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Dana Fraňková

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SUBSTITUTION METHOD FOR GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

DANA FRAŇKOVÁ, Praha

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Summary. The generalized linear differential equation

$$dx = d[A(t)]\, x + df$$

where $A, f \in BV_n^{loc}(J)$ and the matrices $I - \Delta^- A(t)$, $I + \Delta^+ A(t)$ are regular, can be transformed to an ordinary linear differential equation

$$\frac{dy}{ds} = B(s)\, y + g(s)$$

using the notion of a logarithmic prolongation along an increasing function. This method enables to derive various results about generalized LDE from the well-known properties of ordinary LDE. As an example, the variational stability of the generalized LDE is investigated.

Keywords: generalized linear differential equation, logarithmic prolongation, ordinary linear differential equation with a substitution, variational stability.

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INTRODUCTION

The generalized linear differential equation

$$dx = d[A(t)]\, x + df \tag{1}$$

has been investigated many times, e.g. in [S1], [STV]. Equivalently, this equation has the integral form

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} d[A(s)]\, x(s) + f(t_2) - f(t_1) \tag{1'}$$

where the Lebesgue-Stieltjes integral is used. Usually it is assumed that A is a real or complex $n \times n$-matrix valued function on an interval J and f, x are real or complex n-vector valued functions. A very common assumption is

(2) *the matrix-valued function A is locally of bounded variation.*

Let us denote $\Delta^+ A(t) = A(t+) - A(t)$, $\Delta^- A(t) = A(t) - A(t-)$.

When studying the equation (1), one can notice that the irregularity of matrices

$I - \Delta^- A(t), I + \Delta^+ A(t)$ can cause an odd behaviour of the equation (1) and of its solutions. However, if it is assumed that

(3) *for every* $t \in J$ *the matrices* $I - \Delta^- A(t)$, $I + \Delta^+ A(t)$ *are regular*

then many of the properties of the equation (1) are conspicuously similar to the properties of the ordinary linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y + g(s)\,. \tag{4}$$

In this paper it will be shown that this similarity is not accidental because every equation (1) can be transformed to some ordinary linear differential equation (4), provided the conditions (2) and (3) are fulfilled. As an example, a theorem on variational asymptotic stability will be proved.

1. LOGARITHMIC PROLONGATION

1.1. Notation. $\mathbb{R}^n$, $\mathbb{C}^n$ denote the n-dimensional real and complex vector spaces with the Euclidean norm $\|x\| = (\sum_{k=1}^{n} |x_k|^2)^{1/2}$. $M_n(\mathbb{R})$, $M_n(\mathbb{C})$ stand for the spaces of $n \times n$-real or complex matrices with the norm $\|A\| = \sup_{\|x\| \leqq 1} \|Ax\|$.

Let J be an interval and $a_0 \in J$ a given point. If a matrix-valued function A is locally of bounded variation, it can be written as the sum of a continuous function A^c and a break function A^b such that $A^b(a_0) = 0$.

For a function x we denote $x(t-) = \lim_{\tau \to t-} x(\tau)$, $x(t+) = \lim_{\tau \to t+} x(\tau)$ provided the limits exist. $x \circ y$ denotes the composed function $x(y(t))$.

1.2. Throughout this and the next section let us assume that a complex matrix valued function $A\colon J \to M_n(\mathbb{C})$ is given such that the assumption (2) holds. Further, assume that an increasing function $v\colon J \to \mathbb{R}$ is given such that

(5) $\|A(t_2) - A(t_1)\| \leqq v(t_2) - v(t_1)$ holds for every $t_1, t_2 \in J$ such that $t_1 \leqq t_2$.

For instance, the function v can be defined by $v(t) = t + \operatorname{var}_{a_0}^{t} A$ for $t \in J$, $t \geqq a_0$, and $v(t) = t - \operatorname{var}_{t}^{a_0} A$ for $t \in J$, $t < a_0$. Let us denote

$$J' = \operatorname{conv}\{v(J)\} \tag{6}$$

i.e. the convex hull of the set $v(J) = \{\tau \in \mathbb{R};\ \tau = v(t) \text{ for some } t \in J\}$. Evidently, J' is an interval which can be written also in the form

$$J' = \bigcup_{t \in J} [v(t-), v(t+)] = \bigcap_{[\alpha,\beta] \subset J} [v(\alpha), v(\beta)]\,;$$

of course, the interval $[v(t-), v(t+)]$ consists of a single point $v(t)$ provided v is continuous at t. We will assume that

$$\text{(7)} \qquad \begin{array}{ll} \text{if } t \text{ is the left endpoint of } J \text{ then} & v(t-) = v(t)\,; \\ \text{if } t \text{ is the right endpoint of } J \text{ then} & v(t+) = v(t)\,. \end{array}$$

1.3. Definition. a) *A matrix-valued function $\hat{A}$: $J' \to M_n(\mathbb{C})$ will be called a logarithmic prolongation of A along the increasing function v, if*

(i) *$\hat{A}$ is locally absolutely continuous on J';*

(ii) *the continuous part of the composed function $\hat{A} \circ v$ is equal to the continuous part of A;*

(iii) *if $t \in J$ is such a point that $v(t-) < v(t)$ then $\hat{A}$ is linear on the interval $[v(t-), v(t)]$ and*

$$\text{(8)} \qquad I - \exp\left[\hat{A}(v(t-)) - \hat{A}(v(t))\right] = \Delta^- A(t)\,;$$

(iv) *if $t \in J$ is such a point that $v(t) < v(t+)$ then $\hat{A}$ is linear on the interval $[v(t), v(t+)]$ and*

$$\text{(9)} \qquad \exp\left[\hat{A}(v(t+)) - \hat{A}(v(t))\right] - I = \Delta^+ A(t)\,.$$

b) *The matrix-valued function B: $J' \to M_n(\mathbb{C})$ is the derivative of a logarithmic prolongation of A along v, if B is locally Leebesgue integrable and the function*

$$\hat{A}(s) = \int_{v(a_0)}^{s} B(\sigma)\,\mathrm{d}\sigma \quad \textit{satisfies the conditions } \text{(i)}-\text{(iv)}.$$

Remark. (i) The logarithmic prolongation is in general not unique, because the matrices $\hat{A}(v(t-)) - \hat{A}(v(t))$, $\hat{A}(v(t+)) - \hat{A}(v(t))$ given by the relations (8), (9) are not determined uniquely.

(ii) Since the logarithmic prolongation $\hat{A}$ is locally absolutely continuous, evidently it has a derivative

$$B(s) = \frac{\mathrm{d}}{\mathrm{d}s}\,\hat{A}(s) \text{ a.e. on } J'\,.$$

1.4. Proposition. *If a matrix-valued function A has a logarithmic prolongation along v, then the condition* (3) *is satisfied.*

Proof. If $v(t-) < v(t)$ then the matrix $I - \Delta^- A(t) = \exp\left[\hat{A}(v(t-)) - \hat{A}(v(t))\right]$ is evidently regular. If $v(t-) = v(t)$ then the function A is continuous at t due to (5), hence $\Delta^- A(t) = 0$. Similarly the regularity of matrices $I + \Delta^+ A(t)$ can be verified.

