Contraction Mappings in b-metric Spaces

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Abstract. Some generalizations of well known Banach's fixed point theorem in so-called b-metric spaces are presented.

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1. Some problems, particurarly the problem of the convergence of measurable functions with respects to measure lead to a generalization of notion of metric. Using this idea we shall present generalization of some fixed point theorems of Banach type.

Lex X be a spece and let R_+ denotes the set of all nonnegative numbers. A function $d: X \times X \to R_+$ is said to be an b-metric iff for all $x, y, z \in X$ and all $r > 0$ the following conditions are satisfacted:

$$d(x, y) = 0 \text{ iff } x = y \tag{1}$$

$$d(x, y) = d(y, x) \tag{2}$$

$$d(x, y) < r \text{ and } d(x, z) < r \text{ imply } d(y, z) < 2r. \tag{3}$$

A pair (X, d) is called an b-metric space.

Lemma 1. *The condition (3) is equivalent to the following one:*

$$d(x, y) \leq r \text{ and } d(x, z) \leq r \text{ imply } d(y, z) \leq 2r. \tag{4}$$

for all $x, y, z \in X$ and all $r > 0$.

Let us consider the following condition:

$$d(y, z) \leq 2d(x, y) + 2d(x, z) \text{ for all } x, y, z \in X. \tag{5}$$

Of course, the condition (5) is weaker then (3). In the sequel we will call a function $d: X \times X \to R_+$ an b-metric iff the conditions (1) (2) and (5) are satisfied. For $T: X \to X$ we denote by T^n then n-th iterate of T.

2. Now we present following

Theorem 1. *Let (X, d) be e a complete b-metric space and let $T: X \to X$ satisfy*

$$d[T(x), T(y)] \leq \varphi[d(x, y)],\ x, y \in X, \tag{6}$$

where $\varphi : R_+ \to R_+$ is increasing function such that $lim_{n\to\infty}\varphi^n(t) = 0$ for each fixed > 0. Then T has exactly one fixed point u and

$$lim_{n\to\infty} d[T^n(x), u] = 0$$

for each $x \in X$.

PROOF: Take $x \in X$ and $\varepsilon > 0$. Let n be a natural number such that $\varphi^n(\varepsilon) < 4^{-1}$. Put $F = T^n$ and $x_k = F^k(x)$ for $k \in N$ (the set of natural numbers). Then for $x, y \in X$ and $\alpha = \varphi^n$ we have

$$d[F(x), F(y)] \leq \varphi^n[d(x, y)] = \alpha[d(x, y)]. \tag{7}$$

Therefore, for $k \in N$

$$d(x_{k+1}, x_k) \to 0 \text{ as } k \to \infty.$$

Let k be such that $d(x_{k+1}, x_k) < \varepsilon \cdot 4^{-1}$. Then for every $z \in K(x_k, \varepsilon) := \{y \in X : d(x_k, y) \leq \varepsilon\}$ we get

$$d[F(z), F(x_k)] \leq \alpha[d(x_k, z)] \leq \alpha(\varepsilon) = \varphi^n(\varepsilon) < \varepsilon \cdot 4^{-1},$$
$$d[F(x_k), x_k] = d(x_{k+1}, x_k) < \varepsilon \cdot 4^{-1}$$

whence

$$d[F(x_k), x_k] \leq 2(\varepsilon \cdot 4^{-1} + \varepsilon \cdot 4^{-1}) = \varepsilon,$$

which means that F maps $K(x_k, \varepsilon)$ into itself. Consequently

$$d(x_m, x_s) \leq 4\varepsilon \text{ for } m, s \geq k$$

and the sequence $\{x_k\}$ is a Cauchy sequnce, so there exists $u \in X$ such that $x_k \to u$ as $k \to \infty$. Furthermore, by the continuity of F (see (7))

$$u = lim_{k \to \infty} x_{k+1} = lim_{k \to \infty} F(x_{k+1}) = F(u),$$

i.e. u is a fixed point of F. Since $\alpha(t) = \varphi^n(t) < t$ for any $t > 0$, it is clear that F has exactly one fixed point. Moreover, by (6) T is continuous so we have

$$T(u) = lim_{k \to \infty} T[F^k(x)] = lim_{l \to \infty} F^k[T(x)] = u$$

and u is fixed point of T as well. It is obviuos by (6) that such point is only one.

