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Quantification method of classification processes. Concept of structural $a$-entropy

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The aim of this paper is to form a quantificatory theory of classificatory processes. A concept of structural $\alpha$-entropy is defined and its form is derived.

**Definition 1.** Let $B$ be a non-empty set with a normed measure (it is a measure defined on the set of all subsets of $B$ such that the measure of $B$ is 1). Let $\{\mathcal{X}_v\}_{v \in \mathcal{J}}$ be an indexed set of finite families $\mathcal{X}_v$ of propositional functions on $B$ ($\mathcal{X}_v = \{p_1, \ldots, p_N\}$, where $N_v$ is a positive integer) such that

$$
\bigcup_{i=1}^{N_v} M_i(\mathcal{X}_v) = B, \quad M_i(\mathcal{X}_v) \cap M_j(\mathcal{X}_v) = \emptyset \quad \text{for} \quad i \neq j, \quad i, j = 1, 2, \ldots, N_v,
$$

and for every $v \in \mathcal{J}$ where $M_f(\mathcal{X}_v) = \{x : x \in B \text{ and } p_f(x) \text{ holds}\}$. The family $\{M_f(\mathcal{X}_v)\}_{v \in \mathcal{J}}$ is the set $B$, and the family $\mathcal{X}_v$ are said classification, base of classification, and classificatory criteria, respectively.

In the sequel we will denote the classification only by $\mathcal{A}(B) = \{M_i\}$ because we shall not distinguish among classificatory criteria. Let us discuss Definition 1 in more detail: the classification was defined on the sets with normed measure and, consequently, we have simultaneously introduced a quantification of the base of classification. However, it is purposeful to quantitative the classifications of given base. According to this purpose we shall give some formal considerations and denotations: every element of $\mathcal{A}(B)$ we call element of classification; every element $M_i \in \mathcal{A}(B)$ has a measure $\mu(M_i), i = 1, \ldots, N$. The measures $\mu(M_i)$ will serve here as foundation means for quantification of classification and therefore we shall write the classification in the sequel as $\mathcal{A}(B) = \{M_1, \ldots, M_N, \mu_1, \ldots, \mu_N\}$, where $\mu_i = \mu(M_i)$.

In this paper we introduce axiomatically a real function of classifications, so called structural $\alpha$-entropy, which can serve as a quantitative measure of classification. It will be shown, that there is an analogy between $\alpha$-entropy and the usual entropy from information theory.
Definition 2. Let $\mathcal{A}(B) = \{M_1, \ldots, M_N, \mu_1, \ldots, \mu_N\}$ be a classification. A function $S(\mu_1, \ldots, \mu_N; a)$ will be said structural $a$-entropy if

\begin{itemize}
  \item[a)] $S(\mu_1, \ldots, \mu_N; a)$ is continuous in the region $\mu_i \geq 0$, $\sum_{i=1}^{N} \mu_i = 1$, $a > 0$;
  \item[b)] $S(1; a) = 0$, $S(\frac{1}{2}; a) = 1$;
  \item[c)] $S(\mu_1, \ldots, \mu_{i-1}, 0, \mu_{i+1}, \ldots, \mu_N; a) = S(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_N; a)$ for every $i = 1, 2, \ldots, N$;
  \item[d)] $S(\mu_1, \ldots, \mu_{i-1}, v_i, \mu_i, v_i, \mu_{i+1}, \ldots, \mu_N; a) = S(\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_N; a) + \mu_i S\left(\frac{v_i}{\mu_i}, \frac{v_i}{\mu_i}; a\right)$ for every $v_i = \mu_i > 0$, $i = 1, 2, \ldots, N$, $a > 0$.
\end{itemize}

The meaning of axioms a)–c) is clear. What concerns axiom d), an increase of the structural $a$-entropy provided that the classification is "refined" depends on the parameter $a$ which will be said characteristic parameter.

Theorem 1. Axioms a)–d) determine the structural $a$-entropy unambiguously by

\begin{align*}
S(\mu_1, \ldots, \mu_N; a) &= \frac{2^{a-1}}{2^{a-1} - 1}\left(1 - \sum_{i=1}^{N} \mu_i^a\right) \quad \text{for } a > 0, \ a + 1, \\
S(\mu_1, \ldots, \mu_N; 1) &= -\sum_{i=1}^{N} \mu_i \log \mu_i,
\end{align*}

where $\log$ is here and in the sequel taken to the base 2.

Proof of this theorem will be based on the following lemmas:

Lemma 1. $a = 1$.

Proof. According to d)

\begin{align*}
S(1, \frac{1}{2}; a) &= S(1; a) + aS(1, \frac{1}{2}; a),
\end{align*}

which immediately implies the desired assertion (cf. b)).