Before investigating the existence of a logarithmic prolongation, the idea of the prolongation will be enlightened by the following theorem:

1.5. Theorem. *Let $\hat{A}$ be a logarithmic prolongation of A along v and let B be* *its derivative a.e.. Assume that* $[a, b] \subset J$. *The vector-valued function* x *is a solution of the generalized linear differential equation*

$$(10) \qquad \mathrm{d}x = \mathrm{d}[A(t)]\, x$$

on $[a, b]$, *if and only if there is a solution* y *of the ordinary linear differential equation*

$$(11) \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y$$

on $v[(a), v(b)]$ *such that* $x(t) = y(v(t))$ *for every* $t \in [a, b]$.

The essential part of the proof is contained in the following lemma:

1.6. Lemma. *Let two functions* $x: [a, b] \to C^n$, $y: [v(a), v(b)] \to C^n$ *be given such that*

(i) $x(t) = y(v(t))$ *for every* $t \in [a, b]$;
(ii) *the function* x *has bounded variation on* $[a, b]$;
(iii) *if* $t \in (a, b]$ *is such that* $v(t-) < v(t)$ *then* y *is a solution of* (11) *on* $[v(t-), v(t)]$;
(iv) *if* $t \in [a, b)$ *is such that* $v(t) < v(t+)$ *then* y *is a solution of* (11) *on* $[v(t), v(t+)]$.

Then the equality

$$(12) \qquad \int_{t_1}^{t_2} \mathrm{d}[A(s)]\, x(s) = \int_{v(t_1)}^{v(t_2)} B(s)\, y(s)\, \mathrm{d}s$$

holds for all $a \leqq t_1 < t_2 \leqq b$.

Proof. First let us recall that

$$(13) \qquad \int_{s_1}^{s_2} \mathrm{d}[\hat{A}(s)]\, y(s) = \int_{s_1}^{s_2} B(s)\, y(s)\, \mathrm{d}s$$

provided at least one of the integrals exists;
– see [F]. Let us denote

$$(14) \qquad \begin{aligned} C_t^- &= \hat{A}(v(t-)) - \hat{A}(v(t)), \\ C_t^+ &= \hat{A}(v(t+)) - \hat{A}(v(t)) \quad \text{for} \quad t \in J. \end{aligned}$$

Since the function $\hat{A}$ is linear on the intervals $[v(t-), v(t)]$, $[v(t), v(t+)]$, it has there the form

$$(15) \qquad \hat{A}(s) = \hat{A}(v(t)) + \frac{v(t) - s}{\Delta^-\, v(t)}\, C_t^- \quad \text{for} \quad s \in [v(t-), v(t)] \quad \text{and}$$

$$\hat{A}(s) = \hat{A}(v(t)) + \frac{s - v(t)}{\Delta^+\, v(t)}\, C_t^+ \quad \text{for} \quad s \in [v(t), v(t+)]\,.$$

Since $B(s) = (\mathrm{d}/\mathrm{d}s)\,\hat{A}$ a.e. on J', we have

$$B(s) = -\frac{1}{\Delta^- v(t)}\, C_t^- \quad \text{for} \quad s \in (v(t-), v(t))\,, \quad t \in J\,, \tag{16}$$

$$B(s) = \frac{1}{\Delta^+ v(t)}\, C_t^+ \quad \text{for} \quad s \in (v(t), v(t+))\,, \quad t \in J\,.$$

From the assumptions (iii), (iv) of Lemma 1.6 it follows that y has the form

$$y(s) = \exp\left[\frac{v(t) - s}{\Delta^- v(t)}\, C_t^-\right] x(t) = \exp\,[\hat{A}(s) - \hat{A}(v(t))]\, x(t) \quad \text{on} \quad [v(t-), v(t)]\,, \quad \text{and} \tag{17}$$

$$y(s) = \exp\left[\frac{s - v(t)}{\Delta^+ v(t)}\, C_t^+\right] x(t) = \exp\,[\hat{A}(s) - \hat{A}(v(t))]\, x(t) \quad \text{on} \quad [v(t), v(t+)]\,. \tag{18}$$

Using (8), (9), (15), (17) and (18), we get the equalities

$$\begin{aligned}
&\int_{v(t-)}^{v(t)} \mathrm{d}[\hat{A}(s)]\, y(s) = \int_{v(t-)}^{v(t)} B(s)\, y(s)\, \mathrm{d}s = y(v(t)) - y(v(t-)) = \\
&= y(v(t)) - \exp\,[C_t^-]\, y(v(t)) = \\
&= \{I - \exp\,[\hat{A}(v(t-)) - \hat{A}(v(t))]\}\, y(v(t)) = \\
&= \Delta^- A(t)\, x(t) \quad \text{for} \quad t \in (a, b]\,; \\
&\int_{v(t)}^{v(t+)} \mathrm{d}[\hat{A}(s)]\, y(s) = \int_{v(t)}^{v(t+)} B(s)\, y(s)\, \mathrm{d}s = y(v(t+)) - y(v(t)) = \\
&= \exp\,[C_t^+]\, y(t)) - y(v(t)) = \\
&= \{\exp\,[\hat{A}(v(t+)) - \hat{A}(v(t))] - I\}\, y(v(t)) = \\
&= \Delta^+ A(t)\, x(t) \quad \text{for} \quad t \in [a, b)\,.
\end{aligned}$$

Since the functions A and x have bounded variations on $[a, b]$, by Corollary 1.23 in [S1] the integral $\int_{t_1}^{t_2} \mathrm{d}[A(t)]\, x(t)$ exists.

Proposition 4 in [F] which is concerned with a discontinuous increasing substitution in the integral implies that the integral $\int_{v(t_1)}^{v(t_2)} \mathrm{d}[\hat{A}(s)]\, y(s)$ exists and we get the equality

$$\begin{aligned}
&\int_{t_1}^{t_2} \mathrm{d}[A(t)]\, x(t) = \int_{t_1}^{t_2} \mathrm{d}[A^c(t)]\, x(t) + \sum_{t_1 < t \leqq t_2} \Delta^- A(t)\, x(t) + \\
&+ \sum_{t_1 \leqq t < t_2} \Delta^+ A(t)\, x(t) = \int_{t_1}^{t_2} \mathrm{d}[(A \circ v)^c\,(t)]\, y(v(t)) + \\
&+ \sum_{t_1 < t \leqq t_2} \int_{v(t-)}^{v(t)} \mathrm{d}[\hat{A}(s)]\, y(s) + \sum_{t_1 \leqq t < t_2} \int_{v(t)}^{v(t+)} \mathrm{d}[\hat{A}(s)]\, y(s) = \\
&= \int_{v(t_1)}^{v(t_2)} \mathrm{d}[\hat{A}(s)]\, y(s)\,.
\end{aligned}$$

By (13) the integral $\int_{v(t_1)}^{v(t_2)} B(s)\, y(s)\, ds$ exists and we have

$$\int_{v(t_1)}^{v(t_2)} B(s)\, y(s)\, ds = \int_{v(t_2)}^{v(t_1)} d[\hat{A}(s)]\, y(s) = \int_{t_1}^{t_2} d[A(t)]\, x(t)\,.$$

Proof of Theorem 1.5. (i) Assume that y is a solution of the equation (11) on $[v(a), v(b)]$ and that the equality $x(t) = y(v(t))$ holds for $t \in [a, b]$. Then y is absolutely continuous on $[v(a), v(b)]$ and consequently, x has bounded variation on $[a, b]$. By Lemma 1.6 the equality

$$x(t_2) - x(t_1) = y(v(t_2)) - y(v(t_1)) = \int_{v(t_1)}^{v(t_2)} B(s)\, y(s)\, ds =$$
$$= \int_{t_1}^{t_2} d[A(t)]\, x(t)$$

holds for all $t_1, t_2 \in [a, b]$; this means that x is a solution of (10) on $[a, b]$.

