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MODIFIED QUASILINEAR FILTERING METHOD FOR ESTIMATION OF PROCESSES IN MULTIDIMENSIONAL NONLINEAR STOCHASTIC SYSTEMS

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The modified quasilinear filtering method has been proposed. This method produces more accurate filter coefficients than the standard quasilinear filtering method. To compute these coefficients, it suffices to know the distribution of the state vector of a stochastic system. It can be determined by using the software for statistical analysis of multidimensional nonlinear stochastic systems. All computations connected with the determination of coefficients of the modified quasilinear filters do not use the results of observations. Therefore they can be computed before the filter design. An example is also presented.

1. INTRODUCTION

The methods of optimal filtering of processes determined by nonlinear stochastic differential equations usually yield complicated solutions that are difficult to realize. In practical problems, it is common to use suboptimal filtering methods for estimation of state variables and unknown parameters in stochastic systems. In the case of multidimensional nonlinear systems the practical realization of the extended Kalman filter, as well as similar filters [5], involves considerable computational difficulties. The use of a continuous extended Kalman filter is based on solving $n(n+3)/2$, n being the dimension of state vector, differential equations for estimation of the state vector directly during the process of observation results. The quasilinear filtering method is therefore used in the case of multidimensional systems.

This method is based on statistical linearization of nonlinear functions of the original system, followed by Kalman filtering of the thus-obtained linearized system. The statistical linearization coefficients are computed under the assumption of normality of the distribution of the state vector. The practical merit of quasilinear filters is the simple realization of these filters in real time, especially in the case of dynamic systems of large dimensions.

The aim of this paper is to develop alternative methods for more accurate computation of the statistical linearization and quasilinear filter coefficients. These methods can improve the accuracy of quasilinear filters.

2. STATEMENT OF THE PROBLEM

Let us consider a continuous dynamic system determined by a stochastic differential equation

$$\dot{x} = f(x,t) + G(t)\,w(t), \quad t \geq t_0\,. \tag{1}$$

The observed process $y = y(t)$ is determined by

$$y = h(x,t) + v(t)\,. \tag{2}$$

In equations (1) and (2), $x \in \mathbb{R}^\kappa$ is the state vector of the system, $y \in \mathbb{R}^d$ is the vector of observations, $f(x,t)$ and $h(x,t)$ are given functions mapping $\mathbb{R}^n \times \mathbb{R}$ into $\mathbb{R}^n$ and $\mathbb{R}^d$ respectively, $G(t)$ is $(n \times r)$ matrix, $\mathbb{R}^n$ is an n-dimensional Euclidean space, $w \in \mathbb{R}^r$ and $v \in \mathbb{R}^d$ are independent normal white noises, $E[w(t)] = E[v(t)] = 0$, $E[w(t)\,w(\tau)^T] = Q(t)\,\delta(t-\tau)$ and $E[v(t)v(\tau)^T] = R(t)\,\delta(t-\tau)$. Initial value $x_0 = x(t_0)$ is independent of $v(t)$ and $w(t)$, $t \geq t_0$. The matrix $R(t)$ is uniformly nonsingular in t.

On the basis of observations $y_{t_0}^t = \{y(\tau),\ t_0 \leq \tau \leq t\}$ it is required to estimate the state vector $x(t)$ by the minimum mean square error criterion. For approximately solving this nonlinear filtering problem, we shall use statistical linearization method.

3. THE MODIFIED QUASILINEAR FILTERING METHOD

Let us approximate the nonlinear functions $f(x,t)$ and $h(x,t)$ in the equations (1) and (2) by using statistical linearization method [3, 10]. Then we obtain

$$f(x,t) \cong a_0 + a_1(x-m), \quad h(x,t) \cong b_0 + b_1(x-m), \quad m = E[x]\,. \tag{3}$$

