in [7]. The following assertion is straightforward and hence the proof is omitted:

Lemma 2.3 *Let* $\mathbf{C} = (C, F)$ *be a zero-algebra of type* τ, *let* $\mathbf{p}$, $\mathbf{q}$ *be arbitrary nontrivial terms of type* τ. *Then:*

(i) $\mathbf{C} \models \mathbf{p} = \mathbf{q}$;

(ii) every equivalence on C *is a congruence of* $\mathbf{C}$;

(iii) if $\Theta \in \boldsymbol{Con}(\mathbf{C})$ *then* $\mathbf{C}/\Theta \cong \mathbf{B} \in S(\mathbf{C})$; *moreover if* $\mathbf{B} \in S(\mathbf{C})$ *then* $\mathbf{B} \cong \mathbf{C}/\Theta$ *for some* $\Theta \in \boldsymbol{Con}(\mathbf{C})$;

(iv) if $|C| > 1$ *then* $\mathbf{C}$ *contains a two-element zero-algebra* $\mathbf{1}_\tau$.

Applying Lemma 2.1, we conclude immediately:

Lemma 2.4 *Let* $\mathbf{A}$ *be an algebra of type* τ. *The following conditions are equivalent:*

(a) $Id(\mathbf{A}) \neq N(\mathbf{A})$;

(b) $Id(\mathbf{A})$ *are consequences of* $\Sigma \cup \{\mathbf{v}(\mathbf{x}) = \mathbf{x}\}$ *for a set* $\Sigma \subseteq N(\mathbf{A})$ *and for* $\mathbf{v}(\mathbf{x})$ *being an assigned term of* $\mathbf{A}$.

Proof Confront the proof of Theorem 1 of [6]. □

Theorem 2.1 *Let* $\mathbf{A}$ *be an algebra of type* τ *and* $Id(\mathbf{A}) \neq N(\mathbf{A})$. *Let* $\mathbf{B} \in \mathcal{N}(\mathcal{V}(\mathbf{A}))$. *Then* $\mathbf{B}$ *is isomorphic to a subdirect product of algebras* $\mathbf{D}$ *and* $\mathbf{C}$, *where* $\mathbf{D} \models \mathbf{v}(\mathbf{x}) = \mathbf{x}$ *for an assigned term* $\mathbf{v}(\mathbf{x})$ *of* $\mathbf{A}$ *and* $\mathbf{C}$ *is a zero-algebra of type* τ.

Proof Confront [14] and the proof of Theorem 1 of [15]. □

Example 2.2 Let $\mathbf{A} = (\{a, b, c\}, f)$ be a nonunary algebra, where f is defined as in Fig. 1:

Fig. 1

Then $v(c) \neq c$ for every nontrivial term operation v of $\mathbf{A}$. Of course, $Id(\mathbf{A}) = N(\mathbf{A})$ (in fact, $\mathbf{A} \models \mathbf{f}^3(\mathbf{x}) = \mathbf{f}(\mathbf{x})$). $\mathbf{A}$ is isomorphic to a subdirect product of algebras $\mathbf{D} = (\{z, v\}, f)$ and $\mathbf{C} = (\{g, 0\}, f)$ visualized in Fig. 2 and Fig. 3, respectively:

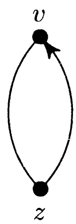

Fig. 2 Fig. 3

$\mathbf{C}$ is a zero-algebra, $a = \langle z, 0\rangle$, $b = \langle v, 0\rangle$, $c = \langle z, g\rangle$. One can verify: $\mathbf{D} \models \mathbf{f}^2(\mathbf{x}) = \mathbf{x}$ (which of course implies $\mathbf{f}^3(\mathbf{x}) = \mathbf{f}(\mathbf{x})$). $\mathbf{A}$ has no zero-algebra as a subalgebra but $\mathbf{A}/\Theta$ is a zero-algebra, where Θ is given by the partition $\{a, b\}, \{c\}$.

Corollary 2.1 *Let* $\mathbf{A}$ *be an algebra of type* τ.

(a) If there exists a homomorphic image or a subalgebra of $\mathbf{A}$ *isomorphic to a nontrivial zero-algebra, then* $Id(\mathbf{A}) = N(\mathbf{A})$.

(b) If there exists an element c *of* $\mathbf{A}$ *which is not a result of any nontrivial term operation of* $\mathbf{A}$, *then* $Id(\mathbf{A}) = N(\mathbf{A})$.