Since for every $x \in X$ and every $r = 0, 1, \ldots, n-1$

$$T^{nk+r}(x) = F^k[T^r(x)] \to u \text{ as } k \to \infty,$$

so $T^m(x) \to u$ as $m \to \infty$ for every $x \in X$. This completes the proof of our theorem. □

For ordinary metric spaces analogous result is contained in [3], p. 12.

Theorem 2. *Let Z be a topological space and let (X, d) be a complete b-metric space. Let $X \times X$ be continuous and satisfy for each $z \in Z$*

$$d[T(x, z), T(y, z)] \leq \alpha d(x, y) \textit{ for all } x, y \in X, \tag{8}$$

where $0 \leq \alpha < 1$. Then for each $z \in Z$ there exists an unique fixed point $x(z)$ of T, i.e. $T[x(z), z] = x(z)$ and the function $z \to x(z)$ is continuous on Z.

PROOF: Put

$$T^1(x,\, z) = T(x,\, z),\; T^{n+1} = T[T^n(x,\, z),\, z],\; n = 1,\, 2,\, \ldots$$

Let us take n such that $\alpha^n < 2^{-1}$. By Theorem 1 for every $z \in Z$, T^n has exactly one fixed point $x(z)$. Since we have

$$T[x(z),\, z] = T[T^n(x(z),\, z),\, z] = T^n[T(x(z),\, z),\, z]$$

so $T[x(z),\, z]$ is also fixed point of T^n but in view of the uniqueness we get

$$T[x(z),\, z] = x(z),$$

i.e. $x(z)$ is a fixed point of T. By (8) one can proof that T has only one fixed point for every $x \in Z$.

Now let $\varepsilon > 0$ be given. The continuity of T implies that T^n is also continuous. Let $z_2 \in Z$ be arbitrarily fixed. Therefore exists a neighbourhood U of z_2 such that

$$d[T^n(z_2),\, z_1),\, T^n(x(z_2),\, z_2] \le \varepsilon \cdot 2^{-1}(1 - 2\alpha^n)$$

for $z_1 \in U$. Consequently we have for $z_1 \in U$

$$\begin{aligned} d[x(z_1),\, x(z_2)] &= d[T^n(x(z_1),\, z_1), T^n(x(z_2),\, z_2)] \le \\ &\le 2 \cdot d[T^n(x(z_1),\, z_1),\, T^n(x(z_2),\, z_1)] + 2 \cdot d[T^n(x(z_2),\, z_1),\, T^n(x(z_2),\, z_2)] \le \\ &\le 2\alpha^n d[x(z_1),\, x(z_2)] + \varepsilon(1 - 2\alpha^n). \end{aligned}$$

Finally we get

$$d[x(z_1),\, x(z_2)] \le \varepsilon \text{ for } z_1 \text{ in } U$$

which proves the continuity of x and completes the proof of the theorem. □

Now we shall prove the following

Theorem 3. *Let $\alpha : (0,\, \infty) \to [0, 2^{-1})$ be decreasing function. Let $(X,\, d)$ be a complete b-metric space and let $T : X \to X$ be a transformation such that*

$$d[T(x),\, T(z)] \le \alpha[d(x,\, z)](d[x,\, T(x)] + d[z,\, T(z)]) \tag{9}$$

for all $x,\, z \in X$, $x \neq z$. If moreover, T is continuous or α is a constant function, then T has a unique fixed point $u \in X$ and $lim_{n\to\infty} d[T^n(x),\, u] = 0$ for each $x \in X$.