(ii) Assume that the function x is a solution of (10) on $[a, b]$. Then x has bounded variation on $[a, b]$. Let us define a function $y: [v(a), v(b)] \to C^n$ such that

$$y(\tau) = x(t) \quad \text{if} \quad \tau = v(t)\,, \quad t \in [a, b]\,;$$

if $t \in (a, b]$ is a point such that $v(t-) < v(t)$ then y has the form (17) on $[v(t-), v(t))$; if $t \in [a, b)$ is such that $v(t) < v(t+)$ then y has the form (18) on $(v(t), v(t+)]$.

From (16), (17), (18) it follows that y is a solution of (11) on every interval of the form $[v(t-), v(t+)]$.

Let $s_1, s_2 \in [v(a), v(b)]$ and $t_1 < t_2$ be given such that $s_i \in [v(t_i-), v(t_i+)]$, $i = 1, 2$. Lemma 1.6 yields

$$y(s_2) - y(s_1) = [y(s_2) - y(v(t_2))] + [x(t_2) - x(t_1)] -$$
$$- [y(s_1) - y(v(t_1))] = \int_{v(t_2)}^{s_2} B(s)\, y(s)\, ds +$$
$$+ \int_{t_1}^{t_2} d[A(t)]\, x(t) - \int_{v(t_1)}^{s_1} B(s)\, y(s)\, ds = \int_{v(t_2)}^{s_2} B(s)\, y(s)\, ds +$$
$$+ \int_{v(t_1)}^{v(t_2)} B(s)\, y(s)\, ds - \int_{v(t_1)}^{s_1} B(s)\, y(s)\, ds = \int_{s_1}^{s_2} B(s)\, y(s)\, ds\,.$$

Consequently, y is a solution of (11) on $[v(a), v(b)]$.

1.7. Theorem. *Assume that the matrix valued function $A: J \to M_n(C)$ and the increasing function v satisfy* (2), (3) *and* (5). *Then there exists a logarithmic prolongation of A along the function v.*

Proof. It is well-known that if M is an $n \times n$ matrix such that $\|M\| < 1$ then the series

$$\ln (I + M) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} M^n$$

is convergent and the equality $\exp(\ln (I + M)) = I + M$ holds. We have an evident estimate

$$\|\ln (I + M)\| \leq \sum_{n=1}^{\infty} \frac{1}{n} \|M\|^n = -\ln (1 - \|M\|)\,. \tag{19}$$

It is easy to find out that the function $\varphi(x) = -\ln(1 - x)$ is convex and consequently,

$$\varphi(x) \leqq \varphi(0) + \frac{x - 0}{x - \frac{1}{2}} [\varphi(\tfrac{1}{2}) - \varphi(0)] = 2 \ln 2 . x \quad \text{for every} \quad x \in [0, \tfrac{1}{2}] .$$

From (19) we get the estimate

(20) $\qquad$ if $\|M\| \leqq \frac{1}{2}$ then $\|\ln(I + M)\| \leqq 2 \ln 2 \|M\|$.

If $t \in J$ is such that $\|\Delta^- A(t)\| < 1$, let us define $C_t^- = \ln(I - \Delta^- A(t))$;

if $\|\Delta^+ A(t)\| < 1$, let us define $C_t^+ = \ln(I + \Delta^+ A(t))$.

It is proved e.g. in [K], Th. 6.1.1 that also in the case $\|\Delta^- A(t)\| \geqq 1$ or $\|\Delta^+ A(t)\| \geqq$ $\geqq 1$ there are such matrices C_t^- or C_t^+ that $\exp C_t^- = I - \Delta^- A(t)$ or $\exp C_t^+ =$ $= I + \Delta^+ A(t)$, respectively. Let us mention that these matrices are in general complex, even if A is a real matrix - valued function.

By (20) we obtain the estimate:

if $\|\Delta^- A(t)\| \leqq \frac{1}{2}$ then $\|C_t^-\| \leqq 2 \ln 2 \|\Delta^- A(t)\|$;

if $\|\Delta^+ A(t)\| \leqq \frac{1}{2}$ then $\|C_t^+\| \leqq 2 \ln 2 \|\Delta^+ A(t)\|$.

Since A is of bounded variation and for every compact interval $[a, b] \subset J$ the set $\{t \in (a, b]; \|\Delta^- A(t)\| \geqq \frac{1}{2}\} \cup \{t \in [a, b); \|\Delta^+ A(t)\| \geqq \frac{1}{2}\}$ is finite, the above estimate implies that

(21) $\qquad$ the series $\sum_{t \in (a,b]} C_t^-$ and $\sum_{t \in [a,b)} C_t^+$ are absolutely convergent for every compact subinterval $[a, b] \subset J$.

Let us define $A_1(\tau) = A^c(t)$ if $\tau \in [v(t-), v(t+)]$, $t \in J$. If $\tau_1, \tau_2 \in J$ are such points that $v(t-) \leqq \tau_1 \leqq \tau_2 \leqq v(t+)$ for some $t \in J$, then

(22) $\qquad \|A_1(\tau_2) - A_1(\tau_1)\| = \|A^c(t) - A^c(t)\| = 0 \leqq \tau_2 - \tau_1$.

If there are such points $t_1, t_2 \in J$ that $t_1 < t_2$ and $v(t_i-) \leqq \tau_i \leqq v(t_i+)$ for $i = 1, 2$, then

(23)
$$\|A_1(\tau_2) - A_1(\tau_1)\| = \|A^c(t_2) - A^c(t_1)\| =$$
$$= \|A^c(t_2-) - A^c(t_1+)\| \leqq v(t_2-) - v(t_1+) \leqq \tau_2 - \tau_1 .$$

From (22), (23) it follows that the function A_1 is lipschitzian on J', consequently it is locally absolutely continuous and has a derivative

(24) $\qquad \dfrac{\mathrm{d}}{\mathrm{d}s} A_1(s) = B_1(s)$ a.e. on J' .

