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ON A SHADOWING LEMMA IN METRIC SPACES

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Summary. In the present paper conditions are studied, under which a pseudo-orbit of a continuous map $f\colon M \to M$, where M is a metric space, is shadowed, in a more general sense, by an accurate orbit of the map f.

Keywords: Pseudo-orbit, shadowing property, shadowing lemma

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0. INTRODUCTION

One of the fundamental technical instruments used in the study of dynamical systems is the shadowing lemma. This lemma permits to prove a number of statements about the so-called axiom A diffeomorphisms ([2]) and about stable properties of diffeomorphisms with hyperbolic chain recurrent sets ([12]). It also can be used to show a chaotic behavior of diffeomorphisms near transversal homoclinic points ([11]). There are some variants of the shadowing lemma for flows ([7]) and for differential equations ([10]).

The shadowing lemma was proved by several authors using several methods. The idea first appears in Anosov's [1] studies of geodesic flows, and later Bowen [2] proved the lemma for axiom A systems. Robinson [12] gave a simple geometric proof of this lemma in general. Franke and Selgrade [7] proved this lemma for flows. The more recent Ekeland's [6] proof is based on a certain inequality. It was shown by Meyer and Sell [10] that the lemma can be proved by the inverse function theorem. Palmer [11] proved the shadowing lemma in $\mathbf{R}^n$ using the exponential dichotomy of difference equations.

The existence of an accurate orbit near a pseudo-orbit has also a great numerical importance. Numerically computed orbits are in fact pseudo-orbits and therefore the problem of their being in a neighborhood of a real orbit for a sufficiently long time arises, i.e., the question whether the numerical calculation has a real meaning. It is shown ([4], [8], [9]) that in some situations we can guarantee the existence of a real orbit for a sufficiently long time. Hence the idea of the search of nonhyperbolic mappings has a self-sufficient sense. Taking mappings defined on metric spaces as the base of our considerations, this is the most general case we can speak about pseudo-orbits. However, we cannot use differential calculus—the basis for the possibility of defining the notion of the hyperbolic set. So this conception must be replaced by a more general one.

The purpose of this paper is to show, for a generalized notion of shadowing, some results concerning φ-Hölderian mappings of metric spaces into themselves. As a consequence we have obtained statements for Lipschitzian mappings.

In the first section necessary notation and definitions are given as well as the text of the classical Shadowing Lemma. In the second section general results for φ-Hölderian mappings are formulated along with one simple application. Sufficient conditions for uniqueness of shadowing orbits are given in the third section, in the last fourth section we state some properties in the case of a compact metric space.

1. Notation and definitions

Let (M, d) be a metric space and $f\colon M \to M$ a map. Denote by I one of the following sets: the set $\mathbb{Z}$ of all integers, the set $\mathbb{N}$ of all natural numbers ($0 \in \mathbb{N}$), or a finite set $\{0, 1, \dots, n\}$. Then a sequence $\{x_k\}_{k\in I}$ of points of M is said to be a ***δ-pseudo-orbit*** of the map f iff

$$d(f(x_k), x_{k+1}) \leqslant \delta, \quad k \in I.$$

Let $\varepsilon > 0$ and let $\{x_k\}_{k\in I}$ be a δ-pseudo-orbit of f. Then the sequence $\{x_k\}_{k\in I}$ is ε-shadowed by an actual orbit iff there is a $y \in M$ such that

$$d(f^k(y), x_k) \leqslant \varepsilon, \quad k \in I.$$

Here $f^0 = \mathrm{id}_M$, the identity map of M, $f^k = f \circ f^{k-1}$ for $k > 0$, and $f^{-k} = (f^{-1})^k$ for $k > 0$, if the map f^{-1} exists.

Before we formulate the classical Shadowing Lemma we define: Let $f\colon M \to M$ be a C^1-diffeomorphism of a smooth manifold M, M being endowed by some Riemannian metric. An invariant set $K \subset M$ is called ***hyperbolic***, if there exists a continuous splitting $TM\big|_K = E^u \oplus E^s$ and constants $C > 0$, $0 < \mu < 1$, such that

$$\|Df^n(x)v\| \leqslant C \cdot \mu^n \|v\| \qquad \text{and} \qquad \|Df^{-n}(x)u\| \leqslant C \cdot \mu^n \|u\|$$

for all $x \in K$, $v \in E_x^s$, $u \in E_x^u$ and $n \geqslant 0$.

