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SOME NEW RESULTS IN STATE SPACE DECOUPLING OF MULTIVARIABLE SYSTEMS I

A Link Between Geometric Approach and Matrix Methods

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A method based on the algorithmization of the geometric approach, using the equivalence between the maximal unobservable subspaces and the maximal (A, B)-invariant subspaces is proposed in order to solve the problem of group decoupling for linear multivariable systems. Necessary and sufficient conditions for compatibility of the maximal controllability subspaces are transformed to the necessary and sufficient conditions for the relevant maximal unobservable subspaces. On that ground, an algorithm for group decoupling, similar to that of Silverman and Payne (1971), is derived in a simple way. Conditions for group decoupling of non-completely output controllable systems are derived. A class of systems with inherent interaction is described.

1. INTRODUCTION

The problem of decoupling linear multivariable systems by state space methods was attacked by many authors during the past decade. Falb and Wolovich [2] and Gilbert [4] solved completely the classical (Morgan's) problem of state feedback decoupling a square system into single input-single output subsystems. Wonham and Morse [7], [10], introducing the new concept of controllability subspaces formulated geometrically the restricted decoupling problem (state feedback decoupling) and the extended decoupling problem (state feedback decoupling in conjunction with dynamic precompensation). The above mentioned authors considered not only decoupling into single input-single output subsystems, but also group decoupling (into multiple input-multiple output subsystems), too.

The main disadvantages of geometric approach were some computational problems involved by an unusual form of the final results. A method for direct transformation of geometric relations to an useful matrix form, proposed by the same authors, needed a special program package and was criticized in a number of papers.

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Silverman and Payne [9], using properties of the structure algorithm (Silverman [8]), solved the group decoupling problem in terms of classical matrix language. Some of their results seemed to be similar to those of Wonham and Morse [10] but nothing was done to compare them. Both mentioned approaches considered only completely output controllable systems.

Silverman and Payne [9] discussed the use of state feedback in conjunction with observer for decoupling but left the problem of static output feedback decoupling open. This problem was completely solved only for the case of single input-single output decoupling of square systems by Howze [5]. More general results using the geometric approach were given by Denham [1] but without practical meaning.

This paper deals with the general group decoupling problem for dynamic stationary linear multivariable systems, not necessarily completely output controllable. Our approach is based on the geometric formulation of decoupling problem given in [10]. A method for transformation of geometric results on group decoupling into a matrix form based on the concept of maximal unobservable subspaces is developed. The results given in this paper seem to be identical to those of [9] for completely output controllable systems, while preserving the advantages of the objective geometric formulation. Conditions for static output feedback decoupling are also considered.

The paper is organized in the following way.

Part I, after an introduction to the geometric approach, discusses the compatibility conditions for certain maximal controllability subspaces derived in [10] and their equivalence to the compatibility conditions for the corresponding maximal unobservable subspaces. This equivalence results into a matrix form for the compatibility conditions of maximal controllability subspaces.

Part II presents the necessary conditions for group decoupling of non-completely output controllable systems. According to the results of Part I, conditions for group decoupling of systems with $D \neq 0$ are derived in a simple way. The problem of static output feedback decoupling is solved.

2. PRELIMINARIES

Consider an n-th order, r-input, m-output linear stationary system S, described by the equations:

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du\,, \tag{1b}$$

state feedback (F, G) of the type:

$$u = Fx + Gv \tag{2}$$

and output feedback (K, G):

(3) $$u = Ky + Gv,$$

where $x \in \mathscr{X}$ is an n-vector of states, $u \in \mathscr{U}$ is an r-vector of inputs, $y \in \mathscr{Y}$ is an m-vector of outputs, $v \in \mathscr{U}$ is an $\bar{r}$-vector of new inputs ($\bar{r}$ being the number of new inputs after the control has been applied) and $\mathscr{X}, \mathscr{U}, \mathscr{Y}$ are linear vector spaces. Matrices A, B, C, D, F, K, G are of appropriate dimensions. Note that linear maps and their matrices with respect to given bases are denoted by identical symbols.

The concept of (A, B)-invariant, controllable, controllability subspaces and their properties are described in details in a number of papers (for example see Wonham [11], MacFarlane and Karcanias [6]). This approach assumes that the reader is familiar with them. For completness some results of the geometric approach useful for our purposes will be further briefly mentioned in the form of statements and definitions, accepted directly from [11] without comments.

Definition 1.1. Let $\Upsilon \subset \mathscr{X}$. Subspace Υ is:
- (A)-invariant if $A\Upsilon \subset \Upsilon$;
- (A, B)-invariant if $A\Upsilon \subset \Upsilon + \langle B \rangle$ where $\langle B \rangle$ denotes Im B.

If $\Upsilon \subset \mathscr{X}$ is (A, B)-invariant, there always exists a map $F : \mathscr{X} \to \mathscr{U}$ such that Υ is $(A + BF)$-invariant, too.

Lemma 1.1. Let $\Upsilon \subset \mathscr{X}$. There exists a map $F : \mathscr{X} \to \mathscr{U}$ such that $(A + BF)\, \Upsilon \subset \Upsilon$ if and only if $A\Upsilon \subset \Upsilon + \langle B \rangle$.

Clearly if Υ is (A, B)-invariant, there exists a nonempty class $\mathbb{F}(\Upsilon) = \{F : (A + BF)\, \Upsilon \subset \Upsilon\}$.

Lemma 2.1. Every subspace Ω contains one and only one maximal (A, B)-invariant subspace $\Upsilon^* \subset \Omega$.

Definition 2.1. (A, B)-invariant subspaces Υ_1, Υ_2 are compatible if $\mathbb{F}(\Upsilon_1) \cap \mathbb{F}(\Upsilon_2) \neq \emptyset$, i.e. if there exists a map $F: \mathscr{X} \to \mathscr{U}$ such that

$$(A + BF)\, \Upsilon_i \subset \Upsilon_i\,, \quad i = 1, 2\,.$$

Notice that disjunction of (A, B)-invariant subspaces does not generally preserve (A, B)-invariance.

