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Convex Sets in Algebras*

Radim Bělohlávek

Department of Computer Science, Faculty of Science,
Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: radim.belohlavek@upol.cz

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Abstract

We introduce the concept of a convex set in universal algebra. It is shown that convex sets generalize some well-known concepts like congruence class or convex sublattice. We study some natural properties of convex sets and outline some problems connected with the new concept.

Key words: Closed subset, term, congruence class.

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1 The concept of convex set

Several special subsets play special roles in classical algebras. For example, subgroups, normal subgroups or cosets of groups, ideals in rings and lattices etc. A considerable part of universal algebra deals with general approaches to various subsets of algebras like those above. Typical examples are subalgebras or congruence classes. In the following we are going to propose a definition of another type of subset of a general algebra. These subsets will be called convex sets.

As we will see, the concept of convex set generalizes some important special subsets. For example: (1) each congruence class is a convex set (thus, e.g. cosets of groups are convex sets); (2) moreover, in permutable varieties, the concepts of a convex set and that of a congruence class coincide (thus, e.g. each convex

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set of a group is, conversely, a coset); (3) in case of lattices, convex subsets are precisely convex sublattices.

We will deal with terms whose variables will be divided into two groups. We will handle this by writing $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$, denoting the variables of the first group by x_i and the variables of the second group by y_j. We will also use the abbreviations $\mathbf{x}$ and $\mathbf{y}$ for $\langle x_1, \ldots, x_n \rangle$ (or $x_1, \ldots, x_n$) and $\langle y_1, \ldots, y_m \rangle$ (or $y_1, \ldots, y_m$), respectively (if n and m are obvious from context).

Definition 1 Let $\mathcal{K}$ be a class of algebras of the same type. An $(n+m)$-ary term $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ $(n, m \geq 0)$ is called a *$\mathcal{K}$-convex term in $y_1, \ldots, y_m$* if the identity $p(x_1, \ldots, x_n, y, \ldots, y) \approx y$ holds in $\mathcal{K}$. For $\mathbf{A} \in \mathcal{K}$, a subset $C \subseteq A$ is called a *$\mathcal{K}$-convex set (in $\mathbf{A}$)* if for each term $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ which is a $\mathcal{K}$-convex term in $y_1, \ldots, y_m$ and every $a_1, \ldots, a_n \in A$, $c_1, \ldots, c_m \in C$, it holds $p(a_1, \ldots, a_n, c_1, \ldots, c_m) \in C$.

Remark 1 (1) If $\mathcal{K}$ is obvious from context (or is not important), we omit the prefix "$\mathcal{K}$-" (e.g. we use only "convex term" instead of "$\mathcal{K}$-convex term"). In similar way we omit any subscripts, superscripts, etc. referring to $\mathcal{K}$.

(2) Note that the fact that $p(x_1, \ldots, x_n, y, \ldots, y) \approx y$ holds in $\mathcal{K}$ means that for each $\mathbf{A} \in \mathcal{K}$ and every $a_1, \ldots, a_n, c \in A$ it holds $p(a_1, \ldots, a_n, c, \ldots, c) = c$. If for every $a_1, \ldots, a_n \in A$, $c_1, \ldots, c_m \in C$, it holds $p(a_1, \ldots, a_n, c_1, \ldots, c_m) \in C$, we say that C is y-closed under p. Therefore, convex sets are just sets which are y-closed under all convex terms.

(3) The set of all $\mathcal{K}$-convex terms in $y_1, \ldots, y_m$ will be denoted by $\mathrm{CT}_{\mathcal{K}}(y_1, \ldots, y_m)$. The set of all $\mathcal{K}$-convex sets in $\mathbf{A} \in \mathcal{K}$ will be denoted by $\mathrm{CS}_{\mathcal{K}}(\mathbf{A})$. If there is no danger of confusion we write only $\mathrm{CT}_{\mathcal{K}}$ instead of $\mathrm{CT}_{\mathcal{K}}(y_1, \ldots, y_m)$. Thus e.g. $p(\mathbf{x}, \mathbf{y}) \in \mathrm{CT}_{\mathcal{K}}$ means that p is a convex term in $\mathbf{y}$. Also, $\mathrm{CT}_{\mathcal{K}}$ will denote the set of all $\mathcal{K}$-convex terms.

(4) The notions of a convex term and a convex set are inspired by the notions of an ideal term and an ideal (see e.g. [1, 2, 8]): Let the class $\mathcal{K}$ contain an equationally definable constant 0. A term $p(\mathbf{x}, \mathbf{y})$ is called a *$\mathcal{K}$-ideal term in* $\mathbf{y}$ if $p(\mathbf{x}, 0, \ldots, 0) \approx 0$ holds in $\mathcal{K}$. For $\mathbf{A} \in \mathcal{K}$, a subset $I \subseteq A$ is called a $\mathcal{K}$-ideal if it is y-closed under all $\mathcal{K}$-ideal terms. Symbols $\mathrm{IT}_{\mathcal{K}}(y_1, \ldots, y_m)$ and $\mathrm{I}_{\mathcal{K}}(\mathbf{A})$ denote the set of all $\mathcal{K}$-ideal terms in $y_1, \ldots, y_m$, and the set of all $\mathcal{K}$-ideals in $\mathbf{A}$.