Theorem (Shadowing Lemma). *Let K be a compact, hyperbolic invariant set of a C^1-diffeomorphism $f\colon M \to M$. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every δ-pseudo-orbit of the map $f|_K$ we can find a $y \in M$ for which this δ-pseudo-orbit is shadowed by the orbit $\{f^k(y)\}_{k\in\mathbf{Z}}$. Moreover, there exists an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the point $y \in M$ is unique.*

In general metric spaces a notion of hyperbolicity is not defined. Thus similar results can be expected only for Lipschitzian maps with a Lipschitz constant $L < 1$. To obtain more delicate and complete outcomes, we will need to extend the usual concepts of δ-pseudo-orbit and ε-shadowing property to the map $f\colon M \to M$, where M is a metric space.

Let $\{\delta_k\}_{k\in I}$ be a sequence of positive real numbers. The sequence $\{x_k\}_{k\in I}$ of points of the space M will be called a (δ_k)-*pseudo-orbit* of f iff for every $k \in I$

(1) $$d(f(x_k), x_{k+1}) \leqslant \delta_k.$$

The usual notion of a δ-pseudo-orbit is obtained by setting $\delta_k = \delta$, $k \in I$.

Let $\{\varepsilon_k\}_{k\in I}$ be another sequence of positive real numbers. Then the (δ_k)-pseudo-orbit $\{x_k\}_{k\in I}$ is (ε_k)-*shadowed* by an actual orbit of the map f if we can find a $y \in M$ such that

(2) $$d(f^k(y), x_k) \leqslant \varepsilon_k, \quad k \in I.$$

Let $\varphi\colon \mathbf{R} \to \mathbf{R}$ be any real function. Then a sequence $\{\alpha_k\}_{k\in I}$ of real numbers is said to be *φ-decreasing* (*φ-increasing*, *φ-nondecreasing*, *φ-nonincreasing*, respectively) iff $\alpha_{k+1} < \varphi(\alpha_k)$ ($\alpha_{k+1} > \varphi(\alpha_k)$, $\alpha_{k+1} \geqslant \varphi(\alpha_k)$, $\alpha_{k+1} \leqslant \varphi(\alpha_k)$, respectively), $k \in I$.

We shall work with φ-Hölderian maps. A map $f\colon M \to M$ is called *φ-Hölderian*, for a given function $\varphi\colon (0,\infty) \to (0,\infty)$, iff

(3) $$d(f(x), f(y)) \leqslant \varphi(d(x,y))$$

for every $x \in M$, $y \in M$, $x \neq y$. In the special case $\varphi(t) = K \cdot t^\alpha$, where $K \geqslant 0$ and $\alpha \in (0, 1]$, one obtains the well-known conception of the α-Hölder continuity; for $\alpha = 1$ this yields the Lipschitz continuity of f.

By $B(y, \varepsilon)$ we denote the closed ball $\{x \in M;\, d(x,y) \leqslant \varepsilon\}$.

2. General case

In this section we will work with a map $f\colon M \to M$, M is an arbitrary metric space. With regard to the fact that f is not assumed to be a homeomorphism we cannot require the uniqueness of the wanted orbit in general. The following lemma represents a simple principle in such situations (cf. also [5]). The index set is either the set $\mathbf{N}$ or the finite set $\{0, 1, \ldots, n\}$.

Lemma 1. *Suppose $f\colon M \to M$ to be φ-Hölderian for some nondecreasing function $\varphi\colon (0,\infty) \to (0,\infty)$. Then for every φ-increasing sequence $\{\varepsilon_k\}$ of positive real numbers there exists a sequence $\{\delta_k\}$ of positive real numbers such that every (δ_k)-pseudo-orbit $\{x_k\}$ is (ε_k)-shadowed by an orbit $\{f^k(y)\}$ for any $y \in B(x_0, \varepsilon_0)$.*