Lemma 3.1. Let $\Upsilon_i \subset \mathscr{X}$, $i = 1, 2$. There exists a map $F : \mathscr{X} \to \mathscr{U}$ such that subspaces Υ_1, Υ_2 are compatible, i.e. $(A + BF)\, \Upsilon_i \subset \Upsilon_i$, $i = 1, 2$, if and only if

$$A\Upsilon_i \subset \Upsilon_i + \langle B \rangle$$

$$A(\Upsilon_1 \cap \Upsilon_2) \subset \Upsilon_1 \cap \Upsilon_2 + \langle B \rangle\,.$$

Special kinds of (A, B)-invariant subspaces are controllable and controllability subspaces.

Definition 3.1. Let $\mathscr{R}_0 \subset \mathscr{X}$. Subspace $\mathscr{R}_0$ is a controllable subspace if

$$\mathscr{R}_0 = \sum_{j \in \bar{n}} A^{j-1} \langle B \rangle ,$$

where the symbol $\bar{n}$ denotes $\{1, 2, \ldots, n\}$.

After applying a control law (2) to system S, the controllable subspace $\mathscr{R}_0$ of the pair (A, B) is identical to or contains the controllable subspace of the pair $(A + BF, BG)$.

Definition 4.1. Let $\mathscr{R} \subset \mathscr{X}$. Subspace $\mathscr{R}$ is a controllability subspace of system S if there exist maps $F : \mathscr{X} \to \mathscr{U}$, $G : \mathscr{U} \to \mathscr{U}$ such that

$$\mathscr{R} = \sum_{j \in \bar{n}} (A + BF)^{j-1} \langle BG \rangle .$$

The following statements are true for controllability subspaces.

Lemma 4.1. Let $\mathscr{R} \subset \mathscr{X}$. Subspace $\mathscr{R}$ is a controllability subspace if and only if there exists a map $F : \mathscr{X} \to \mathscr{U}$ such that

$$\mathscr{R} = \sum_{j \in \bar{n}} (A + BF)^{j-1} (\langle B \rangle \cap \mathscr{R}) .$$

If $\mathscr{R}$ is a controllability subspace, the class $\mathbb{F}(\mathscr{R}) = \{\mathscr{R} : \sum (A + BF)^{j-1} (\langle B \rangle \cap \mathscr{R})\}$ is not empty and, for every map $F \in \mathbb{F}(\mathscr{R})$,

$$\mathscr{R} = \sum_{j \in \bar{n}} (A + BF)^{j-1} (\langle B \rangle \cap \mathscr{R}) .$$

Because of the (A, B)-invariance of the controllability subspaces, Lemmas 2.1, 3.1 and Definition 2.1 apply to them, too.

For the purposes of decoupling, maximal controllability subspaces contained in given subspaces are important. Such controllability subspaces are connected with the maximal (A, B)-invariant subspaces contained in the same subspaces.

Theorem 1.1. Let $\Omega \subset \mathscr{X}$ and $\Upsilon^* \subset \Omega$ is the maximal (A, B)-invariant subspace contained in Ω. If $F \in \mathbb{F}(\Upsilon^*)$, the subspace

$$\mathscr{R}^* = \sum_{j \in \bar{n}} (A + BF)^{j-1} (\langle B \rangle \cap \Upsilon^*)$$

is the maximal controllability subspace contained in Ω.

Some further comments about the subspaces defined above will be given in the next sections.

Finally recall that the concept of group decoupling, introduced in [10], assumes the partition of inputs (resp. new inputs, if a control has been applied) and outputs into l disjoint subsets $u_i(v_i)$ and y_i, each of $r_i(\bar{r}_i)$ and m_i elements, where $\sum_{i \in \bar{l}} r_i = r$, $\sum_{i \in \bar{l}} \bar{r}_i = \bar{r}$, $\sum_{i \in \bar{l}} m_i = m$ (the symbol $\bar{l}$ again denotes $\{1, 2, \ldots, l\}$).

Further we will consider two types of subsystems $S_i(A, B, C_i, D_i)$ and $S_i^*(A, B, C_i^*, D_i^*)$, $i \in \bar{l}$, where dimensions of C_i, D_i, C_i^*, D_i^* are respectively $(m_i \times n)$, $(m_i \times r)$, $((m - m_i) \times n)$, $((m - m_i) \times r)$ and

$$C' = [C_1' \vdots \ldots \vdots C_l'] , \quad D' = [D_1' \vdots \ldots \vdots D_l']$$

$$(4) \quad C_i^{*\prime} = [C_1' \vdots \ldots \vdots C_{i-1}' C_{i+1}' \vdots \ldots \vdots C_l'] , \quad D^{*\prime} = [D_1' \vdots \ldots \vdots D_{i-1}' D_{i+1}' \vdots \ldots \vdots D_l'] , \quad i \in \bar{l} .$$

The $(\bar{r} \times \bar{r}_i)$ column blocks of matrix G will be denoted G_i.

3. THE GEOMETRIC FORMULATION OF DECOUPLING PROBLEM

Decoupled systems are usually defined as systems with diagonal nonsingular transfer function matrix. The concept of group decoupling just mentioned in the introduction seems to be more general and requiring less restrictions on the system.

Definition 5.1. System S whose inputs and outputs are partitioned into disjoint subsets is decoupled if every input group controls one and only one output group without affecting the rest.

Noninteraction between grouped subsystems is the main characteristic of decoupled system. For that reason the decoupling appears as a process of breaking some connections between input and output, possibly reducing output controllability of the system. For the output to be controllable as much as before decoupling, the invariance of output controllability under the decoupling control law is needed.

In square systems the diagonality and the nonsingularity of transfer function matrix are consequences resp. of noninteraction and output controllability of single input-single output subsystems. It is easy to see that the classical definition of a decoupled system is implied by Definition 5.1, but really not vice versa.