(5) Some remarks are now in order. First, the notion of an ideal is not applicable in algebras without (equationally definable) constants. For example, lattice ideals are defined without the need of existence of 0, they are therefore not approachable via the universal algebraic notion of ideal. On the other hand, as we will see, the notion of a convex set, when applied to lattices, yields the notion of a convex (in the sense of order convexity) subuniverse. Therefore, lattice ideals (since they are convex subuniverses) are special cases of convex sets. In lattices with 0, however, a subset is a lattice ideal iff it is an ideal (in the universal algebraic sense). In this case, lattice ideals are precisely convex sets containing 0. Second, even in algebras with 0, other subsets than those with 0 may exhibit behaviour similar to that of ideals. The notion of a convex set seems appropriate in this respect. Moreover, although we come to convex

terms by replacing "0" by "y" in the definition of ideal term, the constants 0 play usually some specific role in classical algebras. This leads to the fact that not each ideal term is a convex term. In this respect, convex sets are to be expected to behave as "ideals unaffected by the special role of 0". Third, there are points of departure from the line pursued in the study of ideals, i.e. there are phenomena which do not exist in the case of ideals (we will see more on this topic later).

As we will see, in several cases the condition "being y-closed under all convex terms" is equivalent to "being y-closed under certain convex terms". This leads to the following definition.

Definition 2 A subset $B \subseteq \mathrm{CT}_{\mathcal{K}}$ is called a *basis* of $\mathrm{CT}_{\mathcal{K}}$ if for each $\mathbf{A} \in \mathcal{K}$ and any $C \subseteq A$ it holds that if C is y-closed under all terms of B then it is $\mathcal{K}$-convex.

2 Relations to other subsets and examples

First, we give some examples of convex terms.

Example 1

1. $p(y) = y$ is a $\mathcal{K}$-convex term (a trivial one) in y for any $\mathcal{K}$.
2. If $\mathcal{K}$ is any class of lattices then
$$p(y_1, y_2) = y_1 \vee y_2, \qquad p(x_1, y_1, y_2) = y_1 \vee (y_2 \wedge x_1)$$
are $\mathcal{K}$-convex terms (in y's).
3. $p(y_1, y_2, y_3) = y_1 y_2^{-1} y_3$ is a $\mathcal{K}$-convex term for any class $\mathcal{K}$ of groups.

Except for a trivial case, there is an infinite number of convex terms.

Proposition 1 *$\mathrm{CT}_{\mathcal{K}}$ is closed under y-composition in that if $p(\vec{x}, y_1, \ldots, y_m)$, $p_j(\mathbf{x}_j, \mathbf{y}_j)$, for $j = 1, \ldots, m$, are from $\mathrm{CT}_{\mathcal{K}}$ then*
$$p(\mathbf{x}, p_1(\mathbf{x}_1, \mathbf{y}_1), \ldots, p_m(\mathbf{x}_m, \mathbf{y}_m)) \in \mathrm{CT}_{\mathcal{K}}.$$
Therefore, if $\mathrm{CT}_{\mathcal{K}}$ contains at least one non-trivial term (i.e. different from y) then $\mathrm{CT}_{\mathcal{K}}$ is infinite.

Proof Let $\mathbf{y}_j = \langle y, \ldots, y \rangle$ (for all $j = 1, \ldots, m$). Then $p_j(\vec{x}_j, \mathbf{y}_j) \approx y$ holds in $\mathcal{K}$ and since also $p(\mathbf{x}, y, \ldots, y) \approx y$ holds in $\mathcal{K}$, we have that
$$p(\mathbf{x}, p_1(\mathbf{x}_1, \mathbf{y}_1) \ldots, p_m(\mathbf{x}_m, \mathbf{y}_m)) \approx p(\mathbf{x}, y, \ldots, y) \approx y$$
holds in $\mathcal{K}$. The rest is clear. □

Proposition 2 *For any $\mathcal{K}$ and each $\mathbf{A} \in \mathcal{K}$, $\emptyset$, $\{a\}$ (for any $a \in A$), and A are convex sets.*

Proof The fact that $\emptyset$ is a convex set follows directly from the definition (this is easier seen if the condition to be satisfied by convex sets is rewritten as $(\forall \mathbf{a}, \mathbf{c})(\mathbf{a} \in A^n, \mathbf{c} \in C^m \Rightarrow p(\mathbf{a}, \mathbf{c}) \in C)$). Any $\{a\}$ is convex since $p(a_1, \ldots, a_n, a, \ldots, a) = a \in \{a\}$ holds for every convex term p (by definition). The last statement is clear. □

Recall that an algebra $\mathbf{A}$ of type $\langle F, \sigma \rangle$ is called *idempotent* if $f(x, \ldots, x) \approx x$ holds in $\mathbf{A}$ for each $f \in F$. A class $\mathcal{K}$ of algebras is idempotent if each $\mathbf{A} \in \mathcal{K}$ has this property.

Proposition 3 *Let $\mathcal{K}$ be a nonempty class of algebras of the same type. Then $\mathcal{K}$ is idempotent iff $\mathrm{CS}_{\mathcal{K}}(\mathbf{A}) \subseteq \mathrm{Sub}(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{K}$.*

Proof Let $\mathcal{K}$ be idempotent, $\mathbf{A} \in \mathcal{K}, C \in \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$. Idempotency of $\mathcal{K}$ yields that for each $f \in F$ ($\langle F, \sigma \rangle$ denotes the type), $f(y_1, \ldots, y_m)$ is a convex term. As a convex set, C is, by definition, closed under f. Thus, $C \in \mathrm{Sub}(\mathbf{A})$. Conversely, if $\mathrm{CS}_{\mathcal{K}}(\mathbf{A}) \subseteq \mathrm{Sub}(\mathbf{A})$, then since $\{a\} \in \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$, we have $\{a\} \in \mathrm{Sub}(\mathbf{A})$. Therefore, for each $f \in F$, $f(a, \ldots, a) \in \{a\}$, i.e. $f(a, \ldots, a) = a$, thus $f(x, \ldots, x) \approx x$ holds in each $\mathbf{A} \in \mathcal{K}$ which means that $\mathcal{K}$ is idempotent. □

Remark 2 In general, neither $\mathrm{CS}_{\mathcal{K}}(\mathbf{A}) \subseteq \mathrm{Sub}(\mathbf{A})$ nor $\mathrm{Sub}(\mathbf{A}) \subseteq \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$ holds. Groups serve as a counterexample: As we will see later on, in the case of groups, convex sets coincide with congruence classes. Now, neither each subgroup is a congruence class, nor each congruence class is a subgroup.