Let us define

$$(25)\qquad B_2(s) = 0 \quad \text{if} \quad s = v(t)\,, \quad t \in J\,;$$

if $t \in J$ is a point such that $v(t-) < v(t)$ then

$$B_2(s) = -C_t^- / \Delta^- v(t) \quad \text{for} \quad s \in [v(t-), v(t))\,;$$

if $t \in J$ is a point such that $v(t) < v(t+)$ then

$$B_2(s) = C_t^+ / \Delta^+ v(t) \quad \text{for} \quad s \in (v(t), v(t+)]\,.$$

Finally, let us define $B(s) = B_1(s) + B_2(s)$ for $s \in J'$.

To find out if the function B_2 is locally integrable, it is sufficient to estimate the integral $\int_{v(a)}^{v(b)} \|B_2(s)\|\,\mathrm{d}s$, because evidently B_2 is a measurable function. For every $[a, b] \in J$ we have the identity

$$\int_{v(a)}^{v(b)} \|B_2(s)\|\,\mathrm{d}s = \sum_{a<t\leqq b} \int_{v(t-)}^{v(t)} \|B_2(s)\|\,\mathrm{d}s + \sum_{a\leqq t<b} \int_{v(t)}^{v(t+)} \|B_2(s)\|\,\mathrm{d}s =$$

$$= \sum_{a<t\leqq b} \|C_t^-\| + \sum_{a\leqq t<b} \|C_t^+\|\,.$$

The last two series are convergent by (21). Hence the function B_2 is locally integrable over J' and we can define

$$A_2(s) = \int_{v(a_0)}^{s} B_2(\sigma)\,\mathrm{d}\sigma\,, \quad \hat{A}(s) = A_1(s) + A_2(s) \quad \text{for} \quad s \in J'\,.$$

Let us prove that $\hat{A}$ is a logarithmic prolongation of A along v. The condition (i) of Definition 1.3 is evidently satisfied. The function

$$A_2 \circ v \quad \text{has for} \quad t \geqq a_0 \quad \text{the form} \quad A_2(v(t)) = \int_{v(a_0)}^{v(t)} B_2(\sigma)\,\mathrm{d}\sigma =$$

$$= \sum_{a_0<\tau\leqq t} \int_{v(\tau-)}^{v(\tau)} B_2(\sigma)\,\mathrm{d}\sigma + \sum_{a_0\leqq\tau<t} \int_{v(\tau)}^{v(\tau+)} B_2(\sigma)\,\mathrm{d}\sigma =$$

$$= \sum_{a_0<\tau\leqq t} C_\tau^- + \sum_{a_0\leqq\tau<t} C_\tau^+\,, \quad \text{hence} \quad \hat{A}(v(t)) = A_1(v(t)) + A_2(v(t)) =$$

$$= A^c(t) + \sum_{a_0\leqq\tau\leqq t} C_\tau^- + \sum_{a_0<\tau\leqq t} C_\tau^+ \quad \text{for} \quad t \in J\,, \quad t \geqq a_0\,. \quad \text{Similarly}$$

$$\hat{A}(v(t)) = A^c(t) - \sum_{t<\tau\leqq a_0} C_t^- - \sum_{t\leqq\tau<a_0} C_t^+ \quad \text{for} \quad t \in J\,, \quad t < a_0\,.$$

We can see that the continuos part of $\hat{A} \circ v$ is equal to A^c.

The conditions (iii), (iv) of Definition 1.3 are obviously satisfied, because

$$\exp\left[\hat{A}(v(t-)) - \hat{A}(v(t))\right] = \exp\left[A_2(v(t-)) - A_2(v(t))\right] =$$

$$= \exp \int_{v(t)}^{v(t-)} B_2(\sigma)\,\mathrm{d}\sigma = \exp C_\tau^- = I - \Delta^- A(t)\,,$$

similarly

$$\exp\left[\hat{A}(v(t+)) - \hat{A}(v(t))\right] = I + \Delta^+ A(t)\,.$$

2. ORDINARY LINEAR DIFFERENTIAL EQUATION WITH A SUBSTITUTION

2.1. Definition. *Assume that a matrix valued function $B: J' \to M_n(C)$, a vector valued function $g: J' \to C^n$ and an increasing function $v: J \to J'$ are given.*

We say that an n-vector valued function x is a solution of the ordinary differential equation with a substitution

$$x(t) = y(v(t)) \,; \quad \frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y + g(s) \tag{26}$$

on an interval $[a, b] \subset J$, if there is a solution y of

$$\frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y + g(s) \tag{27}$$

on the interval $[v(a), v(b)]$ such that the equality $x(t) = y(v(t))$ holds for every $t \in [a, b]$.

Using this definition, we can re-formulate Theorem 1.5 in the following form:

2.2. Corollary. *Let a matrix-valued function B be the derivative a.e. of a logarithmic prolongation of A along v.*

An n-vector valued function x is a solution of the equation (10) *on $[a, b]$ if and only if it is a solution of*

$$x(t) = y(v(t)) \,; \quad \frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y \tag{27$'$}$$

on $[a, b]$.

The aim of this section is to prove an analogous theorem for the equations (1) and (26) and then to obtain some results concerning the equation (1).

2.3. Lemma. *Let the matrix valued function $\hat{A}$ be a logarithmic prolongation of A along v. Let a vector valued function $f: J \to C^n$ be given such that*

$$\|f(t_2) - f(t_1)\| \leqq v(t_2) - v(t_1) \quad \text{for every} \quad t_1, t_2 \in J \,, \quad t_1 < t_2 \,. \tag{28}$$

Let us define a function $g: J' \to C^n$ as follows:

The function f_1 defined by $f_1(s) = f^c(t)$ for $s \in [v(t-), v(t+)]$, $t \in J$, is lipschtzian, which can be verified similarly as in (22), (23). *Hence it has a derivative $g_1(s) = (\mathrm{d}/\mathrm{d}s) f_1(s)$ a.e. on J'. Let us define*

$$g_2(s) = 0 \quad \text{if} \quad s = v(t) \,, \quad t \in J \,; \tag{29}$$

$$g_2(s) = \frac{1}{\Delta^- v(t)} \exp\left[\hat{A}(s) - \hat{A}(v(t-))\right] \Delta^- f(t) \quad \text{if}$$

$$s \in [v(t-), v(t)) \,, \quad t \in J \,;$$

$$g_2(s) = \frac{1}{\Delta^+ v(t)} \exp\left[\hat{A}(s) - \hat{A}(v(t+))\right] \Delta^+ f(t) \quad \textit{if}$$

$$s \in (v(t), v(t+)], \quad t \in J .$$

Further, let us define $g(s) = g_1(s) + g_2(s)$ for $s \in J'$.

Then the function g is locally integrable over J'.