To formulate the decoupling problem, suppose we are given a completely output controllable system S with $D = 0$, enclosed by state feedback (F, G) according to (2), and controllability subspaces $\mathscr{R}_i = \sum_{j \in \bar{n}} (A, BF)^{j-1} \ (\langle B \rangle \cap \mathscr{R}_i)$, $i \in \bar{l}$ generated by new input groups v_i, $i \in \bar{l}$.

From Definition 5.1 for the i-th input group:

– to leave the relevant output groups Y_j, $j \in \bar{l}$, $j \neq i$ unaffected, it has to be satisfied:

$$C_j \mathscr{R}_i = 0 \quad i, j \in \bar{l} , \quad i \neq j ,$$

hence by (4):

$$(5) \qquad \mathscr{R}_i \subset \operatorname{Ker} C_i^* \quad i, j \in \bar{l} , \quad i \neq j ;$$

– to control completely the output group y_i, $i \in l$, the condition

$$C_i \mathscr{R}_i = \langle C_i \rangle \quad i \in l \tag{6}$$

is required. For the control law to exist a map $F : \mathscr{X} \to \mathscr{U}$, constructing simultaneously all subspaces $\mathscr{R}_i$, $i \in l$, satisfying both conditions (5), (6), must exist such that

$$\bigcap_{i \in l} \mathbb{F}(\mathscr{R}_i) \neq \emptyset \tag{7}$$

is true.

The above formulation of the decoupling problem is due to Wonham and Morse [10]. It may be summarized in the following way.

Let us be given a system S completely output controllable and a certain partition (into disjoint subsets) of its outputs. Determine matrices F, G of state feedback constructing controllability subspaces $\mathscr{R}_i$, $i \in l$ with properties (5), (6), (7). These properties are usually named resp. noninteraction, output controllability and compatibility.

The obvious way of solving such a problem is the following. Construct controllability subspaces $\mathscr{R}_i$, $i \in l$ satisfying noninteraction condition (5). If constructed subspaces meet the requirement for compatibility (7), then test output controllability condition (6). Clearly, having a control law satisfying (5), (7), no further problem may appear with the verification of (6), as it is only a geometrical description of the conditions

$$\operatorname{rank}(C_i R_i) = m_i \quad i \in l\,, \tag{8}$$

(R_i being a matrix of dimensions ($n \times \dim \mathscr{R}_i$), $i \in l$, formed by the basis vectors of subspaces $\mathscr{R}_i$). Consequently the decoupling problem may be reduced to the problem of finding a state feedback, which constructs compatible controllability subspaces $\mathscr{R}_i$, $i \in l$ with property (6). As no analytic methods for such a purpose are known, Wonham and Morse [10] used the maximal controllability subspaces $\mathscr{R}_i^*$ contained in Ker C_i, $i \in l$ instead of the arbitrary subspaces $\mathscr{R}_i$, $i \in l$ utilized in the geometric formulation. A recursive method for computation of these subspaces was derived in the above mentioned paper. The necessary and sufficient condition for compatibility of subspaces $\mathscr{R}_i^*$, $i \in l$ was stated to be:

$$A\mathscr{R}^{*\Sigma} \subset \mathscr{R}^{*\Sigma} + \langle B \rangle\,, \tag{9}$$

where $\mathscr{R}^{*\Sigma} = \bigcap_{i \in l} \mathscr{R}_i^{*\Sigma}$ and $\mathscr{R}_i^{*\Sigma} = \sum_{\substack{j \in l \\ i \neq j}} \mathscr{R}_j^*$, $i \in l$, i.e. the (A, B)-invariance of subspace $\mathscr{R}^{*\Sigma}$.

Remark 1.1. Necessary and sufficient condition (9) for compatibility of the maximal controllability subspace $\mathscr{R}_i^*$, $i \in l$ is generally only a sufficient condition for decoupling, as there may (theoretically) exist smaller subspaces than the maximal controllability subspaces $\mathscr{R}_i \subset \operatorname{Ker} C_i^*$, $i \in l$, satisfying also (6). It may be proven that for the majority of cases (for example the classical Morgan's problem, decoupling into single input-multiple output subsystems, decoupling with the restriction on matrix G to be nonsingular) condition (9) is also a necessary condition for decoupling. For that reason the solution of decoupling problem when (9) is true will be referred to as a solution in the sense of maximal controllability subspaces.

Remark 2.1. It was shown in [7] and [9] that if (8) is true, decoupling by state feedback in conjunction with dynamic precompensator is always possible independently of (9).

Remark 3.1. A simple result of the well known connection between controllability and pole assignment is that if the system S is a minimal realisation, all poles of the decoupled system except the modes of subspace $\mathscr{R}^{*\Sigma}$ are freely assignable by state feedback (cf. [10]).

Although the solution of decoupling problem given by Wonham and Morse [10] was objective and well suited to the geometric formulation, it did not confirm itself as a practical method for engineering calculations. This was caused by some computational disadvantages of the recursive algorithm on the one hand and by the needs for a special package for computer implementation of the algorithm on the other hand. The specific form of derived conditions for decoupling did not permit a comparison with the results of another approaches.

4. THE MAXIMAL UNOBSERVABLE SUBSPACES

In this section we derive some properties of maximal unobservable subspaces useful for transformation of geometric compatibility condition (9) into a matrix form. Our approach is based on the connection between the maximal controllability subspaces $\mathscr{R}_i^*$ contained in Ker C_i^*, $i \in \bar{l}$ and the maximal (A, B)-invariant subspaces Υ_i^* contained in Ker C_i^*, $i \in l$, which allows to treat the conditions for compatibility of maximal (A, B)-invariant subspaces Υ_i^* as a conditions for compatibility of $\mathscr{R}_i^*$, $i \in l$.