Let θ be a binary relation on a set A. For any $C \subseteq A$ we put

$$C^\theta = \{a \in A \mid \langle a, c \rangle \in \theta \text{ for some } c \in C\}$$

and call C^θ a *θ-closure* of C. If $C = \{c\}$, C^θ is usually denoted by $[c]_\theta$ and called the class of θ determined by c.

Proposition 4 *Let θ be a reflexive compatible binary relation on an algebra $\mathbf{A} \in \mathcal{K}$. For any $C \in \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$, $C^\theta \in \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$. In particular, each relational class $[a]_\theta$ is a convex set.*

Proof Let $p(x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathrm{CT}_{\mathcal{K}}$, $a_1, \ldots, a_n \in A$, $c_1, \ldots, c_m \in C^\theta$. Then there are $c'_j \in C$ such that $\langle c_j, c'_j \rangle \in \theta$ $(j = 1, \ldots, m)$. Reflexivity and compatibility of θ yields $\langle p(a_1, \ldots, a_n, c_1 \ldots, c_m), p(a_1, \ldots, a_n, c'_1 \ldots, c'_m) \rangle \in \theta$. Furthermore, $p(a_1, \ldots, a_n, c_1 \ldots, c_m) \in C$ by convexity of C. Hence, $p(a_1, \ldots, a_n, c'_1 \ldots, c'_m) \in C^\theta$, proving the convexity of C^θ. The rest follows by Proposition 2. □

Next we describe the basic relation to ideals.

Proposition 5 *Let $\mathcal{K}$ have an equationally definable constant* 0. *Then each convex term is an ideal term, i.e.* $\mathrm{CT}_{\mathcal{K}} \subseteq \mathrm{IT}_{\mathcal{K}}$, *and each ideal is a convex set, i.e.* $\mathrm{I}_{\mathcal{K}}(\mathbf{A}) \subseteq \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$.

Proof Clearly, validity of $p(x_1, \ldots, x_n, y, \ldots, y) \approx y$ implies validity of $p(x_1, \ldots, x_n, 0, \ldots, 0) \approx 0$. The rest is obvious. □

The converse to Proposition 5 (i.e. each ideal term is a convex term, and each convex set containing 0 is an ideal) does not hold. The following is a counterexample.

Example 2 Let $F = \{0, g\}$ where 0 is nullary and g is unary. Consider an algebra $\mathbf{A} = \langle A, F \rangle$, where $A = \{0, a, b\}$ and g is given by

	g
0	0
a	b
b	a

and put $\mathcal{K} = \{\mathbf{A}\}$. The terms are of the form $g^k(y)$ or $g^k(0)$ for $k = 0, 1, 2, 3, \ldots$ (where $g^0(y) = y$). Clearly, each $g^k(y)$ is an ideal term in y ($g^k(0) \approx 0$ holds in $\mathbf{A}$). A moment reflection shows that $g^k(y)$ is a convex term in y iff k is even. Therefore, $\mathrm{IT}_{\mathcal{K}} \not\subseteq \mathrm{CT}_{\mathcal{K}}$. Moreover, $\{0, a\}$ is a convex set containing 0 which is not an ideal (in fact, $g(a) = b \notin \{0, a\}$, i.e. $\{0, a\}$ is not closed under the ideal term g).

Convex sets in "classical algebras" We are now going to show examples of convex sets in some classical algebras.

The first interesting case is that of algebras from permutable varieties: convex sets coincide with congruence classes in algebras from permutable varieties. We will see this (and more) in Section 3.

Let us focus on lattices. Denote by $\mathcal{L}$ the variety of all lattices. If $\mathbf{L} = \langle L, \wedge, \vee \rangle$ is a lattice, a subset $C \subseteq L$ will be called *o-convex* if for every $c_1, c_2 \in C$ and each $a \in L$ we have $a \in C$ whenever $c_1 \leq a \leq c_2$. Denote

$$p_1(x_1, y_1, y_2) = y_1 \vee (y_2 \wedge x_1) \quad (1)$$
$$p_2(y_1, y_2) = y_1 \wedge y_2. \quad (2)$$

Clearly, both p_1 and p_2 are $\mathcal{L}$-convex terms.

Proposition 6 *Let* $\mathbf{L}$ *be a lattice,* $C \subseteq L$. *Then* C *is an* $\mathcal{L}$*-convex set iff* C *is an o-convex subuniverse. Moreover, terms (1) and (2) form a basis of* $\mathrm{CT}_{\mathcal{L}}$.

Proof First, we prove (*): a subset $C \subseteq L$ is an o-convex subuniverse of $\mathbf{L}$ iff it is y-closed under (1) and (2).

Let C be an o-convex subuniverse, $a \in L$, $c_1, c_2 \in C$. Since C is a subuniverse, we have $c_1 \wedge c_2 \in C$, $c_1 \vee c_2 \in C$. Clearly, $c_1 \wedge c_2 \leq p_1(a, c_1, c_2), p_2(c_1, c_2)$ and $p_1(a, c_1, c_2), p_2(c_1, c_2) \leq c_1 \vee c_2$. o-convexity of C then yields $p_1(a, c_1, c_2), p_2(c_1, c_2) \in C$. Thus, C is y-closed under both p_1 and p_2.