Proof. The sequence $\{\varepsilon_k\}$ is φ-increasing and this implies the positiveness of the numbers $\delta_{k-1} = \varepsilon_k - \varphi(\varepsilon_{k-1})$, $k \geqslant 1$. Denote $B_k = B(x_k, \varepsilon_k)$. Then for any (δ_k)-pseudo-orbit $\{x_k\}$ and $k \geqslant 1$ we have

$$f(B_{k-1}) \subset B_k.$$

Indeed, take $z \in B_{k-1}$. Then

$$\begin{aligned} d(f(z), x_k) &\leqslant d(f(z), f(x_{k-1})) + d(f(x_{k-1}), x_k) \\ &\leqslant \varphi(d(z, x_{k-1})) + \delta_{k-1} \leqslant \varphi(\varepsilon_{k-1}) + \varepsilon_k - \varphi(\varepsilon_{k-1}) = \varepsilon_k, \end{aligned}$$

so $f(z) \in B$. Thus we have obtained

$$f^k(B_0) \subset f^{k-1}(B_1) \subset \ldots \subset f(B_{k-1}) \subset B_k,$$

and therefore for any $y \in B_0 = B(x_0, \varepsilon_0)$ we have $d(f^k(y), x_k) \leqslant \varepsilon_k$. This completes the proof. □

Theorem 1. *Let $f\colon M \to M$ be a φ-Hölderian map, where φ is a nondecreasing function for which $\varphi(x) < x$ holds on a subset of the interval $(0, \infty)$ with the cluster point zero. Then for every $\varepsilon > 0$ there exist $\mu \in (0, \varepsilon)$ and $\delta > 0$ such that any δ-pseudo-orbit $\{x_k\}$ is ε-shadowed by the orbit $\{f^k(y)\}$, $y \in B(x_0, \mu)$.*

Proof. For a given $\varepsilon > 0$ choose a $\mu \leqslant \varepsilon$ such that $\varphi(\mu) < \mu$. Then $\{\mu, \mu, \ldots\}$ is a φ-increasing sequence and we can apply Lemma 1 and find $\delta \leqslant \mu - \varphi(\mu)$. □

Corollary 1. *Let a map $f\colon M \to M$ be Lipschitz continuous with a constant $L < 1$ (a contraction). Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed by some orbit of f.*

Proof. Putting $\varphi(t) = L \cdot t$ in the previous theorem we get the statement holding for $\delta \leqslant \varepsilon(1 - L)$. □

Typical situations when the (δ_k)-pseudo-orbits arise are numerical calculations of iterative processes using computers. Here δ_k means the error in the k-th step. If we can estimate these errors by a constant δ, we obtain a δ-pseudo-orbit. Suppose that we numerically calculate iterations of f, where f is a contraction on a complete metric space M, and the error of the calculation in every step is estimated by a constant δ. We know, by Banach contraction principle, that there is a unique fixed point of f to which every sequence of iterations tends. Due to the inaccuracy of calculations we cannot reach this point with an arbitrary precision. If we calculate without errors we can obtain, from the proof of the Banach theorem, the information about the number of iterations being, for a given number $\varepsilon > 0$, in an ε-neighborhood of the fixed point. But also in a real situation it is useful to know beforehand the number of iterations needed.

Let a map $f\colon M \to M$ be a contraction on a complete metric space M. Denote by z_0 the unique fixed point of f and let the sequence $\{x_k\}_{k\geqslant 0}$ be a δ-pseudo-orbit of the map f, obtained, e.g., by numerical iterations of the point x_0. Then we have the estimate

$$d(z_0, x_k) \leqslant \alpha_k \cdot d(x_0, x_1) + \delta \cdot \left(\alpha_k + \frac{1}{1-L}\right), \tag{4}$$

where $\alpha_k = L^k/(1-L)$. To verify this, recall the estimate $d(z_0, y_k) \leqslant \alpha_k \cdot d(y_0, y_1)$ holding for an arbitrary orbit $\{y_k\}$ of f, $y_k = f^k(y_0)$. From Corollary 1 we have $d(f^k(x_0), x_k) \leqslant \delta/(1-L)$. Thus

$$\begin{aligned} d(z_0, x_k) &\leqslant d(z_0, f^k(x_0)) + d(f^k(x_0), x_k) \leqslant \alpha_k d(x_0, f(x_0)) + \frac{\delta}{1-L} \\ &\leqslant \alpha_k(d(x_0, x_1) + d(f(x_0), x_1)) + \frac{\delta}{1-L} \\ &\leqslant \alpha_k d(x_0, x_1) + \alpha_k \delta + \frac{\delta}{1-L} \\ &= \alpha_k d(x_0, x_1) + \delta\left(\alpha_k + \frac{1}{1-L}\right). \end{aligned}$$

Note that the righthand side of (4) tends to $\delta/(1-L)$ for k tending to infinity. Then for a given $\varepsilon > \delta/(1-L)$ we have

$$d(z_0, x_k) \leqslant \varepsilon$$

if

$$k \geqslant \ln \frac{(1-L)-\delta}{d(x_0, x_1) + \delta} \cdot (\ln L)^{-1},$$

which follows from (4) and the definition of α_k.