Consider system S given by (1) and a state feedback by (2):

$$u = Fx . \tag{10}$$

Denote Q the observability matrix

$$Q = [C' \vdots A'C' \vdots \ldots \vdots (A')^{n-1} C']' .$$

We define the unobservable subspace Υ of a system S to be

$$\Upsilon = \text{Ker } Q . \tag{11}$$

It is known that generally observability is not an F-invariant property and it may be proven that there always exists a matrix F_α in (10), not unique, minimizing the rank of the observability matrix $Q(F)$, where

$$Q(F) = [(C + DF)' \vdots \ldots \vdots ((A + BF)')^{n-1} (C + DF)']' . \tag{12}$$

Then if rank $Q(F) = \min$, the system S_{F_α}:

$$\begin{aligned} \dot{x} &= (A + BF)\, x + Bu \\ y &= (C + DF)\, x + Du \end{aligned} \tag{13}$$

will be named maximally unobservable system. Subspace Υ^* defined by

$$\Upsilon^* = \operatorname{Ker} Q(F_\alpha) \tag{14}$$

will be named the maximal unobservable subspace.

Theorem 2.1. The maximal unobservable subspace Υ^* of system S in (1) is identical to the maximal (A, B)-invariant subspace contained in $\operatorname{Ker}(C + DF_\alpha)$, where F_α is the matrix of state feedback for which the system is maximally unobservable.

Proof. From the Calley-Hamilton's theorem:

$$(A + BF_\alpha)' \langle Q(F_\alpha)' \rangle =$$

$$= (A + BF_\alpha)' \sum ((A + BF_\alpha)')^{j-1} \langle (C + DF_\alpha)' \rangle \subset \langle Q(F_\alpha)' \rangle$$

i.e.:

$$\langle Q(F_\alpha)' \rangle \subset ((A + BF_\alpha)')^{-1} \langle Q(F_\alpha)' \rangle ,$$

where -1 denotes functional inversion. For complements:

$$\langle Q(F_\alpha)' \rangle^\perp \supset (((A + BF_\alpha)')^{-1})^\perp \langle Q(F_\alpha)' \rangle = (A + BF_\alpha) \langle Q(F_\alpha)' \rangle^\perp ,$$

i.e.

$$\operatorname{Ker} Q(F_\alpha) \supset (A + BF_\alpha) \operatorname{Ker} Q(F_\alpha) ,$$

which gives with (14):

$$(A + BF_\alpha)\, \Upsilon^* \subset \Upsilon^* .$$

Hence by Lemma 1.1 Υ^* is (A, B)-invariant. (14) also implies $\Upsilon^* \subset \operatorname{Ker}(C + DF_\alpha)$ and the maximality of Υ^* in $\operatorname{Ker}(C + DF_\alpha)$.

Consequently to construct the maximal unobservable subspace it is sufficient to determine a matrix F_α in (10) minimizing the rank of observability matrix and then to compute the kernel. We introduce two constructions for such a purpose. The first may be used only for systems with $D = 0$ and scalar output, the second is general.

Construction 1. Consider system S by (1) with $D = 0$ and $m = 1$. Define an integer d (Gilbert [4]):

$$\begin{aligned} d &= \min \{ j : CA^jB \neq 0,\ j = 0, 1, \ldots, n - 1 \} \\ d &= n - 1 \quad \text{if} \quad CA^jB = 0 , \quad \forall j \geqq 0 . \end{aligned} \tag{15}$$

For arbitrary state feedback by (10) a direct result of (15) is the identity:

$$C(A + BF)^j = \begin{cases} CA^j\,, & j = 0, 1, \ldots, d \\ CA^d(A + BF)^{j-d} & j = d + 1, \ldots, n - 1\,, \end{cases}$$

implying the F-invariance of d and CA^dB. Then the observability matrix $Q(F)$ of system S_F is:

$$Q(F) = \begin{bmatrix} C \\ \vdots \\ C(A + BF)^d \\ \vdots \\ C(A + BF)^{n-1} \end{bmatrix} = \begin{array}{c} \left[\begin{array}{c} C \\ \vdots \\ CA^d \\ \hline CA^d(A + BF) \\ \hline \vdots \\ CA^d(A + BF)^{n-d-1} \end{array}\right] \end{array} \begin{array}{l} 1. \\ \\ \\ 2. \\ \\ 3. \end{array}$$

The rows of the first block of $Q(F)$ are F-invariant. Clearly, if the second block of $Q(F)$ is zero, the observability matrix is of minimal rank, defined by the rank of the F-invariant first block. For the second block to be zero, we have:

(16) $$CA^d(A + BF) = CA^{d+1} + CA^dBF = 0\,,$$

i.e. the class of all matrices F by (10) zeroing the non F-invariant rows of observability matrix and certainly minimizing its rank is given by:

(17) $$F_\alpha = -\bar{D}_\alpha^+ \bar{C}_\alpha\,,$$

where

$$\bar{D}_\alpha = CA^dB\,, \quad \bar{C}_\alpha = CA^{d+1}$$

and $^+$ denotes pseudoinversion.

Matrix F_α in the above construction zeroes all rows of $Q(F)$ except the F-invariant ones. An alternative construction may be involved if the matrix F_α is determinated to make all the rows of the second and the third block to be a linear combination of those of the first block. This case may be easily transformed to the construction described above.

Construction 2. (Structure algorithm – Silverman and Payne [9]).

Let $T_0, T_1, \ldots, T_k$ be nonsingular $(m \times m)$ matrices (not unique) and A, B, C, D be system matrices from (1).