PROOF: Let

$$y_n := d[T^n(x),\, T^{n+1}(x)],\; n = 1,\, 2,\, \ldots,\, x \in X.$$

We may assume that $y_n \neq 0$. Then by (9) we get

$$y_{n+1} \le \alpha(y_n)(y_n + y_{n+1}) \le 2^{-1}(y_n + y_{n+1})$$

whence

$$y_{n+1} \le y_n,\; n = 1,\, 2,\, \ldots$$

So $\{y_n\}$ is a decreasing sequence. Let $y = lim_{n\to\infty} y_n$. We shall prove that $y = 0$. Suppose that $y > 0$. Then

$$y_{n+1} \le \alpha(y)(y_n + y_{n+1})$$

and consequently $y \le 2\alpha(y)y$, which is impossible since $\alpha(y) < 2^{-1}$. This proves that $y = 0$. Now we will show that $\{y_n\}$ is a Cauchy sequence for every $x \in X$. From (9) we get for $m, n \in N$

$$d[T^n(x), T^m(x)] \le \frac{1}{2}(d[T^{n-1}(x), T^n(x)] + d[T^{m-1}(x), T^m(x)]).$$

There exists an n_0 such that for $m, n \ge n_0$

$$d[T^{n-1}(x), T^n(x)] < \varepsilon \text{ and } d[T^{m-1}(x), T^m(x)].$$

and hence

$$d[T^n(x), T^m(x)] < \frac{1}{2}(\varepsilon + \varepsilon) = \varepsilon$$

for all $m, n \ge n_0$. Thus $\{y_n\}$ is a Cauchy sequence and in view of completeness of X there exists an $u \in X$ such that $T^n(x) \to u$. We can check that $T(u) = u$. Really, we have

$$d[u, T(u)] \le 2d[u, T^{n+1}(x)] + 2d[T^{n+1}(x), T(u)].$$

If T is continuous, then the right hand side of the inequality tends to zero as $n \to \infty$, which proves that $T(u) = u$. On the other hand, if $\alpha =$ const., then

$$d[u, T(u)] \le 2d[u, T^{n+1}(x)] + 2\alpha[d[T^n(x), T^{n+1}(x)] + d[u, T(u)]] \le$$
$$\le 2d[u, T^{n+1}(x)] + 2\alpha y_n + 2\alpha d[u, T(u)]].$$

Letting $u \to \infty$ we get

$$d[u, T(u)] \le \alpha d[u, T(u)],$$

i.e. $d[u, T(u)] = 0$.

Finally, to prove the last part of the Theorem, let us assume that

$$T(u_1) = u_1,\ T(u_2) = u_2,\ u_1 \ne u_2,\ u_1,\ u_2 \in X.$$

Therefore we may write

$$d(u_1, u_2) = d[T(u_1), T(u_2)] \le$$
$$\le \alpha[d(u_1, u_2)](d[u_1, T(u_1)] + d[u_2, T(u_2)]) = 0$$

which means that $u_1 = u_2$ and finishes the proof of the Theorem. □

For related problems in metric spaces see [5].

Example

Let

$$T(x) = \begin{cases} \frac{1}{4}x, & x \in [0, 1), \\ \frac{1}{5}, & x = 1. \end{cases}$$

Then

$$|T(x) - T(z)| \leq \frac{1}{3}(|x - T(x)| + |z - T(z)|)$$

for $x, z \in [0, 1]$, i.e. T satisfies the condition (9) but T is not continuous.

3. Let us consider complete b-metric space (X_i, d_i), $i = 1, \ldots, n$. Let $X := X_1 \times \ldots \times X_n$ and let $d : X \times X \to R_+$ be the function defined as follows

$$d(x, z) = \sum_{i=1}^{n} r_i d_i(x_i, z_i), \tag{10}$$

where $x = (x_i, \ldots, x_n)$, $z = (z_1, \ldots, z_n) \in X$ and r_i, $i = 1, \ldots, n$ are given nonnegative real numbers.