The statistical linearization coefficients a_0, a_1, b_0 and b_1 are determined by minimizing the mean square error. The formulae for these coefficients take the form

$$a_0 = E[f(x,t)], \quad a_1 = E[f(x,t)\,(x-m)]\,K^{-1}, \tag{4}$$

$$b_0 = E[h(x,t)], \quad b_1 = E[h(x,t)\,(x-m)]\,K^{-1}\,. \tag{5}$$

Here $m = E[x]$ and $K = E[(x-m)\,(x-m)^T]$ are expectation and covariance matrix of the state vector $x = x(t)$, respectively. Substituting the expressions (3) into (1) and (2), we have

$$\dot{x} = a_1 x + a_0 - a_1 m + Gw, \quad y = b_1 x + b_0 - b_1 m + v\,. \tag{6}$$

The system (5) is a model of system (1) and (2). To this model the Kalman filter equations take the form

$$\dot{\hat{x}} = a_1\hat{x} + a_0 - a_1 m + P b_1^T R^{-1}(y - b_1\hat{x} - b_0 + b_1 m)\,, \tag{7}$$

$$\dot{P} = a_1 P + P a_1^T - P b_1^T R^{-1} b_1 P + GQG^T\,, \tag{8}$$

where $\hat{x} = \hat{x}(t)$ is the estimate of the state vector $x = x(t)$ and $P = P(t)$ is the auxiliary $n \times n$ symmetric matrix. The initial conditions for (6) and (7) are

$\hat{x}_0 = E[x_0]$ and $P_0 = E[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T]$, respectively. We notice that as different from optimal linear Kalman filter the matrix P is not the conditional covariance matrix of the filtering error. It is assumed that observability conditions for the nonlinear system (1), (2) and the linearized system (5) are fulfilled [4, 6].

In the standard case the statistical linearization coefficients a_0, a_1, b_0 and b_1 in (4) are determined approximately under the assumption of normality of the distribution of the state vector $x(t)$. In this case the filter (6) and (7) is called the quasilinear filter (QLF) [2, 3, 10].

To increase the accuracy of computation of the coefficients a_0, a_1, b_0 and b_1 of the QLF (6) and (7), the various approximate methods based on the parametrization of distributions of the state vector are proposed. Approximating the unknown density $p(x;t)$ or characteristic function $g(\lambda;t)$ of the state vector $x(t)$ by some known functions $p^*(x;\theta)$ and $g^*(\lambda;\theta)$ depending on a finite-dimensional vector parameter θ, we reduce the problem of approximate determination of the distribution to the problem of determination of the vector parameter $\theta = \theta(t)$ as function of time t. The vector parameter θ may be represented by the set of moments, semi-invariants(cumulants) and quasi- moments [7]. This vector θ is defined by ordinary differential equations. The filter (6) and (7) with coefficients a_0, a_1, b_0 and b_1 determined by using the various methods based on parametrization of distributions such as the method of moments, the method of semi-invariants, the method of quasi-moments and others, we shall call the modified quasilinear filter (MQLF).

Now we pay attention to two circumstances. First, equations determining the statistical linearization coefficients a_0, a_1, b_0, b_1 and the auxiliary matrix P in (6) and (7) do not contain the results of observations $y(t)$ and consequently, may be determined separately (beforehand when the observations are not yet performed). Then the estimate $\hat{x}$ will be determined by the integration of equation of the MQLF(6) with only the current results of the observations. The equation (6) is sufficiently simple and may be integrated in real time basis during the observation of the studied dynamic system behavior. Secondly, the methods based on parametrization of distributions enable us to determine the true mean square filtering error $E[(x - \hat{x})^T(x - \hat{x})]$ with any degree of accuracy. The calculations of the accuracy of MQLF do not use the results of observations and prior data are only necessary to fulfill them.

4. PARAMETRIZATION OF DISTRIBUTIONS. METHOD OF MOMENTS FOR CALCULATING THE COEFFICIENTS OF THE MQLF

The coefficients a_i and b_i of the MQLF (6) and (7) depend on the expectations in (4). To calculate these expectations it is sufficient to know the one-dimensional distribution (density or characteristic function) of the state vector x determined by (1). In the general case the density of vector x is determined by the multi-dimensional Fokker–Planck–Kolmogorov (FPK) equation. The most general methods to find an approximate solution of FPK equation are based on parametrization of distributions, i.e., representation of unknown density in the form of various truncated series (in particular, the Edgeworth series). Among these are the moments method,

the semi-invariants method and the quasi-moments method [7]. These methods allow a practical solution with any degree of accuracy.