Lemma 2. If $v_k \geq 0$, $k = 1, \ldots, m$, $\sum_{k=1}^{m} v_k = \mu_i > 0$, then

\begin{align*}
S(\mu_1, \ldots, \mu_{i-1}, v_1, \mu_i, v_2, \mu_{i+1}, \ldots, \mu_N; a) &= S(\mu_1, \ldots, \mu_N; a) + \mu_i S\left(\frac{v_1}{\mu_i}, \ldots, \frac{v_n}{\mu_i}; a\right).
\end{align*}

Proof. To prove this Lemma we argue by induction. For $m = 2$ the desired statement holds (cf. d) and Lemma 1). Using Lemma 1, d) and the induction premise we obtain the following result
\[ S(\mu_1, \ldots, \mu_{i-1}, v_i, \ldots, v_{n+1}, \mu_{i+1}, \ldots, \mu_n; a) = \]
\[ = S(\mu_1, \ldots, \mu_{i-1}, v_i, \bar{\mu}, \mu_{i+1}, \ldots, \mu_n; a) + \]
\[ + \beta^i S\left(\frac{v_2}{\bar{\mu}}, \ldots, \frac{v_{n+1}}{\bar{\mu}}; a\right) = S(\mu_1, \ldots, \mu_n; a) + \beta^i S\left(\frac{v_1}{\mu_i}, \frac{\bar{\mu}}{\mu_i}; a\right) + \]
\[ + \beta^{i+1} S\left(\frac{v_2}{\bar{\mu}}, \ldots, \frac{v_n+1}{\bar{\mu}}; a\right), \]

where \( \bar{\mu} = v_2 + \ldots + v_{n+1} \). One more application of the induction premise yields
\[ S\left(\frac{v_1}{\mu_1}, \ldots, \frac{v_{n+1}}{\mu_1}; a\right) = S\left(\frac{v_1}{\mu_1}, \frac{\bar{\mu}}{\mu_1}; a\right) + \left(\frac{\bar{\mu}}{\mu_1}\right)^i S\left(\frac{v_2}{\bar{\mu}}, \ldots, \frac{v_n+1}{\bar{\mu}}; a\right) \]

and hence, in view of the preceding equality, the statement of Lemma 2 holds.

The following Lemma is an obvious consequence of Lemma 2.

**Lemma 3.** If \( \nu_j \geq 0, j = 1, 2, \ldots, m, \) \( \sum_{j=1}^{m} \nu_j = \mu_i > 0, i = 1, 2, \ldots, n, \sum_{i=1}^{n} \mu_i = 1 \), then
\[ S(\nu_1, \ldots, \nu_{mn}, \nu_{n+1}, \ldots, \nu_{mn+1}; a) = \]
\[ = S(\mu_1, \ldots, \mu_n; a) + \sum_{i=1}^{n} \mu_i^i S\left(\frac{v_1}{\mu_i}, \ldots, \frac{v_{mn+1}}{\mu_i}; a\right). \]

If we replace in Lemma 3 \( m_i \) by \( m \) and \( v_{ij} \) by \( l/mn, i = 1, \ldots, m, j = 1, 2, \ldots, n \), where \( m \) and \( n \) are positive integers, then we obtain the following

**Lemma 4.** If \( F(n, a) = S\left(1 - \frac{1}{n}, \ldots, 1 - \frac{1}{n}; a\right) \), then
\[ F(mn, a) = F(m, a) + \frac{1}{n^{m-1}} F(n, a) = F(n, a) + \frac{1}{n^{m-1}} F(m, a), \]

for every positive integers \( m, n \).

This equality implies

**Lemma 5.** If \( a \neq 1 \), then \( F(n, a) = c(a) (1 - 1/n^{m-1}) \), where \( c(a) \) is a function of the characteristic parameter.

The tools are now at hand to prove Theorem 1. If \( n \) and \( r_i 's \) are positive integers, \( \sum_{i=1}^{m} r_i = n \) and if we put \( \mu_i = r_i/n, i = 1, 2, \ldots, m \), then an application of Lemma 3 gives
\[ S\left(\frac{1}{n}, \ldots, \frac{1}{n}, \ldots, \frac{1}{n}, \frac{r_1}{n}, \ldots, \frac{1}{r_m}; a\right) = S(\mu_1, \ldots, \mu_n; a) + \sum_{i=1}^{n} \mu_i^i S\left(\frac{r_1}{r_i}, \ldots, \frac{1}{r_i}; a\right) , \]
or

\[ F(n, a) = S(\mu_1, \ldots, \mu_N; a) + \sum_{i=1}^{N} \mu_i^a F(r_i, \mu_i) \]

this together with Lemma 5 for \( a \neq 1 \) implies that

\[
S(\mu_1, \ldots, \mu_N; a) = c(a)(1 - 1/n^{a-1}) - \sum_{i=1}^{N} \mu_i^a c(a) (1 - 1/r_i^{a-1}) =
\]

\[
c(a)(1 - \sum_{i=1}^{N} \mu_i^a).
\]