Let, on the other hand, C be y-closed under p_1 and p_2. Putting $x_1 = y_2$, we get that C is y-closed under $p_1(y_2, y_1, y_2)$. Since $p_1(y_2, y_1, y_2) \approx y_1 \vee y_2$ holds in $\mathcal{L}$ and $p_2(y_1, y_2) = y_1 \wedge y_2$ we get that C is a subuniverse. Now, let $a \in L$, $c_1, c_2 \in C$ be such that $c_1 \leq a \leq c_2$. Then since C is y-closed under p_1, we get $a = p_1(a, c_1, c_2) \in C$, i.e. C is o-convex, provin (*).

Now, let C be an $\mathcal{L}$-convex subset. By definition, it is y-closed under all $\mathcal{L}$-convex terms, thus also under (1) and (2). By (*), C is an o-convex subuniverse.

Conversely, let C be an o-convex subuniverse. Let $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be an $\mathcal{L}$-convex term. Take any $a_1, \ldots, a_n \in L$, $c_1, \ldots, c_m \in C$. Put $c^\wedge = c_1 \wedge \cdots \wedge c_m$, $c^\vee = c_1 \vee \cdots \vee c_m$. Since C is a subuniverse, we have $c^\wedge, c^\vee \in C$. Furthermore, since p is $\mathcal{L}$-convex in $y_1, \ldots, y_m$ we have $p(a_1, \ldots, a_n, c^\wedge, \ldots, c^\wedge) = c^\wedge$ and $p(a_1, \ldots, a_n, c^\vee, \ldots, c^\vee) = c^\vee$. It is well-known that each term function in a lattice is monotonic, we get from $c^\wedge \leq c_1, \ldots, c_m$ and $c_1, \ldots, c_m \leq c^\vee$ that

$$p(a_1, \ldots, a_n, c^\wedge, \ldots, c^\wedge) \leq p(a_1, \ldots, a_n, c_1, \ldots, c_m) \leq p(a_1, \ldots, a_n, c^\vee, \ldots, c^\vee).$$

Since C is o-convex, we finally get $p(a_1, \ldots, a_n, c_1, \ldots, c_m) \in C$, i.e. C is an $\mathcal{L}$-convex set. □

Remark 3 (1) A direct consequence of Proposition 4 and Proposition 6 is the well-known fact that each congruence class in a lattice is an o-convex sublattice.

(2) We will see later on that in the case of algebras from permutable and regular varieties there is always a finite basis of convex terms. Note that the variety of all lattices is neither permutable nor regular.

(3) Recall that an ideal in a lattice $\mathbf{L}$ is a (nonempty) subset $I \subseteq L$ such that $i_1 \vee i_2 \in I$ for any $i_1, i_2 \in I$, and $a \in I$ whenever $a \leq i$ for some $i \in I$. Clearly, each lattice ideal is an o-convex subuniverse of L, hence by Proposition 6, an $\mathcal{L}$-convex set. It is also clear that if $\mathbf{L}$ has the least element 0 then lattice ideals in $\mathbf{L}$ are precisely $\mathcal{L}$-convex sets containing 0.

3 Convex sets in permutable and regular varieties

Since each congruence class is a convex set, it is natural to ask about the converse: Under what conditions is each convex set a class of *at least* one congruence? It will turn out that a sufficient condition is that the algebra in question is from a so-called permutable variety. Recall that an algebra $\mathbf{A}$ is called *permutable* (or congruence permutable) if

$$\theta \circ \phi = \phi \circ \theta$$

holds for all $\theta, \phi \in \mathrm{Con}(\mathbf{A})$. A variety $\mathcal{V}$ is called permutable if each $\mathbf{A} \in \mathcal{V}$ is permutable. The following characterization of permutable varieties was obtained by Mal'cev [6]: A variety $\mathcal{V}$ is permutable iff there is a ternary term $p(x, y, z)$ such that $x \approx p(x, z, z)$ and $p(x, x, z) \approx z$ hold in $\mathcal{V}$. The term p is called a Mal'cev term of $\mathcal{V}$. The following is another result from [6]: A nonempty subset $C \subseteq A$ of an algebra $\mathbf{A}$ is a congruence class of $\mathbf{A}$ (i.e. there are $\theta \in \mathrm{Con}(\mathbf{A})$ and $a \in A$ such that $C = [a]_\theta$) iff for each unary algebraic function τ it holds $\tau(C) \subseteq C$ or $\tau(C) \cap C = \emptyset$.

Denote by $\mathrm{CC}(\mathbf{A})$ the system of all congruence classes of the algebra $\mathbf{A}$ plus the empty set, i.e.

$$\mathrm{CC}(\mathbf{A}) = \{\emptyset\} \cup \{[a]_\theta \mid a \in A,\, \theta \in \mathrm{Con}(\mathbf{A})\}.$$

For term $t(x_1, \ldots, x_n, y)$ and a Mal'cev term p of a permutable variety $\mathcal{V}$, put

$$p_t(x_1, \ldots, x_n, y_1, y_2, y_3) = p(y_1, t(x_1, \ldots, x_n, y_2), t(x_1, \ldots, x_n, y_3))$$

and

$$B = \{p_t(x_1, \ldots, x_n, y_1, y_2, y_3) \mid t(x_1, \ldots, x_n, y_2) \text{ is a term}\}.$$

Lemma 7 *Let $\mathcal{V}$ be a permutable variety, $\mathbf{A} \in \mathcal{V}$, and $\emptyset \neq C \subseteq A$. Then C is a congruence class of $\mathbf{A}$ iff it is y-closed under all terms of B.*

Proof Let C be y-closed under the terms of B. To prove that C is a congruence class, we use the above Mal'cev characterization of congeuence classes. Let $\tau(x) = t(a_1, \ldots, a_n, x)$ be a unary algebraic function, $c \in C$, and $\tau(c) \in C$. We have to check that for any $d \in C$ it holds $\tau(d) \in C$. Since C is y-closed under p_t, we get $\tau(d) = p(\tau(c), \tau(c), \tau(d)) = p_t(a_1, \ldots, a_n, \tau(c), c, d) \in C$.