We have proved the following

Proposition. *Let a map f be a contraction on a complete metric space M with a Lipschitz constant $L < 1$ and with a fixed point z_0. Let $\varepsilon > 0$ and $\delta > 0$ satisfy the inequality $\delta < \varepsilon(1 - L)$. Then for an arbitrary δ-pseudo-orbit $\{x_k\}$ we have*

$$d(z_0, x_m) \leqslant \varepsilon$$

if

$$m \geqslant \ln \frac{(1 - L) - \delta}{d(x_0, x_1) + \delta} \cdot (\ln L)^{-1}.$$

The following theorem deals with a finite shadowing property for φ-Hölderian maps.

Theorem 2. *Let $f: M \to M$ be a φ-Hölderian map for an increasing function $\varphi: (0, \infty) \to (0, \infty)$ for which $\lim_{t \to 0+} \varphi(t) = 0$. Then for every $\varepsilon > 0$ and for any natural number n there exist $\delta > 0$ and $\gamma > 0$ such that if $I = \{0, \ldots, n\}$ and $\{x_k\}_{k \in I}$ is any δ-pseudo-orbit, then it is ε-shadowed by an orbit $\{f^k(y)\}_{k \in I}$, for $y \in B(x_0, \gamma)$.*

Proof. Take $\varepsilon > 0$ and $n \in \mathbf{N}$. Put $\varepsilon_n = \varepsilon$. Since the function φ is converging to zero together with t, we can surely find a positive number $\varepsilon_{n-1} \leqslant \varepsilon_n$ for which $\varphi(\varepsilon_{n-1}) < \varepsilon_n$. Continuing this process we obtain a finite nondecreasing sequence $\varepsilon_0 \leqslant \varepsilon_1 \leqslant \ldots \leqslant \varepsilon_n = \varepsilon$, where $\varphi(\varepsilon_{k-1}) < \varepsilon_k$, $k = 1, \ldots, n$. Put $\gamma = \varepsilon_0$ and $\delta = \min\{\varepsilon_k - \varphi(\varepsilon_{k-1});\ k = 1, \ldots, n\}$. If $\{x_k\}_{k \in I}$ is any δ-pseudo-orbit, then it is also a (δ_k)-pseudo-orbit, where $\delta_k = \delta$. Then, by Lemma 1, for every $y \in B(x_0, \gamma)$ the sequence $\{x_k\}$ is (ε_k)-shadowed, and hence ε-shadowed, by the orbit $\{f^k(y)\}$. □

3. Uniqueness of shadowing

In this section we will work with homeomorphisms defined on a complete metric space, the index set will be $\mathbf{Z}$, the set of all integers.

Theorem 3. *Let (M, d) be a complete metric space, let $f: M \to M$ be a homeomorphism on M such that f is a φ-Hölderian map for a continuous increasing function φ. Assume there is such an $\alpha \in (0, 1)$ that*

$$\varphi(\alpha t) \leqslant \alpha \cdot \varphi(t), \quad t > 0. \tag{5}$$

Then for an arbitrary $\alpha\cdot\varphi^{-1}$-nonincreasing sequence $\{\varepsilon_k\}$ of positive real numbers there exists a sequence $\{\delta_k\}$ of positive real numbers such that every (δ_k)-pseudo-orbit of f is (ε_k)-shadowed by a unique orbit of f.