1-*st step.* Find matrix T_0 transforming D to the form:

$$T_0 D = \begin{bmatrix} \bar{D}_0 \\ \hdashline 0 \end{bmatrix} \begin{matrix} \}\bar{g}_0 \\ \}m - \bar{g}_0 \end{matrix}\,,$$

where rank $D = \bar{D}_0 = \bar{g}_0$ and the number of rows of matrix $\bar{D}_0$ is $\bar{g}_0$. Then form submatrices $\bar{C}_0, \tilde{C}_0$:

$$T_0 C = \begin{bmatrix} \bar{C}_0 \\ \hdashline \tilde{C}_0 \end{bmatrix}.$$

2-nd step. Find a matrix T_1 of the type:

$$T_1 = \begin{bmatrix} I_{\bar{g}_0} & 0 \\ 0 & \bar{T}_1 \end{bmatrix},$$

such that

$$T_1 \begin{bmatrix} \bar{D}_0 \\ \tilde{C}_0 B \end{bmatrix} = \begin{bmatrix} \bar{D}_1 \\ 0 \end{bmatrix} \begin{matrix} \}\bar{g}_1 \\ \}m - \bar{g}_1 \end{matrix},$$

where

$$\operatorname{rank} \begin{bmatrix} \bar{D}_0 \\ \tilde{C}_0 B \end{bmatrix} = \operatorname{rank} \bar{D}_1 = \bar{g}_1 .$$

Then form submatrices $\bar{C}_1$, $\tilde{C}_1$:

$$T_1 \begin{bmatrix} \bar{C}_0 \\ \tilde{C}_1 A \end{bmatrix} = \begin{bmatrix} \bar{C}_1 \\ \tilde{C}_1 \end{bmatrix}$$

and compose the matrix

$$L_1 = \tilde{C}_1 .$$

$$\vdots$$

k-th step. Find a matrix T_k of the type:

$$T_k = \begin{bmatrix} I_{\bar{g}_{k-1}} & 0 \\ 0 & \bar{T}_{k-1} \end{bmatrix},$$

such that

$$T_k \begin{bmatrix} \bar{D}_{k-1} \\ \tilde{C}_{k-1} \end{bmatrix} = \begin{bmatrix} \bar{D}_k \\ 0 \end{bmatrix} \begin{matrix} \}\bar{g}_k \\ \}m - \bar{g}_k \end{matrix},$$

where

$$\operatorname{rank} \begin{bmatrix} \bar{D}_{k-1} \\ \tilde{C}_{k-1} \end{bmatrix} = \operatorname{rank} \bar{D}_k = \bar{g}_k .$$

Then form submatrices $\bar{C}_k$, $\tilde{C}_k$:

$$T_k \begin{bmatrix} \bar{C}_{k-1} \\ \tilde{C}_{k-1} A \end{bmatrix} = \begin{bmatrix} \bar{C}_k \\ \tilde{C}_k \end{bmatrix}$$

and compose the matrix

$$L_k = \begin{bmatrix} L_{k-1} \\ \tilde{C}_{k-1} \end{bmatrix}.$$

When the rank condition

$$\bar{g}_k < \bar{g}_{k+1}$$

will fail, denote:

$$\bar{D}_k = \bar{D}_\alpha , \quad \bar{C}_k = \bar{C}_\alpha , \quad \tilde{C}_k = \tilde{C}_\alpha , \quad \bar{g}_k = \bar{g}_\alpha .$$

The algorithm will be finished at step $k + 1$ ($k \geqq \alpha$), when

$$\operatorname{rank} L_k = \operatorname{rank} L_{k+1} .$$

We denote $L_k = L_\beta$.

The following Lemma is a simple consequence of the structure algorithm. Its proof is in detail discussed by Silverman and Payne [9].

Lemma 5.1. State feedback of the type (10) with

$$F_\alpha = -\bar{D}_\alpha^+ \bar{C}_\alpha \tag{18}$$

reduces the observability matrix of system (13) to an F-invariant $Q(F_\alpha)$ (having $\langle Q(F_\alpha)' \rangle = \langle L'_\beta \rangle$) with minimal rank.

(i) Integer $\bar{g}_\alpha$ defined at step α of the structure algorithm is identical to the rank of transfer function matrix of system S:

(ii) The matrices $\bar{D}_\alpha$, L_β, B satisfy:

$$\operatorname{Ker} \bar{D}_\alpha = \operatorname{Ker} \begin{bmatrix} D \\ \hline L_\beta B \end{bmatrix} \tag{19}$$

Remark 4.1. Using formally Construction 2 for scalar output subsystems (15) we get the identities:

$$D_\alpha = CA^d B \qquad C_\alpha = CA^{d+1}$$

$$\langle C' \vdots A'C' \vdots \ldots \vdots (A')^d C' \rangle = \langle L' \rangle .$$

Further we will utilize Theorem 2.1 to derive a matrix form of the compatibility conditions for maximal unobservable subspaces Υ_i^* of subsystems S_i^*, $i \in l$.

According to Construction 2 we have, for every subsystem S_i^*, $i \in l$, the following sets of maps constructing the maximal unobservable subspaces:

$$\mathbb{F}(\Upsilon_i^*) = \{F : \bar{D}_{\alpha i}^* F = \bar{C}_{\alpha i}^*\} , \quad i \in l . $$

Clearly, by definition of compatibility $\bigcap_{i \in l} \mathbb{F}(\Upsilon_i^*)$ to be nonempty the consistence of matrix equation

$$\begin{bmatrix} \bar{D}_{\alpha 1}^* \\ \vdots \\ \bar{D}_{\alpha l}^* \end{bmatrix} F = \begin{bmatrix} \bar{C}_{\alpha 1}^* \\ \vdots \\ \bar{C}_{\alpha l}^* \end{bmatrix}$$

is required. We get the Lemma:

Lemma 6.1. Necessary and sufficient condition for compatibility of the maximal unobservable subspaces Υ_i^* of subsystems S_i^*, $i \in l$ is:

$$\operatorname{rank} [\bar{D}_\alpha^* \vdots \bar{C}_\alpha^*] = \operatorname{rank} \bar{D}_\alpha^* , \tag{20}$$

where

$$\bar{D}_\alpha^* = \begin{bmatrix} \bar{D}_{\alpha 1}^* \\ \vdots \\ \bar{D}_{\alpha l}^* \end{bmatrix}, \quad \bar{C}_\alpha^* = \begin{bmatrix} \bar{C}_{\alpha 1}^* \\ \vdots \\ \bar{C}_{\alpha l}^* \end{bmatrix}.$$

Henceforth, if (20) is true, a matrix of state feedback (10) simultaneously constructing the maximal unobservable subspaces Υ_i^*, $i \in \bar{l}$ is given by (21):

$$F_\alpha = -\bar{D}_\alpha^* \bar{C}_\alpha^* . \tag{21}$$

Comparing the foregoing result with Lemma 3.1, it is easy to see that (20) appears as a matrix form of the geometric compatibility condition for maximal unobservable subspaces Υ_i^*, $i \in \bar{l}$.