Conversely, if C is a congruence class then it is, by Proposition 4, y-closed under all $\mathcal{V}$-convex terms. A moment reflection shows that all terms of B are $\mathcal{V}$-convex in y. The conclusion therefore follows. □

Proposition 8 *Let $\mathcal{V}$ be a permutable variety. Then B is a basis of $\mathrm{CT}_{\mathcal{V}}$ and each $\mathcal{V}$-convex set of any algebra $\mathbf{A} \in \mathcal{V}$ is a congruence class.*

Proof If a subset C of an algebra $\mathbf{A} \in \mathcal{V}$ is y-closed under all terms from B, then, by Lemma 7, it is a congruence class and thus, by Proposition 4, a $\mathcal{V}$-convex set. Therefore, B is a basis of $\mathrm{CT}_{\mathcal{V}}$. The rest immediately follows. □

Another question we ask is: Under what conditions is each convex set a class of *at most* one congruence? A sufficient condition is that of regularity. Recall that an algebra $\mathbf{A}$ is called *regular* if $[a]_\theta = [a]_\phi$ implies $\theta = \phi$ for all $\theta, \phi \in \mathrm{Con}(\mathbf{A})$ and each $a \in A$, i.e. two congruences are the same whenever they have a common class. A variety $\mathcal{V}$ is called regular if each $\mathbf{A} \in \mathcal{V}$ is regular. The following claim is immediate.

Proposition 9 *An algebra $\mathbf{A}$ is regular iff each convex set in $\mathbf{A}$ is a class of at most one congruence of $\mathbf{A}$.*

A combination of the two above questions is: Under what conditions is each non-empty convex set a class of *exactly* one congruence? We say that an algebra $\mathbf{A}$ has *congruences determined by convex sets* (CDCS for short) , if each non-empty convex set of $\mathbf{A}$ is a class of exactly one congruence of $\mathbf{A}$. A variety $\mathcal{V}$ has CDCS if each $\mathbf{A} \in \mathcal{V}$ has CDCS.

If $\mathbf{A}$ has CDCS, then we denote the congruence determined by a convex set C by θ_C (i.e. θ_C is the unique congruence satisfying $[c]_{\theta_C} = C$ for some $c \in C$). For a fixed $a \in A$ we denote by $\mathrm{CS}^a(\mathbf{A})$ the set of all convex sets of a given algebra $\mathbf{A}$, i.e.

$$\mathrm{CS}^a(\mathbf{A}) = \{C \in \mathrm{CS}(\mathbf{A}) \mid a \in C\}.$$

Lemma 10 *If $\mathbf{A}$ has CDCS, then $\mathrm{CS}^a(\mathbf{A})$ and $\mathrm{Con}(\mathbf{A})$ are (under set inclusion) isomorphic algebraic lattices for any $a \in A$.*

Proof Since $\mathbf{A}$ has CDCS, the mapping sending C to θ_C is clearly bijective. Furthermore, it is also order-preserving, i.e. $C \subseteq D$ implies $\theta_C \subseteq \theta_D$. Indeed, suppose $\theta_C \not\subseteq \theta_D$. Put $\theta = \theta_C \cap \theta_D$. Then C is a class of θ (since C is both a class of θ_C and a subset of D which is a class of θ_D), but $\theta \neq \theta_C$ (since we assumed $\theta_C \not\subseteq \theta_D$). Therefore, C is a class of two different congruences, a contradiction to CDCS. □

Proposition 11 *Any algebra with CDCS is permutable.*

Proof Let $\theta, \phi \in \mathrm{Con}(\mathbf{A})$, $\mathbf{A}$ be an algebra with CDCS. We have to show that $\theta \circ \phi = \phi \circ \theta$, i.e. that $(\{a\}^\theta)^\phi = (\{a\}^\phi)^\theta$ for any $a \in A$. Since $\{a\}^\theta, \{a\}^\phi \subseteq (\{a\}^\theta)^\phi$, we have $\{a\}^\theta \vee \{a\}^\phi \subseteq (\{a\}^\theta)^\phi$. Conversely, if $b \in (\{a\}^\theta)^\phi$ then for some $c \in \{a\}^\theta$ it holds $\langle b, c\rangle \in \phi = \theta_{\{a\}^\phi} \subseteq \theta_{\{a\}^\theta \vee \{a\}^\phi}$ (we use Lemma 10), i.e. $b \in \{c\}^{\theta_{\{a\}^\theta \vee \{a\}^\phi}} = \{a\}^\theta \vee \{a\}^\phi$. To sum up, $(\{a\}^\theta)^\phi = \{a\}^\theta \vee \{a\}^\phi$. We therefore infer $(\{a\}^\theta)^\phi = \{a\}^\theta \vee \{a\}^\phi = \{a\}^\phi \vee \{a\}^\theta = (\{a\}^\phi)^\theta$, i.e. $\mathbf{A}$ is permutable. □

Proposition 12 *A variety $\mathcal{V}$ has CDCS iff $\mathcal{V}$ is both permutable and regular.*

Proof If $\mathcal{V}$ has CDCS then it is permutable (by Proposition 11) and regular (by Proposition 9). Conversely, permutability and regularity of $\mathcal{V}$ implies (by Proposition 8 and Proposition 9) CDCS. □

By well-known Mal'cev type characterizations of permutability and regularity [4, 6, 9] we get the following corollary.

Corollary 13 *For a variety $\mathcal{V}$, the following conditions are equivalent:*

(1) $\mathcal{V}$ has CDCS.