Proof. Let $\{\varepsilon_k\}$ be an $\alpha\cdot\varphi^{-1}$-nonincreasing sequence of positive real numbers; this by definition means $\varepsilon_{k+1}\leqslant\alpha\cdot\varphi^{-1}(\varepsilon_k)$, $k\in\mathbf{Z}$. Choose a sequence $\{\delta_k\}$ of positive real numbers satisfying

$$0<\delta_k\leqslant(1-\alpha)\cdot\varphi^{-1}(\varepsilon_k). \tag{6}$$

Now, if $\{x_k\}$ is a (δ_k)-pseudo-orbit, denote $B_k=B(x_k,\varepsilon_k)$. Then

$$f^{-1}(B_{k+1})\subset B_k,\quad k\in\mathbf{Z}. \tag{7}$$

As the function φ is increasing and the sequence $\{\varepsilon_k\}$ is $\alpha\cdot\varphi^{-1}$-nonincreasing then from (1) and (6) for every $x\in f^{-1}(B_{k+1})$ we have

$$\begin{aligned} d(x_k,x)&=d(f^{-1}(f(x_k)),f^{-1}(f(x)))\leqslant\varphi(d(f(x_k),f(x)))\\ &\leqslant\varphi(d(f(x_k),x_{k+1})+d(x_{k+1},f(x)))\leqslant\varphi(\delta_k+\varepsilon_{k+1})\\ &\leqslant\varphi((1-\alpha)\varphi^{-1}(\varepsilon_k)+\alpha\varphi^{-1}(\varepsilon_k))=\varepsilon_k.\end{aligned}$$

Thus $x\in B_k$ and the inclusion (7) is valid.

For a fixed integer $m\in\mathbf{Z}$ take the sequence of sets $f^{-n}(B_{m+n})$, $n\geqslant 0$. From the foregoing we have

$$B_m\supset f^{-1}(B_{m+1})\supset\ldots\supset f^{-n}(B_{m+n})\supset\ldots,$$

i.e., a decreasing sequence of closed sets. We want to show that exactly one point is in the intersection of these sets. Because of completeness of M we only need to verify that the diameters of these sets tend to zero.

Let $y\in B_{m+n}$. Then

$$\begin{aligned} d(f^{-n}(y),f^{-n}(x_{m+n}))&\leqslant\varphi^n(d(y,x_{m+n}))\leqslant\varphi^n(\varepsilon_{m+n})\\ &\leqslant\varphi^{n-1}\circ\varphi\left(\alpha\varphi^{-1}(\varepsilon_{m+n-1})\right)\leqslant\alpha\varphi^{n-1}(\varepsilon_{m+n-1})\\ &\leqslant\alpha^2\varphi^{n-2}(\varepsilon_{m+n-2})\leqslant\ldots\leqslant\alpha^n\varepsilon_m,\end{aligned}$$

where we have used (5) and the fact that $\{\varepsilon_k\}$ is an $\alpha\cdot\varphi^{-1}$-nonincreasing sequence. Thus for $y,z\in B$ we have

$$d(f^{-n}(y),f^{-n}(z))\leqslant d(f^{-n}(y),f^{-n}(x_{m+n}))+d(f^{-n}(x_{m+n}),f^{-n}(z))\leqslant 2\alpha^n\varepsilon_m,$$

and therefore

$$\operatorname{diam} f^{-n}(B_{m+n}) = \sup\{d(f^{-n}(y), f^{-n}(z));\ y, z \in B_{m+n}\} \leqslant 2\alpha^n \varepsilon_m.$$

Since $\alpha < 1$, then $\operatorname{diam} f^{-n}(B_{m+n}) \to 0$ for $n \to \infty$. Thus we have proved the existence of a unique $y_m \in \bigcap_{n \geqslant 0} f^{-n}(B_{m+n})$. As $y_m \in f^{-n}(B_{m+n})$ for every $n \geqslant 0$, then $f^n(y_m) \in B_{m+n}$ and hence

$$d(f^n(y_m), x_{m+n}) \leqslant \varepsilon_{m+n}. \tag{8}$$

Now it is sufficient to check $\{y_m\}$ to be an orbit of f, as in that case we have $y_m = f^m(y_0)$ and then

$$d(f^m(y_0), x_m) = d(y_m, x_m) \leqslant \varepsilon_m,$$

which follows from (8) for $n = 0$. So the orbit $\{f^m(y_0)\}$ (ε_m)-shadows (δ_m)-pseudo-orbit $\{x_m\}$.