To be specific, let us consider the method of moments for calculating the coefficients of the MQLF. In this method the vector parameter θ on which depends the function $p^*(x;\theta)$ approximating the unknown density represents the set of initial or central moments of the state vector $x \in \mathbb{R}^n$ up to a given order N :

$$\alpha_{\gamma_1 \cdots \gamma_n}(t) = E[x_1(t)^{\gamma_1} \cdots x_n(t)^{\gamma_n}], \tag{9}$$

$$\begin{aligned} \mu_{\gamma_1 \cdots \gamma_n}(t) &= E[(x_1(t) - m_1)^{\gamma_1} \cdots (x_n(t) - m_n)^{\gamma_n}], \\ m_i &= E[x_i(t)], \ i = 1, \ldots, n, \ \gamma_1 + \cdots + \gamma_n = 1, \ldots, N, \end{aligned} \tag{10}$$

where $x_1, \ldots, x_n$ and $m_1, \ldots, m_n$ are the components of the state vector x and its expectation $m = E[x]$, respectively.

We obtain from (1) the exact ordinary differential equations for the initial moments $\alpha_{\gamma_1 \cdots \gamma_n}$ [7]

$$\begin{aligned} \dot{\alpha}_{\gamma_1 \cdots \gamma_n} &= \sum_{s=1}^{n} \gamma_s E[f_s(x,t) x_1^{\gamma_1} \cdots x_{s-1}^{\gamma_{s-1}} x_s^{\gamma_s - 1} x_{s+1}^{\gamma_{s+1}} \cdots x_n^{\gamma_n}] \\ &+ \frac{1}{2} \sum_{s=1}^{n} \gamma_s (\gamma_s - 1) \sigma_{ss} E[x_1^{\gamma_1} \cdots x_{s-1}^{\gamma_{s-1}} x_s^{\gamma_s - 2} x_{s+1}^{\gamma_{s+1}} \cdots x_n^{\gamma_n}] \\ &+ \sum_{h=2}^{n} \sum_{s=1}^{h-1} \gamma_s \gamma_h \sigma_{sh} E[x_1^{\gamma_1} \cdots x_{s-1}^{\gamma_{s-1}} x_s^{\gamma_s - 1} x_{s+1}^{\gamma_{s+1}} \cdots x_{h-1}^{\gamma_{h-1}} x_h^{\gamma_h - 1} x_{h+1}^{\gamma_{h+1}} \cdots x_n^{\gamma_n}], \\ &\gamma_1 + \cdots + \gamma_n = 1, \ldots, N, \end{aligned} \tag{11}$$

where

$$f(x,t) = [f_1(x,t) \cdots f_n(x,t)]^T, \ \sigma(t) = G(t) Q(t) G(t)^T, \ \sigma = [\sigma_{sh}] \ (s, h = 1, \ldots, n).$$

The method of moments computes the expectations in (10) by replacing the true unknown density $p(x;t)$ of the state vector $x(t)$ with some approximate function $p^*(x;\theta)$, which is entirely determined by the moments up to the Nth order (Edgeworth series). We have

$$p(x;t) \cong p^*(x;\theta) = N(m,K) \left[1 + \sum_{k=3}^{N} \sum_{\nu_1 + \cdots + \nu_n = k} \frac{c_{\nu_1 \ldots \nu_n}}{\nu_1! \cdots \nu_n!} H_{\nu_1 \cdots \nu_n}(x - m) \right], \tag{12}$$

where θ is the vector parameter consisting of the moments up to the Nth order, $N(m,K)$ is normal density with the expectation m and the covariance matrix K and $\{H_\nu(x)\}$ is the sequence of Hermite polynomials which satisfy the biorthogonality condition