(2) There exist a ternary term p, an integer $n \geq 1$, and ternary terms $t_1, \dots, t_n$, such that

$$p(x,z,z) \approx x, \quad p(x,x,z) \approx z$$
$$[t_1(x,y,z) \approx z, \dots, t_n(x,y,z) \approx z] \quad \textit{iff} \quad x \approx y$$

are valid in $\mathcal{V}$.

(3) There exist an integer $n \geq 1$, a (3+n)-ary term q, and ternary terms $t_1, \ldots, t_n$, such that

$$t_1(x,x,z) \approx z, \ldots, t_n(x,x,z) \approx z$$
$$x \approx q(x,y,z,t_1(x,y,z),\ldots,t_n(x,y,z))$$
$$y \approx q(x,y,z,z,\ldots,z)$$

are valid in $\mathcal{V}$.

The following proposition has been obtained in [3] ($t_1, \ldots, t_n, q$ denote the terms of Proposition 13 (3)).

Proposition 14 *Let $\mathcal{V}$ be a regular and permutable variety, let $\mathbf{A} = \langle A, F\rangle \in \mathcal{V}$ and $\emptyset \neq C \subseteq A$. Then C is a congruence class of some $\theta \in \mathrm{Con}(\mathbf{A})$ if and only if C is y-closed under the following terms:*

(a) $t_i(f(q(x_1, x'_1, y, y_{11}, \ldots, y_{1n}), \ldots, q(x_m, x'_m, y, y_{m1}, \ldots, y_{mn})),$
$f(x'_1,, \ldots, x'_m), y)$
for each m-ary $f \in F$ and every $i = 1, \ldots, n$;

(b) $q(x, y, y', y_1, \ldots, y_n)$

(c) $r_i(y_1, y_2) = t_i(y_1, y_2, y_2)$ *for $i = 1, \ldots, n$.*

As a corollary, we get that the following.

Proposition 15 *if $\mathcal{V}$ is a regular and permutable variety of a finite type then the terms in (a)–(c) of Proposition 14 form a finite basis of $\mathrm{CT}_{\mathcal{K}}$.*

Proof Clearly, the terms of Proposition 14 are $\mathcal{V}$-convex in y's. Each $\mathcal{V}$-convex set is therefore y-closed under those terms. Conversely, every subset y-closed under those terms is, by Proposition 14 a congruence class, and therefore, by Proposition 4 a $\mathcal{V}$-convex set. □

4 Some further properties and directions

The lattice of convex sets

Proposition 16 *For any class $\mathcal{K}$ of algebras of the same type and any $\mathbf{A} \in \mathcal{K}$*

(1) $\mathrm{CS}_{\mathcal{K}}(\mathbf{A})$ is an algebraic closure system, the closure operator (sending X to $[X]$) is given by

$$[X] = \{p(\mathbf{a}, \mathbf{c}) \mid p(\mathbf{x}, \mathbf{y}) \in \mathrm{CT}_{\mathcal{K}}(\mathbf{y}), \mathbf{a} \in A, \mathbf{c} \in X\}.$$

(2) $\mathrm{CS}_{\mathcal{K}}(\mathbf{A})$ is under set inclusion an atomic algebraic lattice. The least and the greatest elements are $\emptyset$ and A, the atoms are just the singletons $\{a\}$ $(a \in A)$.

Proof (1) It is routine to check that $\mathrm{CS}_{\mathcal{K}}(\mathbf{A})$ is closed under arbitrary intersections, and hence, that it is a closure system. Take any $X \subseteq A$ and denote

$$(X) = \{p(\mathbf{a},\mathbf{c}) \mid p(\mathbf{x},\mathbf{y}) \in \mathrm{CT}_{\mathcal{K}}(\mathbf{y}),\ \mathbf{a} \in A, \mathbf{c} \in X\}.$$

Since $p(y) = y$ is a convex term, it holds $X \subseteq (X)$. Clearly, any convex set in $\mathbf{A}$ which contains X contains also (X). Furthermore, (X) itself is a convex set. Indeed, if $p(x_1,\dots,x_n,y_1,\dots,y_m)$ is a convex term, $a_1,\dots,a_n \in A$, $c_1,\dots,c_m \in (X)$, i.e. $c_i = p_i(\mathbf{a}_i,\mathbf{c}_i)$ for some convex term p_i, and some $\mathbf{a}_i \in A$, $\mathbf{c}_i \in X$ ($i = 1,\dots,m$), then since $p(x_1,\dots,x_n,p_1(\mathbf{x}_1,\mathbf{y}_1),\dots,p_m(\mathbf{x}_m,\mathbf{y}_m))$ is a convex term (by Proposition 1), we get $p(a_1,\dots,a_n,c_1,\dots,c_m) = p(a_1,\dots,a_n,p_1(\mathbf{a}_1,\mathbf{c}_1),\dots,$ $p_m(\mathbf{a}_m,\mathbf{c}_m)) \in (X)$, i.e. (X) is y-closed under convex terms. To sum up, (X) is the least convex set containing X, i.e. $[X] = (X)$. Algebraicity follows directly from the description of the closure operator $[\,]$: each element $c \in [X]$ is in the form $c = p(a_1,\dots,a_n,c_1,\dots,c_m)$ (p a convex term, $a_1,\dots,a_n \in A$, $c_1,\dots,c_m \in X$), hence $c \in [\{c_1,\dots,c_m\}]$, therefore $[X] = \bigcup\{[X'] \mid X' \subseteq X,\ X' \text{ finite}\}$ which means algebraicity of $[\,]$.