To verify $\{y_m\}$ to be the orbit we shall show that $f^{-1}(y_m) = y_{m-1}$. Because for $n \geqslant 0$ we have $y_m \in f^{-n}(B_{m+n})$, then

$$f^{-1}(y_m) \in f^{-n-1}(B_{m+n}) \subset f^{-n}(B_{m-1+n}),$$

as follows from (7). This implies

$$f^{-1}(y_m) \in \bigcap_{n \geqslant 0} f^{-n}(B_{m-1+n}),$$

which gives the required equality $f^{-1}(y_m) = y_{m-1}$, since the intersection is uniquely determined. This completes the proof. □

Remark 1. The condition (5) in Theorem 3 implies $\lim_{t \to 0+} \varphi(t) = 0$. Indeed, the function φ is increasing and positive, so the limit exists and is nonnegative. For a given $t_0 > 0$ the sequence $\{\alpha^n \cdot t_0\}$ is tending to zero and hence

$$0 \leqslant \lim_{t \to 0+} \varphi(t) = \lim_{n \to \infty} \varphi(\alpha^n \cdot t_0) \leqslant \lim_{n \to \infty} \alpha^n \cdot \varphi(t_0) = 0.$$

The condition (5) is satisfied, for instance, if φ is an increasing convex function for which $\lim_{t \to 0+} \varphi(t) = 0$. Note that (5) is never fulfilled if f is φ-Hölder continuous, except $\alpha = 1$. In that case f is a Lipschitz continuous map with a constant $L > 0$. One can put $\varphi(t) = L \cdot t$ and (5) is valid automatically. In this special case the conditions for $\{\varepsilon_k\}$ and $\{\delta_k\}$ have the form

$$\varepsilon_{k+1} \leqslant \frac{\alpha}{L} \varepsilon_k \qquad \text{and} \qquad \delta_k \leqslant \frac{1-\alpha}{L} \varepsilon_k,$$

for some $\alpha \in (0,1)$. The assertion for a Lipschitz continuous f is summarized in the following

Corollary 2. *Let $f\colon M \to M$ be a homeomorphism of a complete metric space onto itself such that f is a Lipschitz continuous map with a constant $L > 0$. Let $\alpha \in (0,1)$ be an arbitrary real number. Then for any sequence $\{\varepsilon_k\}$ of real positive numbers satisfying $\varepsilon_{k+1} \leqslant \frac{\alpha}{L}\varepsilon_k$ there exists such a sequence $\{\delta_k\}$ of positive real numbers that any (δ_k)-pseudo-orbit of f is (ε_k)-shadowed by a unique orbit of the map f.*

Moreover, if f^{-1} is a contraction, then one can set $\alpha = L$ and then the sequence $\{\varepsilon_k\}$ can be chosen constant:

Corollary 3. *Let $f\colon M \to M$ be a homeomorphism of a complete metric space onto itself such that f is a contraction. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo-orbit of f is ε-shadowed by a unique orbit of the map f.*

The next theorem describes a similar situation as in Theorem 4, but the map f is presumed to be φ-Hölderian instead of the inverse f^{-1}.

Theorem 4. *Let (M,d) be a complete metric space and let $f\colon M \to M$ be a φ-Hölderian homeomorphism of the space M onto itself, where φ is a continuous increasing function satisfying (5) for some $\alpha \in (0,1)$. Then for any $\alpha^{-1}\cdot\varphi$-nondecreasing sequence $\{\varepsilon_k\}$ of real positive numbers there exists a sequence $\{\delta_k\}$ of real positive numbers such that every (δ_k)-pseudo-orbit of f is (ε_k)-shadowed by a unique orbit of the map f.*

Proof. If the sequence $\{\varepsilon_k\}$ is $\frac{1}{\alpha}\cdot\varphi$-nondecreasing, then the sequence $\{\omega_k\}$, where $\omega_k = \varphi(\varepsilon_{-k-1})$, is $\alpha\cdot\varphi^{-1}$-nonincreasing:

$$\omega_{k+1} = \varphi(\varepsilon_{-k-2}) \leqslant \varphi\big(\varphi^{-1}(\alpha\varepsilon_{-k-1})\big) = \alpha\varphi^{-1}\big(\varphi(\varepsilon_{-k-1})\big) = \alpha\varphi^{-1}(\omega_k). \tag{9}$$