Proof. The function g_1 is locally integrable, because f_1 is absolutely continuous. The function g_2 is obviously measurable. If $[a, b] \subset J$ is a compact interval, then

$$\int_{v(a)}^{v(b)} \|g_2(s)\| \, \mathrm{d}s = \sum_{a<t\leqq b} \int_{v(t-)}^{v(t)} \|g_2(s)\| \, \mathrm{d}s + \sum_{a\leqq t<b} \int_{v(t)}^{v(t+)} \|g_2(s)\| \, \mathrm{d}s =$$

$$= \sum_{\substack{a<t\leqq b \\ \Delta^- v(t)>0}} \frac{1}{\Delta^- v(t)} \left\| \int_{v(t-)}^{v(t)} \exp\left[\hat{A}(s) - \hat{A}(v(t-)\right] \mathrm{d}s \, \Delta^- f(t) \right\| +$$

$$+ \sum_{\substack{a\leqq t<b \\ \Delta^+ v(t)>0}} \frac{1}{\Delta^+ v(t)} \left\| \int_{v(t)}^{v(t+)} \exp\left[\hat{A}(s) - \hat{A}(v(t+)\right) \mathrm{d}s \, \Delta^+ f(t) \right\| \leqq$$

$$\leqq \sum_{\substack{a<t\leqq b \\ \Delta^+ v(t)>0}} \left\| \int_{v(t-)}^{v(t)} \exp\left[\hat{A}(s) - \hat{A}(v(t-))\right] \mathrm{d}s \right\| + \sum_{\substack{a\leqq t<b \\ \Delta^+ v(t)>0}} \left\| \int_{v(t-)}^{v(t+)} \exp\left[\hat{A}(s) - \right.\right.$$

$$\left.\left. \hat{A}(v(t+))\right] \mathrm{d}s \right\| .$$

In the last inequality the assumption (28) was used. Using the notation (14) we obtain that the last expression is equal to

$$\sum_{\substack{a<t\leqq b \\ \Delta^- v(t)>0}} \left\| \int_{v(t-)}^{v(t)} \exp\left[\frac{s - v(t-)}{\Delta^- v(t)} C_t^-\right] \mathrm{d}s \right\| +$$

$$+ \sum_{\substack{a\leqq t<b \\ \Delta^+ v(t)>0}} \left\| \int_{v(t)}^{v(t+)} \exp\left[\frac{s - v(t+)}{\Delta^+ v(t)} C_t^+\right] \mathrm{d}s \right\| \leqq$$

$$\leqq \sum_{\substack{a<t\leqq b \\ \Delta^- v(t)>0}} \int_{v(t-)}^{v(t)} \exp\left[\frac{s - v(t-)}{\Delta^- v(t)} \|C_t^-\|\right] \mathrm{d}s +$$

$$+ \sum_{\substack{a\leqq t<b \\ \Delta^+ v(t)>0}} \int_{v(t)}^{v(t+)} \exp\left[\frac{v(t+) - s}{\Delta^+ v(t)} \|C_t^+\|\right] \mathrm{d}s =$$

$$= \sum_{\substack{a<t\leqq b \\ C^-{}_t=0}} \Delta^- v(t) \,.\, 1 + \sum_{\substack{a<t\leqq b \\ C_t{}^- \neq 0}} \Delta^- v(t) \,.\, \frac{e^{\|C_t\|} - 1}{\|C_t^-\|} + \sum_{\substack{a\leqq t<b \\ C_t{}^+ = 0}} \Delta^+ v(t) \,.\, 1 +$$

$$+ \sum_{\substack{a\leqq t<b \\ C_t{}^+ \neq 0}} \Delta^+ v(t) \,.$$

The assumption (i) of Definition 1.3 implies that $\hat{A}$ is bounded on $[v(a), v(b)]$ by a constant $c > 0$. Then $\|C_t^-\| \leqq 2c$, $\|C_t^+\| \leqq 2c$ for every $t \in [a, b]$.

Let us find such $K > 1$ that $(e^x - 1)/x \leqq K$ for every $x \in (0, 2c]$; then the last expression can be estimated by

$$\sum_{a<t\leqq b} \Delta^- v(t) K + \sum_{a\leqq t<b} \Delta^+ v(t) K \leqq K[v(b) - v(a)] .$$

It means that g_2 is integrable over an arbitrary interval $[v(a), v(b)] \subset J'$. Consequently, g is locally integrable over J'.

2.4. Theorem. *Assume that the matrix valued function B is a derivative of a logarithmic prolongation of A along v. Let a vector valued function f be given such that* (28) *holds. If we define a function g as described in Lemma* 2.3, *then the equations* (1) *and* (26) *have the same solutions.*

Remark. If it is assumed that the functions A and f in (1) are locally of bounded variation, then it is possible to find such an increasing function v that both (5) and (28) hold; for instance

$$v(t) = t + \operatorname{var}_{a_0}^t A + \operatorname{var}_{a_0}^t f \quad \text{for} \quad t \in J\,, \quad t \geqq a_0\,,$$

$$v(t) = t - \operatorname{var}_t^{a_0} A - \operatorname{var}_t^{a_0} f \quad \text{for} \quad t \in J\,, \quad t < a_0\,.$$

The proof of Theorem 2.4 is very similar to the proof of Theorem 1.5. In fact, Theorem 1.5 is a special case of Theorem 2.4 and was formulated and proved separately only for better understanding. First, let us prove a lemma:

2.5. Lemma. *Assume that $[a, b] \subset J$. Let vector valued functions x: $[a, b] \to \mathbb{C}^n$ and y: $[v(a), v(b)] \to \mathbb{C}^n$ be given such that*

(i) $x(t) = y(v(t))$ *for* $t \in [a, b]$;

(ii) *the function x has bounded variation on* $[a, b]$;

(iii) *if $t \in (a, b]$ is such a point that $v(t-) < v(t)$ then y is a solution of* (27) *on* $[v(t-), v(t)]$;

(iv) *if $t \in [a, b)$ is such a point that $v(t) < v(t+)$ then y is a solution of* (27) *on* $[v(t), v(t+)]$.

Then $\int_{t_1}^{t_2} \mathrm{d}[A(t)]\, x(t) + f(t_2) - f(t_1) = \int_{v(t_1)}^{v(t_2)} [B(s)\, y(s) + g(s)]\, \mathrm{d}s$ *for every* $a \leqq t_1 < t_2 \leqq b$.

Proof. Denote $\hat{A}(s) = \int_{v(a_0)}^s B(\sigma)\, \mathrm{d}\sigma$, $\hat{f}(s) = \int_{v(a_0)}^s g(\sigma)\, \mathrm{d}\sigma$ for $s \in J'$. If we define matrices C_t^-, C_t^+ by (14), then B has the form (16) on the intervals $(v(t-), v(t))$, $v(t), v(t+))$, and

$$g(s) = \frac{1}{\Delta^- v(t)} \exp\left[\frac{s - v(t-)}{\Delta^- v(t)}\, C_t^-\right] \Delta^- f(t) \quad \text{for} \quad s \in [v(t-), v(t)) \quad \text{where}$$

$t \in J$ and $v(t-) < v(t)$, and

$$g(s) = \frac{1}{\Delta^+ v(t)} \exp\left[\frac{s - v(t+)}{\Delta^+ v(t)}\, C_t^+\right] \Delta^+ f(t) \quad \text{for} \quad s \in (v(t), v(t+)]$$

where $t \in J$, $v(t) < v(t+)$.