$$\int_{\mathbb{R}^n} w(x) H_\nu(x) G_\mu(x) \, dx = \nu_1! \cdots \nu_n! \delta_{\nu\mu}$$

with another sequence of polynomials $\{G_\nu(x)\}$. Here $w(x) = N(0,K)$ is the weight function, $\nu = [\nu_1 \ldots \nu_n]$ and $\mu = [\mu_1 \ldots \mu_n]$ are multi-indexes, $\delta_{\nu\mu}$ is the Kronecker

symbol, $\delta_{\nu\nu} = 1$ and $\delta_{\nu\mu} = 0$ $(\nu \neq \mu)$. Let $\overline{w}(x)$ be some density in the n-dimensional space $\mathbb{R}^n$. A system of pairs of polynomials $\overline{p}_i(x)$ and $\overline{q}_i(x)$ $(i = 0, 1, 2, \ldots)$ is called a biorthogonal system with the weight $\overline{w}(x)$ if

$$\int_{\mathbb{R}^n} \overline{w}(x)\,\overline{p}_i(x)\,\overline{q}_j(x)\,\mathrm{d}x = \begin{cases} 0\,, & i \neq j \\ \text{constant}\,, & i = j\,. \end{cases} \tag{13}$$

In special case where $\overline{p}_i(x) = \overline{q}_i(x)$ $(i = 0, 1, 2, \ldots)$ the condition (12) takes the form

$$\int_{\mathbb{R}^n} \overline{w}(x)\,\overline{p}_i(x)\,\overline{p}_j(x)\,\mathrm{d}x = \begin{cases} 0\,, & i \neq j \\ \text{constant}\,, & i = j\,. \end{cases} \tag{14}$$

In this case the system of polynomials $\{\overline{p}_i(x)\}$ is orthogonal if it satisfies the condition (13). The sequences of Hermite polynomials $\{\overline{p}_i(x), \overline{q}_i(x)\} = \{H_i(x), G_i(x)\}$ may serve as examples of biorthogonal systems [7]. Hermite polynomials are often used for an approximate representation of densities.

The coefficients $c_\nu(t) = c_{\nu_1 \cdots \nu_n}(t)$ in (11) are given by

$$c_\nu = \int_{\mathbb{R}^n} p(x;t)\,G_\nu(x - m)\,\mathrm{d}x = E[G_\nu(x - m)] = G_\nu(\mu)\,, \tag{15}$$

where $G_\nu(\mu)$ is a linear combination of central moments of the state vector $x = x(t)$ obtained as a result of replacement in $G_\nu(x - m)$ of every monomials

$$(x_1 - m_1)^{h_1} \cdots (x_n - m_n)^{h_n}$$

by the corresponding central moment $\mu_{h_1 \cdots h_n}$ (9).

Substituting approximate density $p^*(x;\theta)$ into the expectations in (10) and evaluating that for the coefficients c_ν (14) expressed in terms of initial moments, we obtain the differential equations for all initial moments up to the Nth order. These equations are numerically integrated with the initial conditions

$$\alpha_{\gamma_1 \cdots \gamma_n}(t_0) = E[x_1(t_0)^{\gamma_1} \cdots x_n(t_0)^{\gamma_n}]\,, \quad \gamma_1 + \cdots + \gamma_n = 1, \ldots, N\,.$$

In particular case of $N = 2$ and normal density $p^*(x;\theta) = N(m, K)$, (10) represents the equations for the first- and second-order moments m and K of the state vector x. And in this case the equations of the MQLF coincide with those of the QLF. Thus the coefficients of the MQLF (6) and (7) may be determined by the method of moments. Software for determining the distributions of the state vector x of the continuous and discrete stochastic systems has been developed for various classes of computers. It contains various methods of statistical analysis of the dynamic systems based on parametrization of distributions [1, 8, 9]. This computer software include standard programs for automatic derivation and solution of the differential and difference equations for the distribution parameters (such as moments, semi-invariants and quasi-moments) of the state vector of stochastic systems of arbitrary dimensions with polynomial and some nonpolynomial nonlinearities.