The inequality is obtained from the following observation: if $\varepsilon_{n+1} \geqslant \frac{1}{\alpha}\varphi(\varepsilon_n)$ then

$$\varepsilon_n \leqslant \varphi^{-1}(\alpha\varepsilon_{n+1}), \tag{10}$$

since φ^{-1} is also increasing. Let $\{\delta_k\}$ be a sequence of real numbers such that

$$0 < \delta_k \leqslant (1-\alpha)\cdot\varepsilon_{k+1}, \tag{11}$$

and further let $\{x_k\}$ be a (δ_k)-pseudo-orbit of the map f. Then denoting $y_k = f(x_{-k-1})$ and $\varrho_k = \delta_{-k-2}$, the sequence $\{y_k\}$ is a (ϱ_k)-pseudo-orbit of the map f^{-1}:

$$d(f^{-1}(y_{k-1}), y_k) = d(x_{-k}, f(x_{-k-1})) \leqslant \delta_{-k-1} = \varrho_{k-1}.$$

Moreover, from (11) we have

$$\varrho_k = \delta_{-k-2} \leqslant (1-\alpha)\cdot\varepsilon_{-k-1} = (1-\alpha)\cdot\varphi^{-1}(\omega_k).$$

Thus for $\{y_k\}$, $\{\varrho_k\}$ and $\{\omega_k\}$ one can use the statement of Theorem 4 and obtain a unique orbit $\{(f^{-1})^k(y)\}$, (ω_k)-shadowing $\{y_k\}$. Writing $-k$ instead of k this yields

$$d(f^k(y), y_{-k}) \leqslant \omega_{-k},$$

i.e.,

$$d(f^k(y), f(x_{k-1}) \leqslant \omega_{-k}.$$

Hence, from (9) and (11),

$$\begin{aligned} d(f^k(y), x_k) &\leqslant d(f^k(y), f(x_{k-1})) + d(f(x_{k-1}), x_k) \\ &\leqslant \omega_{-k} + \delta_{k-1} \leqslant \alpha\varepsilon_k + (1-\alpha)\varepsilon_k = \varepsilon_k, \end{aligned}$$

so that the orbit $\{f^k(y)\}$ (ε_k)-shadows the (δ_k)-pseudo-orbit $\{x_k\}$. But we must distinguish this case from the previous one in Theorem 3 because we now have no guarantee for the existence of only one such orbit.

Let there be two orbits, $\{f^k(y)\}$ and $\{f^k(z)\}$, which shadow $\{x_k\}$. Then for $k \geqslant 0$

$$\begin{aligned} d(y,z) &\leqslant d(y, f^k(x_{-k})) + d(z, f^k(x_{-k})) \\ &= d(f^k(f^{-k}(y)), f^k(x_{-k})) + d(f^k(f^{-k}(z)), f^k(x_{-k})) \\ &\leqslant \varphi^k(d(f^{-k}(y), x_{-k})) + \varphi^k(d(f^{-k}(z)), x_{-k})) \\ &\leqslant 2\varphi^k(\varepsilon_{-k}) \leqslant 2\varphi^k(\varphi^{-k}(\alpha^k\varepsilon_0)) = 2\alpha^k\varepsilon_0 \end{aligned}$$

(the last inequality follows from (10) and (5) by the induction). Since $\alpha < 1$ we must have $d(y,z) = 0$ and the proof is complete. □

As a consequence of Theorem 4 we obtain for Lipschitzian homeomorphisms

Corollary 4. *Let (M,d) be a complete metric space, $f\colon M \to M$ a Lipschitzian homeomorphism with a constant $L > 0$. Let $\alpha \in (0,1)$. Then for every sequence $\{\varepsilon_k\}$ of positive real numbers for which $\varepsilon_{k+1} \geqslant \frac{L}{\alpha}\cdot\varepsilon_k$, there exists a sequence of positive real numbers $\{\delta_k\}$ such that every (δ_k)-pseudo-orbit of f is (ε_k)-shadowed by a unique orbit of the map f.*

In addition, if f is a contraction map, then by putting $\alpha = L$ we can choose the sequences $\{\varepsilon_k\}$ and $\{\delta_k\}$ to be constant, so we immediately have

Corollary 5. *Let $f\colon M \to M$ be a homeomorphism of a complete metric space onto itself which is a contraction. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit of f is ε-shadowed by a unique orbit of the map f.*