Proof A nontrivial zero-algebra does not satisfy the identity $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ for any nontrivial unary term $\mathbf{v}(\mathbf{x})$ of $\mathbf{A}$. Hence, if either a homomorphic image or a subalgebra of $\mathbf{A}$ is a nontrivial zero-algebra, then also $\mathbf{A}$ cannot satisfy any identity of the form $\mathbf{v}(\mathbf{x}) = \mathbf{x}$ for a nontrivial term $\mathbf{v}$. By virtue of Lemma 2.1, $Id(\mathbf{A}) = N(\mathbf{A})$.

If $c \in A$ and c is not a result of any nontrivial term operation of $\mathbf{A}$, then the equivalence Θ given by the partition $\{c\}$, $A - \{c\}$ is a congruence of $\mathbf{A}$ and $\mathbf{A}/\Theta$ is a two-element zero-algebra (whose zero is the class $A - \{c\} = 0$). By the formerly proved assertion (a), $Id(\mathbf{A}) = N(\mathbf{A})$. □

Remark 2.2 The converse of Corollary 2.1 does not hold in general. There exists an algebra $\mathbf{A}$ with $Id(\mathbf{A}) = N(\mathbf{A})$ but $\mathbf{A}$ has no subalgebra and no homomorphic image isomorphic with a zero-algebra, moreover, every element of $\mathbf{A}$ is a result of some nontrivial operation of $\mathbf{A}$.

Example 2.3 Let $\mathbf{A} = (\{a, b, c\}, f, g)$ be an algebra with two unary operations defined as in Fig. 4:

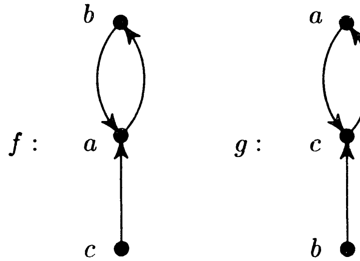

Fig. 4

Then, as can be easily checked, $Id(\mathbf{A}) = N(\mathbf{A})$. Of course, $\mathbf{A} \models \mathbf{f}^3(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, $\mathbf{g}^3(\mathbf{x}) = \mathbf{g}(\mathbf{x})$. $\mathbf{A}$ has no proper subalgebra, $\mathbf{A}$ is not a zero-algebra and $\mathbf{A}$ has only one nontrivial congruence Θ defined by the partition $\{a, b\}, \{c\}$. Moreover, $\mathbf{A}/\Theta$ is not a zero-algebra.

Remark 2.3 The property $Id(\mathbf{A}) \neq N(\mathbf{A})$ is not preserved by direct products.

Example 2.4 Let $\mathbf{A}_1 = (\{a, b\}, f, g)$, $\mathbf{A}_2 = (\{c, d\}, f, g)$ be algebras with two unary operations given in Fig. 5 and Fig. 6:

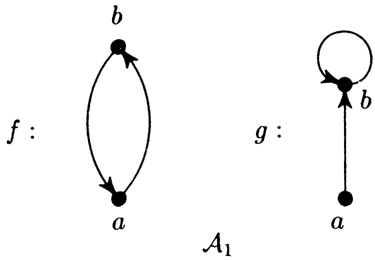

Fig. 5

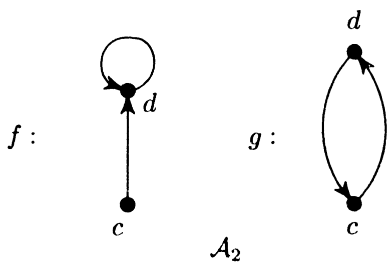

Fig. 6

Then $Id(\mathbf{A}_1) \neq N(\mathbf{A}_1)$ and $Id(\mathbf{A}_2) \neq N(\mathbf{A}_2)$ since $\mathbf{A}_1 \models \mathbf{f}^2(\mathbf{x}) = \mathbf{x}$ and $\mathbf{A}_2 \models \mathbf{g}^2(\mathbf{x}) = \mathbf{x}$. Consider the direct product $\mathbf{A} = \mathbf{A}_1 \times \mathbf{A}_2$. Then the equivalence Θ given by the partition $\{\langle a,c\rangle\}, \{\langle a,d\rangle, \langle b,c\rangle, \langle b,d\rangle\}$ is a congruence on $\mathbf{A}$ and $\mathbf{A}/\Theta$ is a two-element zero-algebra. By Corollary 2.1, $Id(\mathbf{A}) = N(\mathbf{A})$.