Since B is constant on intervals of the form $(v(t-), v(t))$, $(v(t), v(t+))$ and y is a solution of (27) on these intervals, we can easily compute that

$$y(s) = \exp\left[\hat{A}(s) - \hat{A}(v(t))\right] x(t) + \left[s - v(t)\right] g(s) \quad \text{for} \tag{30}$$
$$s \in [v(t-), v(t)) , \quad t \in (a, b] \quad \text{and} \quad s \in (v(t), v(t+)] , \quad t \in [a, b) .$$

If $t \in (a, b]$, $v(t-) < v(t)$ then

$$\int_{v(t-)}^{v(t)} [B(s)\, y(s) + g(s)]\, \mathrm{d}s = y(v(t)) - y(v(t-)) =$$
$$= x(t) - \exp\left[\hat{A}(v(t-)) - \hat{A}(v(t))\right] x(t) - \left[v(t-) - v(t)\right] g(v(t-)) =$$
$$= \Delta^- A(t)\, x(t) + \Delta^- f(t) ;$$

if $t \in [a, b)$, $v(t) < v(t+)$ then

$$\int_{v(t)}^{v(t+)} [B(s)\, y(s) + g(s)]\, \mathrm{d}s = y(v(t+)) - y(v(t)) =$$
$$= \exp\left[\hat{A}(v(t+)) - \hat{A}(v(t))\right] x(t) + \left[v(t+) - v(t)\right] g(v(t+)) - x(t) =$$
$$= \Delta^+ A(t)\, x(t) + \Delta^+ f(t) .$$

Using Proposition 4 in [F] about increasing substitution in the integral, we get the equality

$$\int_{t_1}^{t_2} \mathrm{d}[A(t)]\, x(t) + f(t_2) - f(t_1) = \int_{t_1}^{t_2} \mathrm{D}[A(t)\, x(\tau) + f(t)] =$$
$$= \int_{t_1}^{t_2} \mathrm{D}[A^c(t)\, x(\tau) + f^c(t)] + \sum_{t_1 < t \leqq t_2} [\Delta^- A(t)\, x(t) + \Delta^- f(t)] +$$
$$+ \sum_{t_1 \leqq t < t_2} [\Delta^+ A(t)\, x(t) + \Delta^+ f(t)] =$$
$$= \int_{t_1}^{t_2} \mathrm{D}[(\hat{A} \circ v)^c (t)\, y(v(\tau)) + (f \circ v)^c (t)] +$$
$$+ \sum_{t_1 < t \leqq t_2} \int_{v(t-)}^{v(t)} \mathrm{D}[\hat{A}(s)\, y(\sigma) + \hat{f}(s)] +$$
$$+ \sum_{t_1 \leqq t < t_2} \int_{v(t)}^{v(t+)} \mathrm{D}[\hat{A}(s)\, y(\sigma) + \hat{f}(s)] =$$
$$= \int_{v(t_1)}^{v(t_2)} \mathrm{D}[\hat{A}(s)\, y(\sigma) + \hat{f}(s)] = \int_{v(t_1)}^{v(t_2)} [B(s)\, y(s) + g(s)]\, \mathrm{d}s .$$

The proof of Theorem 2.4 is very similar to the proof of Theorem 1.5 and is omitted.

2.6. Theorems 1.7 and 2.4 give us a tool for deriving various results about generalized linear differential equations from the well-known properties of ordinary linear differential equations.

If for an equation (1) the assumption (2) holds but the assumption (3) is not satisfied, we can make use of the fact that the points t at which the matrices $I - \Delta^- A(t)$ or $I + \Delta^+ A(t)$ are not regular are isolated. Hence the interval of definition can be divided into subintervals at which (3) holds and our theory can be used.

Immediately we get some basic theorems. For instance, we know that for $s_0 = v(t_0)$ the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y + g(s) ; \quad y(s_0) = x_0 \tag{31}$$

has a unique maximal solution ψ and this maximal solution is defined on J'. Then, evidently, also the initial value problem

$$x(t) = y(v(t)) ; \quad \frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y + g(s) ; \quad x(t_0) = x_0 \tag{32}$$

has a unique maximal solution $\varphi(t) = \psi(v(t))$ on J. According to Theorem 2.4 the same holds for the initial value problem

$$\mathrm{d}x = \mathrm{d}[A(t)]\, x + \mathrm{d}f ; \quad x(t_0) = x_0 . \tag{33}$$

2.7. As an example, an explicit formula for solutions of (1) will be found provided $n = 1$, under the assumptions (2), (3).

For given functions A, f let us find such an increasing function v that (5), (28) hold. Let us find a logarithmic prolongation $\hat{A}$ of A along v by Theorem 1.7; further, let us define the function g by Lemma 2.3 and denote $\hat{f}(s) = \int_{v(a_0)}^{s} g(\sigma)\, \mathrm{d}\sigma$, $s \in J$. Finally, let us denote $\tilde{A}(t) = \hat{A}(v(t))$, $t \in J$.

Since every maximal solution of (11) has the form

$$y(s) = \exp\left[\hat{A}(s) - \hat{A}(s_0)\right] y(s_0) , \quad s \in J' ,$$

the equation with a substitution (27)′ as well as the equation (10) have maximal solutions $x(t) = \exp\left[\hat{A}(v(t)) - \hat{A}(v(t_0))\right] x(t_0)$, $t \in J$, i.e.

$$x(t) = \exp\left[\tilde{A}(t) - \tilde{A}(t_0)\right] x(t_0) , \quad t \in J .$$

The variation-of-constants formula yields the maximal solutions of (27):

$$\begin{aligned} y(s) &= \exp\left[\hat{A}(s) - \hat{A}(v(t_0)\right] y(v(t_0)) + \\ &+ \textstyle\int_{v(t_0)}^{s} \exp\left[\hat{A}(s) - \hat{A}(\sigma)\right] g(\sigma)\, \mathrm{d}\sigma ; \end{aligned}$$

hence every maximal solution of (1) has for $n = 1$ the form

$$\begin{aligned} x(t) &= \exp\left[\hat{A}(v(t)) - \hat{A}(v(t_0))\right] x(t_0) + \\ &+ \textstyle\int_{v(t_0)}^{v(t)} \exp\left[\hat{A}(v(t)) - \hat{A}(\sigma)\right] g(\sigma)\, \mathrm{d}\sigma = \\ &= \exp\left[\tilde{A}(t) - \tilde{A}(t_0)\right] x(t_0) + \\ &+ \textstyle\int_{t_0}^{t} \exp\left[\hat{A}(v(t)) - \hat{A}(v(\tau))\right] \mathrm{d}(f \circ v)^c(\tau) + \\ &+ \sum_{t_0 < \tau \leqq t} \textstyle\int_{v(\tau-)}^{v(\tau)} \exp\left[\hat{A}(v(t)) - \hat{A}(\sigma)\right] . \\ &\cdot \frac{1}{\Delta^- v(\tau)} \exp\left[\hat{A}(\sigma) - \hat{A}(v(\tau-)\right) \Delta^- f(\tau) + \\ &+ \sum_{t_0 \leqq \tau < t} \textstyle\int_{v(\tau)}^{v(\tau+)} \exp\left[\hat{A}(v(t)) - \hat{A}(\sigma)\right] . \\ &\cdot \frac{1}{\Delta^+ v(\tau)} \exp\left[\hat{A}(\sigma) - \hat{A}(v(\tau+)\right] \Delta^+ f(\tau) = \\ &= \exp\left[\tilde{A}(t) - \tilde{A}(t_0)\right] x(t_0) + \textstyle\int_{t_0}^{t} \exp\left[\tilde{A}(t) - \tilde{A}(\tau)\right] \mathrm{d}f^c(\tau) + \end{aligned}$$

$$+ \sum_{t_0 < \tau \leqq t} \exp [\tilde{A}(t) - \tilde{A}(\tau -)] \, \Delta^- f(\tau) +$$
$$+ \sum_{t_0 \leqq \tau < t} \exp [\tilde{A}(t) - \tilde{A}(\tau +)] \, \Delta^+ f(\tau)$$

provided $t_0 < t$; if $t_0 > t$ we put $\sum_{t_0 < \tau \leqq t} = - \sum_{t < \tau \leqq t_0}$ etc.