In view of axiom a), the later equality holds also for irrational \( \mu_i \)'s. Using axiom b) we get

\[
c(a) = \frac{2^{a-1}}{2^{a-1} - 1},
\]

That is, for \( a \neq 1 \) we have obtained the desired result

\[
S(\mu_1, \ldots, \mu_N; a) = \frac{2^{a-1}}{2^{a-1} - 1} (1 - \sum_{i=1}^{N} \mu_i^a).
\]

The equality

\[
S(\mu_1, \ldots, \mu_N; 1) = - \sum_{i=1}^{N} \mu_i \log \mu_i
\]

is a consequence of the fact that the structural \( a \)-entropy is a continuous function of \( a \).

Remark. It is to be noted that the validity of Theorem 1 does not depend ultimately on the assumption of continuity of \( S \) in variable \( a \). If this continuity is not required, the proof of Theorem 1 remains unaltered if \( a \neq 1 \) and for \( a = 1 \) it can be modified by means of results of [1]. Consequently, the requirement of the continuity mentioned above is not necessary (cf. axiom a)).

In the sequel we list some basic properties of the structural \( a \)-entropy.

**Theorem 2.** \( S(\mu_1, \ldots, \mu_N; a) \) is in the region \( \mu_i \geq 0, \ i = 1, 2, \ldots, N, \sum_{i=1}^{N} \mu_i = 1 \), concave function achieving maximum for \( \mu_i = 1/N, \ i = 1, 2, \ldots, N \).

Proof. Concavity follows from the fact that the matrix of second derivatives of \( S(\mu_1, \ldots, \mu_N; a) \) is in the given region negative semidefinite. The proof of the second assertion will be given in the following two steps:

1. Suppose first that \( a \neq 1 \). As \( (2^{x-1} - 1) x^a / (2^{x-1} - 1) \) is for \( 0 \leq x \leq 1 \) convex function, we can write for \( \mu_i \) under consideration

\[
\frac{2^{x-1} - 1}{2^{x-1} - 1} \left( \sum_{i=1}^{N} \mu_i^a \right) \leq \frac{2^{x-1} - 1}{2^{x-1} - 1} \sum_{i=1}^{N} \mu_i^a,
\]

which yields the desired result.
2. Let now $a = 1$. As $x \log x$ is for $0 \leq x \leq 1$ convex function, we can write

$$\left( \sum_{i=1}^{N} \frac{1}{N} \mu_i \right) \log \left( \sum_{i=1}^{N} \frac{1}{N} \mu_i \right) \leq \sum_{i=1}^{N} \frac{1}{N} \mu_i \log \mu_i,$$

and the conclusion of the proof is clear.

The following property of the structural $a$-entropy seems to be useful for applications:

**Theorem 3.** If $\mu_j \geq 0, j = 1, \ldots, N$, $\sum_{j=1}^{N} \mu_j = 1$, $\mu_{i-1} < \mu_i$ for $i = 2, \ldots, N$ and if $0 < \varepsilon < (\mu_j - \mu_{j-1})/2$, then

$$S(\mu_1, \ldots, \mu_N; a) < S(\mu_1, \ldots, \mu_{i-1} + \varepsilon, \mu_i - \varepsilon, \ldots, \mu_N; a).$$

**Proof.** This Theorem obviously follows from Theorem 2.

In closing this paper let us note that the normed measure used in our considerations does not need to be interpreted as a probability measure. The structural $a$-entropy may be considered as a new generalization of the Shannon's entropy which differs from the generalization given by Rényi [2].

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Kvantifikační metoda klasifikačních procesů
Pojem strukturální a-entropie

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Práce je věnována vytvoření jisté kvantifikační metody klasifikačních procesů, přičemž pojem klasifikace je zaveden v definici 1. Problém kvantifikace klasifikace spočívá v axiomatickém zavedení jisté funkce, tzv. strukturální a-entropie na množině všech klasifikací dané množiny s normovanou mírou.

Axiomatické zavedení strukturální a-entropie uvedeným způsobem vede k jednoznačnému určení tvaru strukturální a-entropie. Dále jsou uvedeny základní vlastnosti strukturální a-entropie a ukázána možnost pravděpodobnostní interpretace získaných výsledků, která vede k jistému zobecnění Shannonovy entropie.

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