Theorem 2.2 *Let* $\mathbf{A}$ *be an algebra of type* τ *with* $Id(\mathbf{A}) \neq N(\mathbf{A})$. *Let* $\mathbf{B} = (B, F) \in \mathcal{N}(\mathcal{V}(\mathbf{A}))$ *and let an assigned term* $\mathbf{v}$ *of* $\mathbf{A}$ *be given. Then* $B^* = \{b \in B : v(b) = b\}$ *is a subuniverse of* $\mathbf{B}$ *and:*

(a) $\mathbf{B}^* = (B^*, F) \models Id(\mathbf{A})$;

(b) $\mathbf{B}^*$ *is a maximal subalgebra of* $\mathbf{B}$ *satisfying* $Id(\mathbf{A})$.

Proof By Lemma 2.2, the assigned term $\mathbf{v}(\mathbf{x})$ of $\mathbf{A}$ is an endomorphism of $\mathbf{B}$ thus $\mathbf{B}^*$ is a subalgebra of $\mathbf{B}$. Since $\mathbf{B} \in \mathcal{N}(\mathcal{V}(\mathbf{A}))$, we have: $\mathbf{B}^* = (B^*, F) \models N(\mathbf{A})$. By the definition of B^*, also $B^* \models \mathbf{v}(\mathbf{x}) = \mathbf{x}$, thus $\mathbf{B}^* \models Id(\mathbf{A})$ by Lemma 2.4. If $\mathbf{D}$ is a subalgebra of $\mathbf{A}$, $B^* \subseteq D \subseteq B$ and $\mathbf{D} \models Id(\mathbf{A})$, then $\mathbf{D} \models \mathbf{v}(\mathbf{x}) = \mathbf{x}$, whence $D \subseteq B^*$. Therefore, $\mathbf{B}^*$ is a maximal subalgebra of $\mathbf{B}$ satisfying $Id(\mathbf{A})$. □

Remark 2.4 The subalgebra $\mathbf{B}^*$ of $\mathbf{B}$ in Theorem 2.2 consists just of all results of all operations of all elements of $\mathbf{B}$. In other words, these elements are in the rang of at least one nontrivial term operation of $\mathbf{B}$. Of course, if $b = f(b_1, \ldots, b_n)$ for some $b_1, \ldots, b_n \in B$ then, by Lemma 2.2, $v(f(b_1, \ldots, b_n)) = f(b_1, \ldots, b_n)$. Thus $b \in B^*$.

3 Birkhoff's theorems for normal varieties

Our aim is to present a variation of Birkhoff's type theorems for *normal varieties*, via a suitable notion of *normal congruences.*

First Birkhoff's Theorem of 1935, see [2], asserts that a set Σ of *identities* of a given type τ can be represented in the form $\Sigma = Id(\mathcal{K})$, (i.e. is an *equational class*) if and only if Σ is closed under rules (1)–(5) of derivation (called Birkhoff's rules of derivation), [13], [18]. For a given set Σ of identities of type τ, $E(\Sigma)$ denotes the set of all consequences of the set Σ by means of the derivation rules.

Second Birkhoff's Theorem [2] asserts that a nonempty class of algebras of a given type τ is a *variety* (i.e. is closed under operation of passing to subalgebras, homomorphic images and also arbitrary direct products) if and only if it is an *equational class.*

Our aim is to present a variation of Birkhoff's type theorems for *normal varieties,* via a suitable notion of *normal congruence.*

Normal congruences are closely related with the notion of *normally presented variety.*

Therefore we begin with the following:

Definition 3.1 Given an algebra $\mathbf{A} = (A, F)$, a congruence Θ of $\mathbf{A}$ is called *normal* if the factor algebra $\mathbf{A}/\Theta$ is normal or trivial. Otherwise, it is called *non-normal.*

Note, that a non-normal algebra $\mathbf{A}$ has always the trivial normal congruence $A \times A$. The notion of *normal congruence* is eventually interesting only for *non normal algebras.*

Theorem 3.1 *The set of all normal congruences $\boldsymbol{NCon}(\mathbf{A})$ of an algebra $\mathbf{A}$ is a complete lattice $\boldsymbol{NCon}(\mathbf{A})$. It is a complete meet-subsemilattice of the congruences lattice $\boldsymbol{Con}(\mathbf{A})$. Moreover, the set of all normal congruences $\boldsymbol{NCon}(\mathbf{A})$ of an algebra $\mathbf{A}$ is an algebraic closure system over $A \times A$.*