Now, let us express the value $\exp [\tilde{A}(t) - \tilde{A}(\tau)]$ in a more detailed form: For $\tau, t \in J$, $\tau < t$ we have

$$\tilde{A}(t) - \tilde{A}(\tau) = \hat{A}(v(t)) - \hat{A}(v(\tau)) = A^c(t) - A^c(\tau) +$$
$$+ \sum_{\tau \leqq s < t} \hat{A}(v(s+)) - \hat{A}(v(s))] - \sum_{\tau < s \leqq t} [\hat{A}(v(s-)) - \hat{A}(v(s))] ;$$

hence

$$\exp [\tilde{A}(t) - \tilde{A}(\tau)] = \exp [A^c(t) - A^c(\tau)] \prod_{\tau \leqq s < t} \exp [\hat{A}(v(s+)) - \hat{A}(v(s))] \, .$$
$$. \prod_{\tau < s \leqq t} [\exp [\hat{A}(v(s-)) - \hat{A}(v(s))]]^{-1} =$$
$$= \exp [A^c(t) - A^c(\tau)] \prod_{\tau \leqq s < t} (1 + \Delta^+ A(s)) \prod_{\tau < s \leqq t} (1 - \Delta^- A(s))^{-1} \, .$$

2.8. In [FS] a Sturm-Liouville theorem was proved for a system

$$\mathrm{d}x = y \, \mathrm{d}P \, ,$$
$$\mathrm{d}y = x \, \mathrm{d}R \, .$$

The method used there is in fact the same as described in this paper.

2.9. The well-known variation-of-constants formula yields the maximal solutions of (27) in the form

$$y(s) = W(s, s_0) \, y(s_0) + \int_{s_0}^{s} W(s, \sigma) \, g(\sigma) \, \mathrm{d}\sigma \, ; \quad s, s_0 \in J'$$

where $W(s, s_0)$ is the fundamental matrix of the equation (11).

Theorem 1.5 implies that $U(t, t_0) = W(v(t), v(t_0))$; $t_0, t \in J$ is the fundamental matrix of the equation (10). By Theorem 2.4 the maximal solutions of (1) are of the form

$$x(t) = y(v(t)) = W(v(t), v(t_0)) \, y(v(t_0)) + \int_{v(t_0)}^{v(t)} W(v(t), \sigma) \, g(\sigma) \, \mathrm{d}\sigma =$$
$$= U(t, t_0) \, x(t_0) + \int_{v(t_0)}^{v(t)} W(v(t), \sigma) \, \mathrm{d}\hat{f}(\sigma) \, .$$

Proposition 4 in [F] about substitution in the integral yields

$$x(t) = U(t, t_0) \, x(t_0) + \int_{t_0}^{t} W(v(t), v(\tau)) \, \mathrm{d}(f \circ v)^c \, (\tau) +$$
$$+ \sum_{t_0 < \vartheta \leqq t} \int_{v(\vartheta -)}^{v(\vartheta)} W(v(t), \sigma) \, g(\sigma) \, \mathrm{d}\sigma + \sum_{t_0 \leqq \vartheta < t} \int_{v(\vartheta)}^{v(\vartheta +)} W(v(t), \sigma) \, g(\sigma) \, \mathrm{d}\sigma =$$
$$= U(t, t_0) \, x(t_0) + \int_{t}^{t_0} U(t, \tau) \, \mathrm{d}f^c(\tau) + \sum_{t_0 < \vartheta \leqq t} W(v(t), v(\vartheta_-)) \, .$$
$$. \int_{v(\vartheta -)}^{v(\vartheta)} W(v(\vartheta_-), \sigma) \, g(\sigma) \, \mathrm{d}\sigma + \sum_{t_0 \leqq \vartheta < t} W(v(t), v(\vartheta_+)) \, .$$
$$. \int_{v(\vartheta)}^{v(\vartheta +)} W(v(\vartheta_+), \sigma) \, g(\sigma) \, \mathrm{d}\sigma =$$

$$= U(t, t_0)\, x(t_0) + \int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f^c(\tau) + \sum_{t_0 < \vartheta \leqq t} U(t, \vartheta) \, .$$

$$. \int_{v(\vartheta-)}^{v(\vartheta)} \exp\left[\hat{A}(v(\vartheta_-)) - \hat{A}(\sigma)\right] . \frac{1}{\Delta^- v(\vartheta)} \exp\left[\hat{A}(\sigma) - \hat{A}(v(\vartheta-))\right] .$$

$$. \Delta^- f(\vartheta)\, \mathrm{d}\sigma + \sum_{t_0 \leqq \vartheta < t} U(t, \vartheta_+) \, . \int_{v(\vartheta)}^{v(\vartheta+)} \exp\left[\hat{A}(v\vartheta_+)) - \hat{A}(\sigma)\right] .$$

$$. \frac{1}{\Delta^+ v(\vartheta)} \exp\left[\hat{A}(\sigma) - \hat{A}(v(\vartheta+))\right] \Delta^+ f(\vartheta)\, \mathrm{d}\sigma =$$

$$= U(t, t_0)\, x(t_0) + \int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f^c(\tau) +$$

$$+ \sum_{t_0 < \vartheta \leqq t} U(t, \vartheta-) \, . \int_{v(\vartheta-)}^{v(\vartheta)} \frac{1}{\Delta^- v(\vartheta)} \Delta^- f(\vartheta)\, \mathrm{d}\sigma +$$

$$+ \sum_{t_0 \leqq \vartheta < t} U(t, \vartheta+) \int_{v(\vartheta)}^{v(\vartheta+)} \frac{1}{\Delta^+ v(\vartheta)} \Delta^+ f(\vartheta)\, \mathrm{d}\sigma =$$

$$= U(t, t_0)\, x(t_0) + \left[\int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f(\tau) - \sum_{t_0 < \vartheta \leqq t} U(t, \vartheta)\, \Delta^- f(\vartheta) - \right.$$

$$\left. - \sum_{t_0 \leqq \vartheta < t} U(t, \vartheta)\, \Delta^+ f(\vartheta)\right] + \sum_{t_0 < \vartheta \leqq t} U(t, \vartheta-)\, \Delta^- f(\vartheta) +$$

$$+ \sum_{t_0 \leqq \vartheta < t} U(t, \vartheta+)\, \Delta^+ f(\vartheta) \, .$$