One can easily see that

$$d(x, z) = 0 \text{ iff } x = z \text{ and } d(x, z) = d(z, x) \text{ for every } x, z \in X.$$

Moreover, we have for $x, y, z, \in X$

$$d(x, z) \leq \sum_{i=1}^{n} 2r_i[d_i(x_i, y_i) + d_i(y_i, z_i)] = 2d(x, y) + 2d(y, z),$$

which gives then inequality

$$d(x, z) \leq 2d(x, y) + 2d(y, z),$$

for all $x, y, z, \in X$. This means that the the function d is an b-metric in X. If all spaces (X_i, d_i), $i = 1, \ldots, n$ are complete then the space (X, d) is also complete with respect to the b-metric d.

Using this idea we get the following theorem for system of transformations.

Theorem 4. *Let (X_i, d_i), $i = 1, \ldots, n$ be complete b-metric spaces. Let $a_{i,k}$, $i, k = 1, \ldots, n$ be nonnegative real numbers such that transformations $T_i : X \to X_i$, $i = 1, \ldots, n$ fulfill the inequalities*

$$d_i[T_i(x), T_i(z)] \leq \sum_{i=1}^{n} a_{i,k} d_k(x_k, z_k) \tag{11}$$

for all $x = (x_1, \ldots, x_n)$, $z = (z_1, \ldots, z_n) \in X$. If, moreover, the absolute values of the characteristic roots fo the matrix $[a_{i,k}]_{i,k=}^{n}$ are less then one, then the system of equations

$$T_i(x_1, \ldots, x_n) = x_i, \quad i = 1, \ldots, n$$

has exactly one solution $u = (u_1, \ldots, u_n) \in X$ *given by the formulas*

$$u_i = lim_{s\to\infty} x_i^s, \quad i = 1, \ldots, n, \tag{12}$$

$$x_i^{s+1} = T_i(x_i^s, \ldots, x_n^s), \quad i = 1, \ldots, n, \quad s = 0, 1, \ldots, \tag{13}$$

where $x_I^0 \in X_i$, $i = 1, \ldots, n$ *are arbitrarily fixed.*

PROOF: From Perron's Theorem ([2], p. 354) we conclude that there exists positive numbers r_i, $i = 1, \ldots, n$ satisfying the system of inequalities

$$\sum_{i=1}^{n} r_i a_{i,k} < r_k, \quad k = 1, \ldots, n \tag{14}$$

(see also [1] and [4]). Take

$$v := max_{k\in\{1,\ldots,n\}} \left(r_k^{-1} \sum_{i=1}^{n} r_i a_{i,k} < r_k \right),$$

then (14) implies

$$0 \le v < 1 \tag{15}$$

$$\sum_{i=1}^{n} r_i a_{i,k} < r_k \le v r_k, \quad k = 1, \ldots, n. \tag{16}$$

Let d be defined by formula (10). It has been mentioned that (X, d) is a complete b-metric space. Now let us consider the mapping $T : X \to X$ defined by $T(x) = (T_1(x), \ldots, T_n(x))$ for $x \in X$. We are able to check that T si a contraction map. Indeed, in view of (10), (11) and (16) for $x, z \in X$ we get

$$\begin{aligned} d[T(x), T(z)] &= \textstyle\sum_{i=1}^n r_i d_i[T_i(x), T_i(z)] \le \sum_{i=1}^n r_i \sum_{k=1}^n a_{i,k} d_k(x_k, z_k) = \\ &= \textstyle\sum_{i=1}^n (\sum_{k=1}^n r_i a_{i,k}) d_k(x_k, z_k) = \sum_{i=1}^n v r_k a_{i,k} d_k(x_k, z_k) = v d(x, z) \end{aligned}$$

Taking into account (15) and applying Theorem (1) for $\varphi(t) = v \cdot t$ we obtain our assertion. □

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