Proof Obviously the intersection of two normal congruences Θ and Ψ in $\mathbf{A}$ is a normal congruence. The same statement can be easily proved for a family$\{\Theta_i; i \in I\}$ of normal congruences in $\mathbf{A}$. Therefore $\mathbf{NCon}(\mathbf{A})$ is a complete meet-subsemilattice of the lattice $\mathbf{Con}(\mathbf{A})$. Therefore the partially ordered set: $(\mathbf{NCon}(\mathbf{A}), \subseteq)$ is a complete lattice. We will show, that the join of given two normal congruences Θ and Ψ, in the lattice $\mathbf{Con}(\mathbf{A})$ i.e. $\Theta \vee \Psi$ is a normal congruence in $\mathbf{A}$.

To show this we shall associate with every normal congruence Θ of $\mathbf{A}$ the normal variety $HSP(\mathbf{A}/\Theta)$. This variety will be denoted by $\mathcal{V}_\Theta$. Given two normal congruences Θ and Ψ in $\mathbf{A}$. We will show, that:

$$\mathcal{V}_\Theta \cap \mathcal{V}_\Psi \supseteq \mathcal{V}_{\Theta \vee \Psi} .$$

Namely:

$$\begin{aligned} Id(\mathcal{V}_\Theta \cap \mathcal{V}_\Psi) &= E(Id(\mathcal{V}_\Theta) \cup Id(\mathcal{V}_\Psi)) = E(Id(\mathbf{A}/\Theta) \cup Id(\mathbf{A}/\Psi)) \\ &\subseteq Id(\mathbf{A}/(\Theta \vee \Psi)) = Id(\mathcal{V}_\Theta \vee \mathcal{V}_\Psi). \end{aligned}$$

Similarly one can show the dual:

$$\mathcal{V}_\Theta \vee \mathcal{V}_\Psi = \mathcal{V}_{\Theta \cap \Psi} .$$

The proof that $\mathbf{NCon}(\mathbf{A})$ is an algebraic closure system is basically those of Lemma 3 and Theorem 1 of [13], p. 51. Namely, it is easy to see that for a given directed family $\{\Theta_i : i \in I\}$ of normal congruences of $\mathbf{A}$:

$$\bigcup(\Theta_i : i \in I) = \bigvee(\Theta_i : i \in I).$$

Therefore $\mathbf{NCon}(\mathbf{A})$ is an algebraic closure system over $A \times A$. □

Example 3.1 We can show that the inclusion $V_\Theta \cap V_\Psi \supseteq V_{\Theta \vee \Psi}$ (mentioned in the proof of Theorem 3.1) cannot be converted in a general case. Counsider a two-element zero-algebra of type τ : $\mathbf{A} = \mathbf{1}_\tau = (\{0, 1\}, \mathrm{F})$ and its direct power $\mathbf{A} \times \mathbf{A}$. Let Θ, Φ be congruences on $\mathbf{A} \times \mathbf{A}$ given by their partitions

$$\Theta : \ \{(0,0),(0,1)\},\{(1,0),(1,1)\}$$

$$\Phi : \ \{(0,0),(1,0)\},\{(0,1),(1,1)\}.$$

Then $\Theta \vee \Phi = A \times A$ in $\mathbf{Con}(\mathbf{A} \times \mathbf{A})$ whence $V_{\Theta \vee \Phi}$ is trivial. On the other hand, V_Θ, V_Φ are not trivial since $V_\Theta = V_\Phi = \mathrm{HSP}(\mathbf{1}_\tau)$ thus also $V_\Theta \cap V_\Phi$ is non-trivial, i.e.

$$V_{\Theta \vee \Phi} \subseteq V_\Theta \cap V_\Phi \quad \text{but} \quad V_{\Theta \vee \Psi} \neq V_\Theta \cap V_\Psi.$$

Theorem 3.2 *The lattice $\boldsymbol{NCon}(\mathbf{A})$ is an algebraic lattice.*

Proof Similarly as in the proof of Theorem 10.2 [13], p. 52, for congruences, our proof follows from Theorem 6.5 of [13] and Theorem 3.1. □

Given an algebra $\mathbf{A} = (A, F)$, $E(\mathbf{A})$ denotes the set of all endomorphisms of an algebra $\mathbf{A}$. A congruence $\Theta \in \mathbf{Con}(\mathbf{A})$ is called *fully invariant* in $\mathbf{A}$ if $\Theta \in \mathbf{Con}(\mathbf{A}^+)$, where $\mathbf{A}^+ = (A, F \cup E(\mathbf{A}))$.