(2) Follows directly from (1) and Proposition 2. □

Remark 4 Although $\mathrm{CC}(\mathbf{A})$ (the system of all congruence classes plus $\emptyset$) is a subset of $\mathrm{CS}(\mathbf{A})$, it is in general not a sublattice. For example, take the five element lattice $\mathbf{L}$ called diamond (i.e. one of the prototypical non-distributive lattices), denote its elements by $0,a,b,c,1$ ($0 < a,b,c < 1$, every two of a, b, and c are non-comparable). Now, $\{a\},\{b\} \in \mathrm{CC}(\mathbf{L}) \subseteq \mathrm{CS}(\mathbf{L})$. Since $\{0,a,b,1\}$ is an o-convex subuniverse, we get by Proposition 6 that $\{a\} \vee \{b\} = \{0,a,b,1\}$ in $\mathrm{CS}(\mathbf{L})$. However, $\{0,a,b,1\}$ is not a congruence class: in $\mathrm{CC}(\mathbf{L})$ we have $\{a\} \vee \{b\} = \{0,a,b,c,1\}$.

Closure operators offer a general approach to geometry (the following is due to Maeda, see e.g. [9]): The primitive notions are the set of points and the operator assigning to each set of points a subspace of points determined by the set. A *geometry* is a pair $\langle G, \mathrm{C}\rangle$ where

(1) G is a nonempty set (of so called points),

(2) C is a closure operator on G,

(3) $\mathrm{C}(\{p\}) = \{p\}$ for each $p \in G$ (a subspace determined by a point consists just of the point),

(4) $\mathrm{C}(\emptyset) = \emptyset$,

(5) $\mathrm{C}(X) = \bigcup\{\mathrm{C}(Y) \mid Y \text{ is a finite subset of } X\}$ (the closure operator is algebraic).

As one may easily see, $\mathrm{CC}(\mathbf{A})$ (precisely: A as the set of points and the closure operator induced by $\mathrm{CC}(\mathbf{A})$) forms a geometry in the above sense. This geometry has been investigated by Wille in [9] (Wille calls it "Kongruenzklassengeometrie", i.e. the geometry of congruence classes). The following proposition which is a direct corollary of Proposition 16 shows that convex sets offer a more general approach.

Corollary 17 *For any class $\mathcal{K}$ of algebras of the same type and any $\mathbf{A} \in \mathcal{K}$, $\langle A, [\ ]\rangle$ (where $[\ ]$ is the closure operator induced by $\mathrm{CS}_{\mathcal{K}}(\mathbf{A})$) is a geometry.*

Therefore, the geometry of congruence classes is a special case of the "geometry of convex sets". It seems to be an interesting problem to ask whether the converse is true: is each geometry a geometry of convex sets of some algebra? For the case of geometry of congruence classes the answer is negative [9, p. 28].

Convex equivalences

Definition 3 Let $\mathbf{A}$ be an algebra of a class $\mathcal{K}$ of algebras of the same type, θ be an equivalence on A. θ is called a *$\mathcal{K}$-convex equivalence* on $\mathbf{A}$ if each class of θ is a $\mathcal{K}$-convex set, i.e. $[a]_\theta \in \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$ for any $a \in A$.

Remark 5 (1) We denote by $\mathrm{CEqv}_{\mathcal{K}}(\mathbf{A})$ the set of all $\mathcal{K}$-convex equivalences on $\mathbf{A}$.

(2) Since each congruence class is a convex set, each congruence is a convex equivalence, i.e. $\mathrm{Con}(\mathbf{A}) \subseteq \mathrm{CEqv}_{\mathcal{K}}(\mathbf{A})$.

(3) Clearly, there is no counterpart of the notion of convex equivalence in the theory of ideals. Namely, each ideal contains 0, so ideals are located by around 0.

Example 3 Lattices provide non-trivial examples of convex equivalences that are not congruences. By Proposition 6, an equivalence on a lattice is convex iff each of its classes is an o-convex subuniverse. Therefore, the equivalence on a four-element Boolean lattice with precisely one non-singleton class containing one atom and the least element 0 is a convex equivalence which is not a congruence.

Proposition 18 *For any $\mathcal{K}$ and $\mathbf{A} \in \mathcal{K}$, $\mathrm{CEqv}_{\mathcal{K}}(\mathbf{A})$ is under set inclusion a complete lattice with the least element ω (identity) and the greatest element ι (the full relation $A \times A$).*

Proof Consider the intersection of any family of convex equivalences. Each class of the intersection (the intersection is an equivalence) is an intersection of classes of the equivalence relations of the family, thus, an intersection of convex sets. The claim therefore follows from Proposition 16. □

Although $\mathrm{Con}(\mathbf{A})$ is a sublattice of $\mathrm{Eqv}(A)$, $\mathrm{CEqv}(\mathbf{A})$ is not. The following is a counterexample.

Example 4 Let $\mathbf{L}$ be the four-element Boolean lattice, i.e. $L = \{0, a, b, 1\}$, $0 < a, b < 1$, a and b are mutually non-comparable. The partitions

$$\pi_1 = \{\{0, a\}, \{b\}, \{1\}\} \quad \text{and} \quad \pi_2 = \{\{0\}, \{b\}, \{a, 1\}\}$$

represent convex equivalences $\theta_{\pi_1}, \theta_{\pi_2}$ from CEqv(**L**). Clearly, the join $\theta_{\pi_1} \vee \theta_{\pi_2}$ in Eqv(L) is represented by $\{\{0, a, 1\}, \{b\}\}$. However, in CEqv(**L**), the join is the full relation $L \times L$.

However, the following is true.

Proposition 19 *For any* **A**, Con(**A**) *is a (complete) sublattice of* CEqv(**A**).

Proof The join $\bigvee_{i \in I} \theta_i$ in Con(**A**) of $\theta_i \in$ Con(**A**) coincides with the join in Eqv(A). Since $\bigvee_{i \in I} \theta_i \in$ CEqv(**A**), the conclusion follows. □

The concepts of a convex set and a convex equivalence naturally generate problems studied for congruence classes and congruence relations. As an example, we focus on the problem of direct decomposability.