Theorem 5. *Let $f: M \to M$ be a continuous map of a compact metric space M into itself. Then for any $\varepsilon > 0$ and for an arbitrary natural number n there exist positive real numbers $\delta > 0$ and $\gamma > 0$ such that every δ-pseudo-orbit $\{x_k\}_{k=1}^n$ is ε-shadowed by the orbit $\{f^k(y)\}_{k=1}^n$ for an arbitrary $y \in B(x_0, \gamma)$.*

Proof. Consider the function $\psi(t) = \sup\{d(f(u), f(v));\ d(u,v) \leqslant t\}$. It can be easily verified that ψ is a nondecreasing function continuous at zero. Moreover, for every $x, y \in M$

$$d(f(x), f(y)) \leqslant \sup\{d(f(u), f(v));\ d(u,v) \leqslant d(x,y)\} = \psi(d(x,y)),$$

so that (3) holds for ψ. In addition, there is an increasing function φ: $[0,\infty) \to [0,\infty)$ such that $\varphi(0) = \psi(0) = 0$ and $\varphi(t) \geqslant \psi(t)$ for every $t > 0$. Then the map f is also φ-Hölderian and we can apply Theorem 2. □

Suppose that the continuous map $f: M \to M$ has the property as in the classical Shadowing Lemma, i.e., for every ε, $0 < \varepsilon \leqslant \varepsilon_0$, there exists $\delta > 0$ such that every δ-pseudo-orbit of f is uniquely shadowed by some orbit of the map f. We shall call it the *unambiguous shadowing property*. Now, we can state the following theorem conserning the existence and isolation of fixed points of the map f.

Theorem 6. *Let a continuous map $f: M \to M$ of a compact metric space M have the unambiguous shadowing property. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d(x, f(x)) \leqslant \delta$ for some $x \in M$, then in the open ε-neighborhood of the point x there is a unique fixed point of f.*

Proof. For a given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if for some $x \in M$ we have $d(x, f(x)) \leqslant \delta_1$, then in the ε-neighborhood of x there exists a fixed point of the map f. Indeed, if not, then for every $n \in \mathbb{N}$ we can find a sequence $\{x_k\}$ in M such that $d(x_n, f(x_n)) \leqslant \frac{1}{n}$ and no fixed point is in the ε-neighborhood of x_n. Due to the compactness of M we can choose $\{x_{n_i}\}$, a convergent subsequence of $\{x_n\}$, tending to some $x \in M$. Since

$$\begin{aligned} 0 \leqslant d(x_0, f(x_0)) &\leqslant d(x_0, x_{n_i}) + d(x_{n_i}, f(x_{n_i})) + d(f(x_{n_i}), f(x_0)) \\ &\leqslant d(x_0, x_{n_i}) + \frac{1}{n_i} + d(f(x_{n_i}), f(x_0)), \end{aligned}$$

and the last expression converges to zero for $i \to \infty$ due the continuity of the map f, x_0 is a fixed point of f. Since $x_{n_i} \to x_0$, this is a contradiction to the assumption that in the ε-neighborhood of x there is no fixed point.

From the unambiguous shadowing property we obtain for $\varepsilon > 0$ the existence of a $\delta_2 > 0$ such that every δ_2-pseudo-orbit is uniquely ε-shadowed by an orbit of f. Set $\delta = \min\{\delta_1, \delta_2\}$ and assume that there is a point $x \in M$ for which $d(x, f(x)) \leqslant \delta$. Then in a δ-neighborhood of x there exists a fixed point x_0. The sequence $\{\dots, x, x, x, \dots\}$ is a δ-pseudo-orbit of f and hence in $B(x, \varepsilon)$ there is exactly one orbit of f, namely $\{\dots, x_0, x_0, x_0, \dots\}$. □

Remark 2. Note that we can use neither Corollary 3 nor 5 to obtain the unambiguous shadowing property, since the homeomorphism of a compact space onto itself cannot be a contractive map.

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S ú h r n

LEMA O TIENENÍ V METRICKÝCH PRIESTOROCH

TIBOR ŽÁČIK

V článku sa skúmajú podmienky na to, aby bola pseudo-orbita spojitého zobrazenia $f: M \to M$, kde M je metrický priestor, tienená v zovšeobecnenom zmysle skutočnou orbitou zobrazenia f.

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