In conclusion, the maximal unobservable subspaces may be easily computed and it is possible to state the condition for their compatibility in a convenient matrix form. These subspaces are (A, B)-invariant, hence according to Theorem 1.1, they are useful for the determination of the relevant controllability subspaces $\mathscr{R}_i^*$, $i \in \bar{l}$.

Example 1.1. Given the system S by (1) with matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

whose outputs are partitioned such that $m_1 = 1$, $m_2 = 2$. For subsystems S_1^*, S_2^* defined in this way, compatibility of the maximal unobservable subspaces Υ_1^*, Υ_2^* will be tested.

Applying Construction 2 for subsystem S_1^*, we have:

1. $$C_2 B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}, \quad \begin{bmatrix} {}^1D_2 \\ \hdashline 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_2 A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} {}^1\bar{C}_2 \\ \hdashline {}^1\bar{\bar{C}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

2. $$\bar{D}_{\alpha 1}^* = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{C}_{\alpha 1}^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix},$$

$$L_\beta^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}.$$

As $m_1 = 1$, Construction 1 gives for S_2^*:

$$\bar{D}_{\alpha 2}^* = C_1 A^{d_1} B = [1\ 0\ 0], \quad \bar{C}_{\alpha 2}^* = C_1 A^{d_1 + 1} = [0\ 1\ 0\ 0].$$

Then matrix $\bar{D}_\alpha^*$ is:

$$\bar{D}_\alpha^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

hence $\bar{D}_\alpha^*$ is of full rank and compatibility condition (20) is satisfied. Matrix F_α

of state feedback constructing the subspaces Υ_1^*, Υ_2^* is computed by (21) to be:

$$F_\alpha = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Clearly the observability matrices $Q_1^*(F_\alpha)$; $Q_2^*(F_2)$ of subsystems S_1^*, S_2^* are F-invariant and hence maximally reduced.

5. AN ALGORITHMIC FORM OF THE GEOMETRIC FORMULATION OF THE DECOUPLING PROBLEM

We will show that if compatibility condition is satisfied, no further complications can appear with the determination of the maximal controllability subspaces $\mathscr{R}_i^*$, $i \in l$.

Theorem 3.1. The maximal controllability subspaces $\mathscr{R}_i^*$ of subsystems S_i^*, $i \in l$ according to (4) are compatible if and only if the relevant maximal unobservable subspaces Υ_i^*, $i \in l$ are compatible.

Proof. If: From $\mathscr{R}_i^* \subset \Upsilon_i^*$, $i \in l$ it follows:

$$\mathscr{R}^{*\Sigma} \subset \Upsilon^*,$$

where $\mathscr{R}^{*\Sigma} = \bigcap_{i \in l} \mathscr{R}_i^{*\Sigma}$ and $\mathscr{R}_i^{*\Sigma} = \sum_{\substack{j \in l \\ i \neq j}} \mathscr{R}_j^*$, $i \in l$ (Υ^* being the maximal unobservable subpsace of system), hence (A, B)-invariance of $\mathscr{R}^{*\Sigma}$ results.

Only if: According to Remark 3.1, (n − dim $\mathscr{R}^{*\Sigma}$) poles of the decoupled system may be freely assigned by state feedback. Let these poles be assigned to correspond to (n − dim $\mathscr{R}^{*\Sigma}$) of the modes of the maximal unobservable subspaces Υ_i^*, $i \in l$. The modes of remnant $\mathscr{R}^{*\Sigma}$ are fixed for any matrix F of state feedback, but because of $\mathscr{R}^{*\Sigma} \subset \Upsilon^*$ they are also modes of the maximal unobservable subspaces Υ^*. So if $\mathscr{R}_i^*$, $i \in l$ are compatible, there always exists a state feedback by (10) constructing all the maximal unobservable subspaces Υ_i^*, $i \in l$, i.e. the maximal unobservable subspaces are compatible. □

Remark 5.1. An evident conclusion of Theorem 3.1 and Lemma 6.1 is that (20) appears as a necessary and sufficient condition for subspaces $\mathscr{R}_i^*$, $i \in l$ to be compatible.

Since by Theorem 2.1 the maximal unobservable subspaces are identical to the maximal (A, B)-invariant subspaces contained in Ker C_i^*, $i \in l$, Theorem 1.1 may be used for the determination of relevant maximal controllability subspaces $\mathscr{R}_i^*$, $i \in l$.