Let $\mathcal{K}$ be a class of algebras of the same type. It is easy to see that if C_A and C_B are $\mathcal{K}$-convex sets in $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B} \in \mathcal{K}$, respectively, then $C_A \times C_B$ is a $\mathcal{K}$-convex set in $\mathbf{A} \times \mathbf{B}$. Therefore, if θ_A and θ_B are $\mathcal{K}$-convex equivalences on algebras **A** and **B** from $\mathcal{K}$, respectively, then the relation $\theta_A \times \theta_B$ on $\mathbf{A} \times \mathbf{B}$ defined by $\langle\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\rangle \in \theta_A \times \theta_B$ iff $\langle a_1, a_2\rangle \in \theta_A$ and $\langle b_1, b_2\rangle \in \theta_B$, is a convex equivalence. We say that a class $\mathcal{K}$ of algebras has *directly decomposable convex equivalences* (DDCE) if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, the algebra $\mathbf{A} \times \mathbf{B}$ has DDCE in that for any $\theta \in \mathrm{CEqv}_{\mathcal{K}}(\mathbf{A} \times \mathbf{B})$ there are $\theta_A \in \mathrm{CEqv}_{\mathcal{K}}(\mathbf{A})$ and $\theta_B \in \mathrm{CEqv}_{\mathcal{K}}(\mathbf{B})$ such that $\theta = \theta_A \times \theta_B$. Unlike direct decomposability of congruences (see [5]), DDCE is too strong for varieties: no variety $\mathcal{V}$ except for the trivial one given by the identity $x \approx y$ has DDCE (in fact, we may replace "variety" by "class closed w.r.t. direct products"). Indeed, let $\mathbf{A} \in \mathcal{V}$, $|A| > 1$, and take $a \in A$. The equivalence on $A \times A$ given by the partition $\{\{a\} \times A\} \cup \{\langle b, c\rangle \mid b \neq a\}$ is convex but cannot be decomposed into a direct product of convex equivalences on **A**. Hence $\mathcal{V}$ does not have DDCE.

However, there is a weaker condition: We say that a class $\mathcal{K}$ of algebras has *directly decomposable convex sets* (DDCS) if for any $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, the algebra $\mathbf{A} \times \mathbf{B}$ has DDCS, i.e. each $C \in \mathrm{CS}_{\mathcal{K}}(\mathbf{A} \times \mathbf{B})$ is of the form $C = C_A \times C_B$ for some $C_A \in \mathrm{CS}_{\mathcal{K}}(\mathbf{A})$ and $C_B \in \mathrm{CS}_{\mathcal{K}}(\mathbf{B})$.

Proposition 20 *The following conditions are equivalent for any variety* $\mathcal{V}$*:*

1. *(1)* $\mathcal{V}$ *has DDCS.*

2. *(2) There is an* $(n+2)$*-ary term* p *and binary terms* $r_1, s_1, \ldots, r_n, s_n$ *such that*

$$\begin{aligned} y &\approx p(x_1, \ldots, x_n, y, y) \\ x &\approx p(r_1(x, y), \ldots, r_n(x, y), x, y) \\ y &\approx p(s_1(x, y), \ldots, s_n(x, y), x, y) \end{aligned}$$

holds in $\mathcal{V}$.

Proof Observe first the following Claim: For algebras $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and $C \subseteq A \times B$, C is directly decomposable iff $\langle a, b\rangle \in C$ implies $\langle a, b'\rangle \in C$ for any $b' \in \mathrm{pr}_B(C)$.

"(1) $\Rightarrow$ (2):" Take $F_{\mathcal{V}}(x, y)$ (the free algebra in $\mathcal{V}$ generated by x and y) and consider the $\mathcal{V}$-convex set $[\langle x, x\rangle, \langle y, y\rangle]$ generated by $\{\langle x, x\rangle, \langle y, y\rangle\}$ in $F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$. By Claim we have $\langle x, y\rangle \in [\langle x, x\rangle, \langle y, y\rangle]$. Therefore, by Proposition 16 we infer that there is a $\mathcal{V}$-convex term $p(x_1, \ldots, x_n, y_1, y_2)$ and elements $\langle r_i, s_i\rangle = \langle r_i(x, y), s_i(x, y)\rangle \in F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$ such that $\langle x, y\rangle = p(\langle r_1, s_1\rangle, \ldots, \langle r_n, s_n\rangle, \langle x, x\rangle, \langle y, y\rangle)$. Rewriting this identity coordinatewise and the fact that p is $\mathcal{V}$-convex yields (2).

"(2) $\Rightarrow$ (1):" Let $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, $C \in \mathrm{CS}_{\mathcal{V}}(\mathbf{A} \times \mathbf{B})$. By Claim, we have to prove that if $\langle a, b\rangle, \langle a', b'\rangle \in C$ then $\langle a, b'\rangle \in C$. By (2) and the fact that C is $\mathcal{V}$-convex we have $\langle a, b'\rangle = p(\langle r_1(a, a'), s_1(b, b')\rangle, \ldots, \langle r_n(a, a'), s_n(b, b')\rangle, \langle a, b\rangle, \langle a', b'\rangle) \in C$, completing the proof. $\square$

Example 5 (1) Rings with unit have DDCS: Put $n = 1$, $p(x_1, y_1, y_2) = y_2 + x_1(y_1 - y_2)$, $r_1(x, y) = 1$, $s_1(x, y) = 0$.

(2) Lattices have DDCS: Put $n = 1$, $p(x_1, y_1, y_2) = (y_1 \wedge (y_2 \vee x)) \vee (y_2 \wedge (y_1 \vee x))$, $r_1(x, y) = x$, $s_1(x, y) = y$.

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