Note that while using a truncated orthogonal expansion small negative values of the density are sometimes obtained. It is natural that any truncated series gives

only an approximate value of the function. In using the approximate methods in the form of various truncated series, it is recommended to vary both the number of terms of the series and the form of series. For this purpose the special software is used [1, 8, 9].

We summarize these results in the following theorem.

Theorem 4.1. (Modified Quasilinear Filter) Given $\hat{x}$ and P, the modified quasilinear filter for the model (1) and (2) is obtained by the following relations:

$$\dot{\hat{x}} = a_1\hat{x} + a_0 - a_1 m + Pb_1^T R^{-1}(y - b_1\hat{x} - b_0 + b_1 m), \tag{16}$$

$$\dot{P} = a_1 P + Pa_1^T - Pb_1^T R^{-1} b_1 P + GQG^T, \tag{17}$$

where coefficients a_0, a_1, b_0 and b_1 are determined by using the various methods based on parametrization of distributions. In the case of the method of moments, a_0 and a_1 are obtained by (10).

5. EXAMPLE

Let the dynamic system be described by scalar equation

$$\dot{x} = -x^3 + w, \quad t \geq 0 \tag{18}$$

and the observed process $y(t)$ be determined by

$$y = x + v. \tag{19}$$

Here $w = w(t)$ and $v = v(t)$ are independent normal white noises, $E[w(t)\,w(\tau)] = Q\delta(t-\tau)$ and $E[v(t)\,v(\tau)] = R\delta(t-\tau)$, where Q and R are known constants. Initial state $x_0 = x(0)$ is normal random variable with known expectation $m_0 = E[x_0]$ and variance $\mathcal{D}_0 = E[(x_0 - m_0)^2]$. The statistical linearization of the function $f(x,t) = -x^3$ takes the form (3)

$$-x^3 \cong a_0 + a_1(x - m), \quad m = m(t) = E[x(t)]. \tag{20}$$

The coefficients a_0 and a_1 are determined by the formulae (4)

$$a_0 = -E[x^3], \quad a_1 = -E[x^3(x-m)]/\mathcal{D}, \quad \mathcal{D} = \mathcal{D}(t) = E[(x(t) - m(t))^2]. \tag{21}$$

For finding the estimate of the state variable x, we shall apply the QLF and MQLF. The equations of the QLF take the form

$$\begin{aligned} \dot{\tilde{x}} &= \overline{a}_1\tilde{x} + \overline{a}_0 - \overline{a}_1 m + \overline{P}(y - \tilde{x})/R, \quad \tilde{x}_0 = m_0, \\ \dot{\overline{P}} &= 2\overline{a}_1\overline{P} - \overline{P}^2/R + Q, \quad \overline{P}_0 = \mathcal{D}_0, \end{aligned} \tag{22}$$

where $\tilde{x} = \tilde{x}(t)$ is the estimate of the state variable $x = x(t)$ and the statistical linearization coefficients $\overline{a}_0$ and $\overline{a}_1$ are evaluated for normal density $p^*(x;\theta) = N(m, \mathcal{D})$

by using the recursive formula for high-order moments $\alpha_k = E[x^k]$ of normal distributed random variable,

$$\alpha_k = \alpha_1\alpha_{k-1} + (k-1)\mathcal{D}\alpha_{k-2}, \quad \alpha_1 = E[x], \quad \mathcal{D} = \alpha_2 - \alpha_1^2, \quad k = 3, 4, \ldots . \tag{23}$$

We have

$$\begin{aligned} a_0 &\cong \overline{a}_0 = -(\alpha_1\alpha_2 + 2\mathcal{D}\alpha_1) = 2\alpha_1^3 - 3\alpha_1\alpha_2 , \\ a_1 &\cong \overline{a}_1 = -3\left(\alpha_1^2 + \mathcal{D}\right) = -3\alpha_2 . \end{aligned}$$