Construction 3. According to Theorem 1.1 we have for the i-th subspace $\mathscr{R}_i^*$:

$$\mathscr{R}_i^* = \sum_{j \in l} (A + BF_{\alpha i})^{j-1} (\langle B \rangle \cap \Upsilon_i^*), \quad i \in l .$$

This may be written also in the form:

$$\mathscr{R}_i^* = \sum_{j \in l} (A + BF_{\alpha i})^{j-1} \langle BG_i \rangle , \quad i \in l .$$

For both expressions of $\mathscr{R}_i^*$ to be equivalent, validity of

$$\langle B \rangle \cap \Upsilon_i^* = \langle BG_i \rangle , \quad i \in l ,$$

is required, $F_{\alpha i}$ being a state feedback by (10) maximally reducing the rank of observability matrix $Q^*(F_{\alpha i})$ of system S_i^*, $i \in l$. Then we get for G_i:

$$\langle G_i \rangle = \operatorname{Ker} (Q^*(F_{\alpha i}) B) , \quad i \in l . \tag{22}$$

Applying Lemma 5.1:

$$\langle G_i \rangle = \operatorname{Ker} (L_{\beta i}^* B) = \operatorname{Ker} \bar{D}_{\alpha i}^* , \quad i \in l , \tag{23}$$

hence the maximal controllability subspace contained in Ker C_i^* is defined to be:

$$\mathscr{R}_i^* = \sum_{j \in l} (A + BF_{\alpha i})^{j-1} \langle BG_i \rangle , \quad i \in l , \tag{24}$$

where G_i is given by (23).

We generalize the foregoing results in an algorithm for group decoupling of multivariable systems with $D = 0$, suitable for computer implementation.

Algorithm 1.

1. Determine subsystems S_i^*, $i \in l$ according to desired configuration of output blocks.

2. Apply Construction 2 (resp. Construction 1) to every subsystem S_i^*, $i \in l$ and derive matrices $\bar{D}_{\alpha i}^*$, $\bar{C}_{\alpha i}^*$.

3. Compute matrices $F_{\alpha i}$ reducing maximally the observability matrices of subsystems S_i^*, $i \in l$.

4. Construct maximal controllability subspaces R_i^*, $i \in l$ contained in Ker C_i^* according to (24), (23).

5. Test necessary condition (8). If (8) fails, decoupling is not possible and the algorithm terminates.

6. Test compability condition (20). If it fails, state feedback decoupling in the sense of maximal controllability subspaces is not possible, see also Remarks 1.1, 2.1. In the contrary compute matrices F, G of the state feedback which decouples the system:

$$
\begin{aligned}
F_\alpha &= -\bar{D}_\alpha^{*+}\bar{C}_\alpha^* , \\
G &= [G_1 \vdots \dots \vdots G_l] .
\end{aligned} \tag{25}
$$

Remark 6.1. For practical use step 6 of the algorithm may be executed before step 4 and really, if matrix G is nonsingular, (8) need not to be tested.

Example 2.1. Given the system S by (1) with matrices:

$$
A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}
$$

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},
$$

whose output is partitioned such that $m_1 = 2$, $m_2 = 6$. To solve the decoupling problem, Algorithm 1 will be applied.

According to Construction 2, we have for every subsystem:

$$
\bar{D}_{\alpha 1}^* = [1 \;\; 0 \;\; 0] \qquad C_{\alpha 1}^* = [1 \; -4 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0]
$$

$$
\bar{D}_{\alpha 2}^* = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad C_{\alpha 2}^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 292 & 125 & 216 \\ 0 & 0 & -0{\cdot}5 & 4{\cdot}5 & 7{\cdot}875 & 12{\cdot}5 & 9{\cdot}75 \end{bmatrix}
$$

Clearly (20) is satisfied. Matrices F_α, G of state feedback which decouples the system are:

$$
F_\alpha = \begin{bmatrix} -1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0{\cdot}5 & -4{\cdot}5 & -7{\cdot}875 & -12{\cdot}5 & -9{\cdot}75 \\ 0 & 0 & 0 & 0 & -292 & -125 & -216 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
$$

Because of nonsingularity of G, condition for output controllability (8) will not be tested.

The transfer function of the decoupled system is:

$$S_{(p)}^{(*)} = \begin{bmatrix} \frac{p+2}{p^2} & 0 & 0 \\ \frac{p+1}{p^2} & 0 & 0 \\ 0 & -\frac{p^2+10p+25}{p^3} & 0 \\ 0 & -\frac{p^3+10{\cdot}125p^2+23{\cdot}25p-6{\cdot}875}{p} & -\frac{p+1}{p^2} \\ 0 & -\frac{0{\cdot}875p^2+9{\cdot}5p+20{\cdot}625}{p^4} & \frac{p+3}{p} \\ 0 & \frac{-1}{p^3} & 0 \\ 0 & -\frac{1{\cdot}75p-13{\cdot}75}{p^4} & \frac{2}{p^2} \\ 0 & -\frac{2p^2+21p+55}{p^3} & 0 \end{bmatrix}$$

Algorithm 1 was derived directly from the geometric formulation of decoupling problem given by Wonham and Morse [10], utilizing the concept of maximal unobservable subspaces. This concept enables one to treat the condition for compatibility of the maximal controllability subspaces contained in Ker C_i^*, $i \in l$ like condition for compatibility of relevant maximal unobservable subspaces and this way to state it into a matrix form. Using the equivalence between the maximal unobservable subspaces and the maximal (A, B)-invariant subspaces contained in Ker C_i^*, $i \in l$, the maximal controllability subspaces $\mathscr{R}_i^*$, $i \in l$ are found without the recursive algorithm given by Wonham and Morse [10]. Algorithm 1 seems to give the same results as algorithm derived by Silverman and Payne [9] but is obtained in a more objective and simpler way. The necessary and sufficient conditions for decoupling in Algorithm 1 and the final expressions (25), (26) for control law matrices F, G that are formally identical to those of Silverman and Payne [9] prove the equivalence between the approaches of Wonham and Morse [10] and Silverman and Payne [9] and show the correctness of the present method for algorithmisation.

6. DECOUPLING OF NON-COMPLETELY OUTPUT CONTROLLABLE SYSTEMS

Both the geometric formulation of decoupling problem and Algorithm 1 consider only completely output controllable systems. The extension of the formulation of decoupling problem to non-completely output controllable systems really involves some restrictions on the class of decouplable systems. Before stating the conditions for decoupling of non-completely output controllable systems, we will discuss the nature of output controllability. Similarly to the (state) controllable subspaces we define output controllable subspaces.