Theorem 3.3 (G. Grätzer) *Let $\mathbf{F}(X)$ denotes a free algebra with a free generating set X, let Θ be a congruence of $\mathbf{F}(X)$. Then the algebra $\mathbf{F}(X)/\Theta$ is a free algebra with the set $\{[x]_\Theta : x \in X\}$ of free generators if and only if Θ is fully invariant in $\mathbf{F}(X)$.*

Theorem 3.4 *Let $\mathbf{F}(X)$ denotes a free algebra with a free generating set X, let Θ be a congruence of $\mathbf{F}(X)$. Then the algebra $\mathbf{F}(X)/\Theta$ is either trivial or a normal free algebra with the set $\{[x]_\Theta : x \in X\}$ of free generators if and only if Θ is normal and fully invariant congruence in $\mathbf{F}(X)$.*

Proof The proof follows from Theorem 3.3 and the fact that the algebra $\mathbf{F}(X)/\Theta$ is either trivial or satisfies only normal identities iff Θ is normal. □

For a given class $\mathcal{K}$ of algebras of type τ, denote by $\mathcal{K}^*$ the class $\mathcal{K} \cup \{\mathbf{1}_\tau\}$.

From ([2], [13]) and he fact that algebra $\mathbf{1}_\tau$ satisfies only normal identities, we conclude immediately:

Lemma 3.1 *For every class $\mathcal{K}$ of algebras of the same type τ, all the classes:*

$$\mathcal{K}^*,\ P(\mathcal{K}^*),\ S(\mathcal{K}^*),\ H(\mathcal{K}^*),\ \mathcal{V}(\mathcal{K}^*)$$

satisfy the normal identities $N(\mathcal{K})$.

Given a class $\mathcal{K}$ of algebras of type τ, we denote by $F_\mathcal{K}(X)$ the free algebra of $\mathcal{K}$ generated by the set X of free generators.

In the sequel, a set Σ consisting only of *normal identities* will be called *normal.*

Theorem 3.5 (First G. Birkhoff's Theorem for normal identities) *A set Σ of identities of type τ can be represented in the form: $\Sigma = Id(\mathcal{K}^*)$ if and only if Σ is closed under the Birkhoff's rules of derivation (1)–(5) and Σ is normal.*

Proof The proof is similar to that of First Birkhoff's theorem, presented in [12] for regular identities. The main difference is that one should concentrate on *normal identities* instead of *regular*. Moreover, we should notice, that for a given type τ, the zero-algebra $\mathbf{1}_\tau$ satisfies exactly all the normal identities of type τ. Finally we can see that the consequences via the rules (1)–(5) of inferences of a normal set Σ of identities of type τ is normal. □

Theorem 3.6 *Let $\mathcal{K}$ be a nonempty class of algebras of type τ, $\mathbf{A}$ be an algebra of type τ and X be a set such that $|X| \geq |A|$. The following condition is satisfied:*

$$\mathbf{A} \models Id(\mathcal{K}^*) \textit{ if and only if } \mathbf{A} \in H(\mathbf{F}_{\mathcal{K}^*}(X)).$$

The proof is analogous to that of A. Tarski [17] taking in account that $Id(\mathcal{K}^*)$ are exactly all normal identities of $\mathcal{K}$

Theorem 3.7 (Second G. Birkhoff's Theorem on H,S,P,*) *For every nonempty class $\mathcal{K}$ of algebras of the same type τ,*

$$HSP(\mathcal{K}^*) = Mod(Id(\mathcal{K}^*)).$$

Proof The inclusion $\subseteq$. Given a nonempty class $\mathcal{K}$ of algebras. Let $\Sigma = Id(\mathcal{K}^*)$. From Lemma 3.1: $\mathcal{V}(\mathcal{K}^*) \models \Sigma$, i.e. $\mathcal{V}(\mathcal{K}^*) \subseteq Mod(\Sigma)$, therefore:

$$HSP(\mathcal{K}^*) = \mathcal{V}(\mathcal{K}^*) \subseteq Mod(Id(\mathcal{K}^*)).$$