The differential equations for α_1 and α_2 take the form (10)

$$\dot{\alpha}_1 = -\alpha_3 , \quad \dot{\alpha}_2 = -2\alpha_4 + Q , \tag{24}$$

where the high-order moments α_3 and α_4 are evaluated for normal density by using the formula (22), i.e.,

$$\alpha_3 = \alpha_1\alpha_2 + 2\mathcal{D}\alpha_1 = 3\alpha_1\alpha_2 - 2\alpha_1^3, \quad \alpha_4 = \alpha_1\alpha_3 + 3\mathcal{D}\alpha_2 = 3\alpha_2^2 - 2\alpha_1^4 .$$

The equations of the MQLF take the form

$$\begin{aligned} \dot{\hat{x}} &= a_1\hat{x} + a_0 - a_1 m + P(y - \hat{x})/R , \quad \hat{x}_0 = m_0 , \\ \dot{P} &= 2a_1 P - P^2/R + Q , \quad P_0 = \mathcal{D}_0 , \end{aligned} \tag{25}$$

where $\hat{x} = \hat{x}(t)$ is the estimate of the state variable $x = x(t)$ and the statistical linearization coefficients a_0 and a_1 are determined by the formulae (4)

$$a_0 = -\alpha_3, \; a_1 = \left(-\alpha_4 + \alpha_1\alpha_3\right) / \left(\alpha_2 - \alpha_1^2\right) . \tag{26}$$

For determining the initial moments $\alpha_k = \alpha_k(t)$, $k = 1, 2, 3, 4$, we use the method of moments of the fourth order ($N = 4$). According to (10) the differential equations for α_k take the form

$$\dot{\alpha}_1 = -\alpha_3, \quad \dot{\alpha}_2 = -2\alpha_4 + Q, \quad \dot{\alpha}_3 = -3\alpha_5 + 3Q\alpha_1, \quad \dot{\alpha}_4 = -4\alpha_6 + 6Q\alpha_2 . \tag{27}$$

The approximation of the density $p(x;t)$ of the state variable x take the form (11)

$$p(x;t) \cong p^*(x;\theta) = N(m, \mathcal{D}) \left[1 + \frac{c_3}{3!}H_3(x-m) + \frac{c_4}{4!}H_4(x-m)\right] , \tag{28}$$

where Hermite polynomials $H_\nu(x)$, $G_\nu(x)$ and the coefficients c_3 and c_4 are determined by the formulae

$$\begin{aligned} H_\nu(x) &= (-1)^\nu \exp\left(\frac{x^2}{2\mathcal{D}}\right) \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\left[\exp\left(-\frac{x^2}{2\mathcal{D}}\right)\right] , \\ G_\nu(x) &= (-1)^\nu \exp\left(\frac{x^2}{2\mathcal{D}}\right) \left\{\frac{\mathrm{d}^\nu}{\mathrm{d}y^\nu}\left[\exp\left(-\frac{\mathcal{D}y^2}{2}\right)\right]\right\}_{y=x/\mathcal{D}} , \\ c_3 &= E\left[G_3(x-m)\right] , \quad c_4 = E\left[G_4(x-m)\right] . \end{aligned}$$

These formulae give the following recursive expressions for Hermite polynomials

$$H_{\nu+1}(x) = \frac{x}{\mathcal{D}} H_\nu(x) - \frac{\nu}{\mathcal{D}} H_{\nu-1}(x), \quad G_{\nu+1}(x) = xG_\nu(x) - \nu \mathcal{D} G_{\nu-1}(x).$$

In this particular case, we have

$$\begin{aligned} &H_0 = 1, \quad H_1 = x/\mathcal{D}, \quad H_2 = \left(x^2 - \mathcal{D}\right)/\mathcal{D}^2, \\ &H_3 = (x^3 - 3\mathcal{D}x)/\mathcal{D}^3, \quad H_4 = (x^4 - 6\mathcal{D}x^2 + 3\mathcal{D}^2)/\mathcal{D}^4, \\ &G_0 = 1, \quad G_1 = x, \quad G_2 = x^2 - \mathcal{D}, \\ &G_3 = x^3 - 3\mathcal{D}x, \quad G_4 = x^4 - 6\mathcal{D}x^2 + 3\mathcal{D}^2 \end{aligned} \tag{29}$$

and

$$c_3 = \alpha_3 - 3\alpha_1\alpha_2 + 2\alpha_1^3, \quad c_4 = \alpha_4 - 4\alpha_1\alpha_3 + 12\alpha_1^2\alpha_2 - 3\alpha_2^2 - 6\alpha_1^4. \tag{30}$$

Thus the vector parameter θ in (27) represents a 4-dimensional vector $\theta = [\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]^T$.