Definition 6.1. Subspace $\mathscr{V}_0 \subset \mathscr{Y}$ is the output controllable subspace if

$$\mathscr{V}_0 = C(\sum_{j \in \bar{l}} A^{j-1}\langle B \rangle) + \langle D \rangle \, .$$

Definition 7.1. Subspace $\mathscr{V} \subset \mathscr{Y}$ is the output controllability subspace if there exist maps $F : \mathscr{X} \to \mathscr{U}$ and $G : \mathscr{U} \to \mathscr{U}$ such that

$$\mathscr{V} = (C + DF)(\sum_{j \in \bar{l}} (A + BF)^{j-1} \langle BG \rangle + \langle DG \rangle \, .$$

Clearly Definition 6.1 determines a subspace of output space $\mathscr{Y}$ reachable (controllable) by input. Subspace $\mathscr{V}_0$ contains every subspace $\mathscr{V}$ because of F-invariance of controllability and its dependence on G. If system S is completely output controllable, $\mathscr{V}_0$ coincides with $\mathscr{Y}$. The rank of the output controllability matrix

$$V = [D \mid CB \mid \ldots \mid CA^{n-1}B]$$

defines the dimension of output controllable subspace $\mathscr{V}_0$.

The ultimate purpose of every synthesis is to control the system output at some nontrivial values. For that reason the output controllable subspace is required to be maximal. As control law may only restrict the output controllable subspace, desirable maximality of output controllable subspace appears as invariance of output controllability under state feedback. Henceforth the ranks of output controllability matrix V of given system, V_i of subsystems S_i, $V^{(a)}$ of decoupled system, and $V_i^{(a)}$, $i \in \bar{l}$ of its subsystems must satisfy the following necessary conditions:

$$\text{rank } V = \text{rank } V^{(a)} \, , \tag{27}$$

$$\text{rank } V_i = \text{rank } V_i^{(a)} \, , \quad i \in \bar{l} \, . \tag{28}$$

Notice that these conditions are equivalent to (8) and to the desired complete output controllability in the geometric formulation.

Evidently the output controllability matrix $V^{(a)}$ of decoupled system is block diagonal, hence it is true for its blocks:

$$\text{rank } V^{(a)} = \sum_{i \in \bar{l}} \text{rank } V_i^{(a)} \, . \tag{29}$$

Finally, from (27), (28), (29), we get the following restriction on the class of all decouplable (not necessarily completely output controllable) systems:

$$\sum_{i \in l} \operatorname{rank} V_i = \operatorname{rank} V . \tag{30}$$

It is not difficult to show that (30) does not depend only on the given system, but also on the way in which output groups are selected. Consequently, for completely output controllable systems, (30) is always true and (27) implies (8).

Definition 8.1. If for a given partition of system outputs the condition (30) fails, the system will be called a system with inherent interaction.

Clearly systems with inherent interaction cannot be decoupled. It will be shown that a system may be with inherent interaction for one partition of outputs and without inherent interaction for another.

Example 3.1. Given system by (1) with matrices:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let the outputs be partitioned into 2 subsets, such that $m_1 = 2$ and $m_2 = 1$. Then we have

$$C_1 = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_2 = [0 \ 1 \ 0],$$

and the system is without inherent interaction. For another partition of outputs $m_1 = 1$, $m_2 = 2$ we get:

$$C_1 = [2 \ 0 \ -1], \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

i.e., there is inherent interaction.

Inherent interaction expresses physically nonrealistic requirements on control. So it will be further assumed that for a given partition of outputs the system has no inherent interaction. For systems without inherent interaction clearly conditions (27) and (28) are equivalent. For non-completely output controllable systems, necessary condition (8) has to be modified into the form (28):

$$\operatorname{rank}(C_i \bar{R}_i) = \operatorname{rank}(C_i[B \vdots \ldots \vdots A^{n-1}B]), \quad i \in l . \tag{31}$$

Because of evident independence between compatibility condition and output controllability, Algorithm 1 may be applied, with (31) in place of (8), also for the case of non-completely output controllable systems.

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REFERENCES

[1] M. J. Denham: A necessary and sufficient condition for decoupling by output feedback. IEEE Trans. Automat. Control *AC-18* (1973), 5, 535—537.

[2] P. L. Falb, W.A. Wolovich: Decoupling in the design and synthesis of multivariable control systems. IEEE Trans. Automat. Control *AC-12* (1967), 6, 651—659.

[3] D. F. Filev: State-Space Approach to Synthesis of Autonomous Multi Input-Multi Output Systems (in Czech). Ph. D. Dissertation, Czech Technical University, Prague 1979.

[4] E. G. Gilbert: The decoupling of multivariable systems by state feedback. SIAM J. Control *7* (1969), 1, 50—63.

[5] J. W. Howze: Necessary and sufficient conditions for decoupling using output feedback. IEEE Trans. Automat. Control *AC-18* (1973), 1, 44—46.

[6] A. G. J. MacFarlane, N. Karcanias: Relationships between state space and frequency-response concepts. Proc. 7-th IFAC Congres, Helsinki 1978.

[7] A. S. Morse, W. M. Wonham: Decoupling and pole assignment by dynamic compensation. SIAM J. Control *8* (1970), 3, 317—337.

[8] L. M. Silverman: Inversion of multivariable linear systems. IEEE Trans. Automat. Control *AC-14* (1969), 270—276.

[9] L. M. Silverman, H. J. Payne: Input-output structure of linear systems with application to the decoupling problem. SIAM J. Control *9* (1971), 2, 199—233.

[10] W. M. Wonham, A. S. Morse: Decoupling and pole assignment in linear multivariable systems: a geometric approach. SIAM J. Control *8* (1970), 1, 1—18.

[11] W. M. Wonham: Linear Multivariable Control: A Geometric Approach. Springer-Verlag, Berlin—Heidelberg—New York 1974.

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