The inclusion $\supseteq$. From the other hand, by Theorem 3.6, every element of $Mod(Id(\mathcal{K}^*))$ is a homomorphic image of a free algebra $\mathbf{F}_{\mathcal{K}^*}(X)$, which is an element of $\mathcal{V}(\mathcal{K}^*)$, therefore:

$$Mod(Id(\mathcal{K}^*)) \subseteq \mathcal{V}(\mathcal{K}^*) = HSP(\mathcal{K}^*).$$

Finally we conclude:

$$\mathcal{V}(\mathcal{K}^*) = HSP(\mathcal{K}^*) = Mod(Id(\mathcal{K}^*)).$$

This proves the equality $=$ in the H,S,P,* theorem of Birkhoff, because $\mathcal{K}^*$ is a (normal) variety of algebras if and only if $\mathcal{K}^* = HSP(\mathcal{K}^*)$, which follows from the well known inclusions bellow:

$$\begin{array}{ll} (1) & SH(\mathcal{K}^*) \subseteq HS(\mathcal{K}^*); \\ (2) & PH(\mathcal{K}^*) \subseteq HP(\mathcal{K}^*); \\ (3) & PS(\mathcal{K}^*) \subseteq SP(\mathcal{K}^*); \end{array}$$

and the fact that $HSP(\mathcal{K}^*)$ is obviously closed under * operation. □

References

[1] Ash, C. J.: *Pseudovarieties, Generalized Varieties and Similarly Described Classes.* Journal of Algebra **92** (1985), 104–115.

[2] Birkhoff, G.: *On the structure of abstract algebras.* Proc. Cambr. Philos. Soc. **31** (1935), 433–454.

[3] Chajda, I.: *Congruence properties of algebras in nilpotent shifts of varieties.* In: General Algebra and Discrete Mathematics, ed: K. Denecke and O. Lüders, Heldermann Verlag (1995), 35-46.

[4] Chajda, I.: *Normally presented varieties.* Algebra Universalis **34** (1995), 327–335.

[5] Chajda, I., Rosenberg, I. G.: *Remarks on Jónsson's Lemma.* Houston Journal of Mathematics **22**, 2 (1996), 249–262.

[6] Graczyńska, E.: *On bases for normal identities.* Studia Scientiarum Mathematicarum Hungarica **19** (1984), 317–319.

[7] Graczyńska, E.: *On normal and regular identities.* Algebra Universalis **27** (1990), 387–397.

[8] Graczyńska, E.: *On some operators on pseudovarities.* Contributions to General Algebra **7**, Verlag Hölder–Pichler–Tempsky, Wien, 1991, 177–184.

[9] Graczyńska, E.: *EIS for nilpotent shifts of algebras.* In: General Algebra and its Applications, ed: K. Denecke, H.-J. Vogel, Heldermann Verlag, Berlin, 1993, 116–121.

[10] Graczyńska, E.: *On some operators on pseudovarieties II.* Contributions to General Algebra **9**, Verlag Hólder–Pichler–Tempsky, Wien, 1995, 197–202.

[11] Graczyńska, E., Chajda, I.: *Jónsson's lemma for regular and nilpotent shifts of pseudovarieties.* Bulletin of the Section of Logic **26**, 2 (1997), University of Łódź, 85–93.

[12] Graczyńska, E.: *G. Birkhoff's theorems for regular varieties.* Bulletin of the section of Logic **26**, 4 (1997), University of Łódź, 210–219.

[13] Grätzer, G.: *Universal Algebra.* 1st ed., D. VAN NOSTRAND COMPANY, INC., printed in the USA, 1968.

[14] Mel'nik, I. I.: *Nilpotent shifts of varieties.* Mat. Zametki. **14**, 5 (1973), 703–712 (in Russian). English translation: Math. Notes **14** (1973), 962–966.

[15] Płonka, J.: *On the subdirect product of some equational classes of algebras.* Mathematische Nachrichten **63** (1974), 303–305.

[16] Płonka, J.: *On varieties of algebras defined by identities of some special forms.* Houston J. Math. **14** (1988), 253–263.

[17] Tarski, A.: *A remark on functionally free algebras.* Ann. of Math. **47** (1946), 163–167.

[18] Taylor, W.: *Equational logic.* Houston J. Math. **5** (1979), 1–83.