To close the equations (26) it is necessary to represent the high-order moments α_5 and α_6 in terms of lower-order moments α_k $(k = 1, 2, 3, 4)$. According to the method of moments and using the approximation of density $p^*(x;\theta)$ (27), (28) and (29), we obtain

$$\begin{aligned} \alpha_k &= \int_{-\infty}^{\infty} x^k p^*(x;\theta)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} x^k N(m,\mathcal{D})\,\mathrm{d}x + \frac{c_3}{6}\int_{-\infty}^{\infty} x^k H_3(x-m)N(m,\mathcal{D})\,\mathrm{d}x \\ &+ \frac{c_4}{24}\int_{-\infty}^{\infty} x^k H_4(x-m)N(m,\mathcal{D})\,\mathrm{d}x \quad (\text{at } k = 5, 6). \end{aligned} \tag{31}$$

Denote by

$$I_k = \int_{-\infty}^{\infty} x^k N(m,\mathcal{D})\,\mathrm{d}x, \quad J_{k,\nu} = \int_{-\infty}^{\infty} x^k H_\nu(x-m)\,N(m,\mathcal{D})\,\mathrm{d}x. \tag{32}$$

Then the integrals I_k and $J_{k,\nu}$ are calculated by the recursive formulae

$$\begin{aligned} &I_k = mI_{k-1} + (k-1)\mathcal{D}I_{k-2}, \quad I_1 = m = \alpha_1, \quad I_2 = \mathcal{D} + m^2 = \alpha_2, \\ &J_{k,\nu+1} = \frac{1}{\mathcal{D}}\left(J_{k+1,\nu} - mJ_{k,\nu} - \nu J_{k,\nu-1}\right), \; J_{k,0} = I_k, \quad J_{k,1} = \frac{1}{\mathcal{D}}\left(I_{k+1} - mI_k\right). \end{aligned} \tag{33}$$

Thus the system of ordinary differential equations for initial moments α_k (26) in which the high-order moments α_5 and α_6 are calculated by the formulae (30), (31) and (32) completely determine the method of moments of the fourth-order.

All the differential equations for the moments (23) and (26) have been numerically integrated by the Euler and Adams–Moulton method, respectively, with a step $\Delta t = 0.01$. Initial values are given by $Q = 0.05$, $R = 0.05$, $m_0 = 4.0$ and $D_0 = 0.05$. The results of Monte–Carlo simulation, performed with identical noise sequences for the QLF algorithm (21) and the MQLF algorithm (24), are compared in Figure 1.

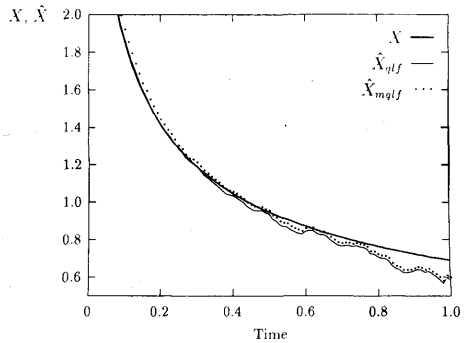

Fig. 1. Performance of the QLF and MQLF.

Figure 2 shows the true mean square filtering errors

$$P_{qlf} = E(x - \tilde{x})^2\,, \quad P_{mqlf} = E(x - \hat{x})^2$$

for the QLF and MQLF, respectively. The mean square filtering errors P_{qlf} and P_{mqlf} have been calculated by solving the corresponding statistical analysis problems of the 2-dimensional random processes $[x\ \tilde{x}]^T$ and $[x\ \hat{x}]^T$ determined by the stochastic differential equations (17), (18) and (21) and (17), (18) and (24), respectively.

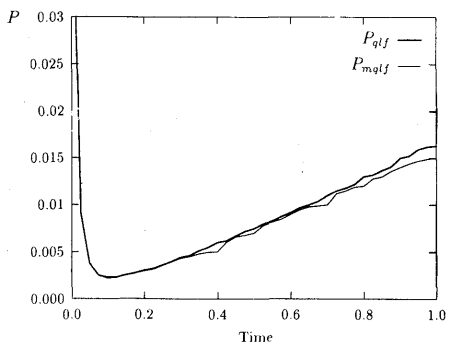

Fig. 2. Mean Square Filtering Errors.

The joint density of x and $\tilde{x}$ (x and $\hat{x}$) have been calculated by using the method of moments of the fourth-order. The mean square errors P_{qlf} and P_{mqlf} estimate the accuracy of filters. As illustrated in Figure 2 the MQLF is better than the QLF. At $t = 1.0$ we have $P_{mqlf}(1.0) = 0.015$, $P_{qlf}(1.0) = 0.0163$ and relative error $\epsilon = \epsilon(1.0) = |(P_{mqlf} - P_{qlf})/P_{mqlf}| \times 100\,\% = 8.6\,\%$. It is a good result especially for high-accuracy control system design (the oscillations of an aircraft in a tubulent air, the gyroscope drift and others). We note that the increasing of accuracy of the MQLF during the filtering is achieved without increasing complexity of computations since the calculations of a_i and b_i of the MQLF (6) and (7) don't depend on

measurements and therefore they can be calculated before the filter design. Thus the results of a comparative analysis show that the MQLF are more accurate than the QLF.

6. CONCLUSION

The MQLF can be used for a wide range of control problems of large dimensional systems. As a linear Kalman filtering theory, all the computations connected with the determination of a filter gains (a_0, a_1 and P) are based on a priori data about the model (1) and (2) and do not use the results of observations. Therefore they can be carried out before the MQLF design. The determination of the estimate $\hat{x}(t)$ from the results of observations reduces to integration of the differential equation (6), which can be done in a real time.

The MQLF also enable us to estimate the state of the system under conditions of uncertainty, when the functions f and h in (1) and (2) depend on unknown parameters. For this purpose it is necessary to extend the state vector x by including in it the unknown parameters as additional components. The MQLF can be used for the discrete and continuous-discrete filtering problems.

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REFERENCES

[1] M. L. Dashevskii: Technical realization of the moment-semi-invariant method for analysis of random processes. Automat. Remote Control *37* (1976), 10, 1490–1492.

[2] M. L. Dashevskii, I. D. Siluyanova and V. I. Shin: On error evaluation of the quasilinear filtering method. Automat. Remote Control *45* (1984), 6, 731–737.

[3] A. Gelb: Applied Optimal Estimation. MIT Press, Cambridge, MA 1974.

[4] R. Hermann and A. J. Krener: Nonlinear controllability and observability. IEEE Trans. Automat. Control *22* (1977), 5, 728–740.

[5] A. H. Jazwinski: Stochastic Processes and Filtering Theory. Academic Press, New York 1970.

[6] H. Kwakernaak and R. Sivan: Linear Optimal Control Systems. Wiley–Interscience, New York 1972.

[7] V. S. Pugachev and I. N. Sinitsyn: Stochastic Differential Systems: Analysis and Filtering. Wiley, New York 1987.

[8] V. S. Pugachev, I. N. Sinitsyn and V. I. Shin: Program implementation of the normal approximation method for nonlinear stochastic system analysis. Automat. Remote Control *48* (1987), 2, 189–194.

[9] V. S. Pugachev, I. N. Sinitsyn, A. A. Cherednichenko, V. I. Shin and V. I. Sinitsyn: Mathematical software for analysis of multidimensional nonlinear stochastic systems. Automat. Remote Control *52* (1991), 1, 73–80.

[10] P. F. Stendel: Stochastic Optimal Control: Theory and Application. Wiley, New York 1986.

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