Hence we get the following formulae for the maximal solutions of (1):

(34)
$$x(t) = U(t, t_0)\, x(t_0) + \int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f^c(\tau) +$$

$$+ \sum_{t_0 < \vartheta \leqq t} U(t, \vartheta-)\, \Delta^- f(\vartheta) + \sum_{t_0 \leqq \vartheta < t} U(t, \vartheta+)\, \Delta^+ f(\vartheta) =$$

$$= U(t, t_0)\, x(t_0) + \int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f(\tau) -$$

$$- \sum_{t_0 < \vartheta \leqq t} \left[U(t, \vartheta) - U(t, \vartheta-)\right] \Delta^- f(\vartheta) +$$

$$+ \sum_{t_0 \leqq \vartheta < t} \left[U(t, \vartheta+) - U(t, \vartheta)\right] \Delta^+ f(\vartheta) \quad \text{for} \quad t_0, t \in J \, , \quad t_0 \leqq t \, .$$

Similarly, for $t < t_0$ it can be proved that

(35)
$$x(t) = U(t, t_0)\, x(t_0) + \int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f^c(\tau) -$$

$$- \sum_{t < \vartheta \leqq t_0} U(t, \vartheta-)\, \Delta^- f(\vartheta) - \sum_{t \leqq \vartheta < t_0} U(t, \vartheta+)\, \Delta^+ f(\vartheta) =$$

$$= U(t, t_0)\, x(t_0) + \int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f(\tau) +$$

$$+ \sum_{t < \vartheta \leqq t_0} \left[U(t, \vartheta) - U(t, \vartheta-)\right] \Delta^- f(\vartheta) -$$

$$- \sum_{t \leqq \vartheta < t_0} \left[U(t, \vartheta+) - U(t, \vartheta)\right] \Delta^+ f(\vartheta) \, .$$

We will be concerned with the generalized linear differential equation

$$(36) \qquad dx = d[A(t)]\,x$$

where A is a real $n \times n$-matrix valued function which is defined on the interval $J = [0, \infty)$. Throughout this section we assume that (2), (3) hold.

3.1. Following [S2], we will say that

(i) the solution $x_0 \equiv 0$ of (36) is variationally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x: [t_0, t_1] \to R^n$, $0 \leqq t_0 < t_1 < \infty$, is a function of bounded variation with

$$\|x(t_0)\| < \delta \quad \text{and} \quad \operatorname{var}_{t_1}^{t_2} \{x(t) - \int_{t_0}^{t} d[A(s)]\,x(s)\} < \delta\,,$$

then $\|x(t)\| < \varepsilon$ for $t \in [t_0, t_1]$;

(ii) the solution $x_0 \equiv 0$ of (36) is variationally attracting if there exists $\delta_0 > 0$ and for every $\varepsilon > 0$ there exist $T > 0$ and $\gamma > 0$ such that if $x: [t_0, t_1] \to R^n$, $0 \leqq$ $\leqq t_0 < t_1 < \infty$, is a function of bounded variation and

$$\|x(t_0)\| < \delta_0 \quad \text{and} \quad \operatorname{var}_{t_0}^{t_1} [x(t) - \int_{t_0}^{t} d[A(s)]\,x(s)] < \gamma\,,$$

then $\|x(t)\| < \varepsilon$ for all $t \in [t_0, t_1] \cap [t_0 + T, \infty)$;

(iii) the zero solution $x_0 \equiv 0$ is variationally-asymptotically stable if it is variationally stable and variationally attracting.

The aim of this section is to find some sufficient conditions for the equation (36) to have the zero solution variationally stable or variationally-asymptotically stable.

3.2. Let us define

$$A_0(t) = A(t) - \sum_{\substack{0 < \tau \leqq t \\ \|\Delta^- A(\tau)\| \geqq 1}} \Delta^- A(\tau) - \sum_{\substack{0 \leqq \tau < t \\ \|\Delta^+ A(\tau)\| \geqq 1}} \Delta^+ A(\tau) \quad \text{for} \quad t \in [0, \infty).$$

The function A_0 has the same continuous part as A, further $\Delta^- A_0(\tau) = \Delta^- A(\tau)$ provided $\|\Delta^- A(\tau)\| < 1$ and $\Delta^- A_0(\tau) = 0$ provided $\|\Delta^- A(\tau)\| \geqq 1$, similarly for $\Delta^+ A_0(\tau)$. The set $\{\tau \in [0, \infty);\ \|\Delta^- A(\tau)\| \geqq 1 \text{ or } \|\Delta^+ A(\tau)\| \geqq 1\}$ consists of isolated points in $[0, \infty)$. Again we will denote $A(0-) = A_0(0-) = A(0)$.

Let an arbitrary increasing function $v: [0, \infty) \to [0, \infty)$ be given such that the conditions (5) and

$$(37) \qquad v(0) = 0\,, \quad \lim_{t \to \infty} v(t) = \infty$$

hold. Let us define a logarithmic prolongation $\hat{A}_0$ of A_0 along v in the same way as in the proof of Theorem 1.7, $B_0(s) = (d/ds)\,\hat{A}_0(s)$ a.e. on $[0, \infty)$.

Since $\|\Delta^- A_0(\tau)\| < 1$ and $\|\Delta^+ A_0(\tau)\| < 1$ for every $\tau \in [0, \infty)$, the matrices $C_t^- = \ln(I - \Delta^- A_0(t))$, $C_t^+ = \ln(I + \Delta^+ A_0(t))$ are real; consequently, $\hat{A}_0$ and B_0 are real matrix valued functions.

3.3. Lemma. (i) *Let us define*

(38) $$\mu(s) = 0 \quad \text{if} \quad s = v(t) \quad \text{for some} \quad t \in [0, \infty) ;$$

if $t \in (0, \infty)$, $v(t-) < v(t)$ *then*

$$\mu(s) = \frac{\mathrm{d}}{\mathrm{d}s} \ln \|\exp [\hat{A}_0(s) - \hat{A}_0(v(t-))]\| \quad \text{for} \quad s \in [v(t-), v(t)) ;$$

if $t \in [0, \infty)$, $v(t) < v(t+)$ *then*

$$\mu(s) = \frac{\mathrm{d}}{\mathrm{d}s} \ln \|\exp [\hat{A}_0(s) - \hat{A}_0(v(t))] \exp [\hat{A}_0(v(t)) - \hat{A}_0(v(t-))]\| \quad \text{for}$$

$s \in (v(t), v(t+)]$.

Then

(39) $$\int_{v(t-)}^{v(t+)} \mu(s)\, \mathrm{d}s = \ln \|(I + \Delta^+ A_0(t))(I - \Delta^- A_0(t))^{-1}\| \quad \text{for} \quad t \in [0, \infty)$$

and the function μ *is locally integrable over* $[0, \infty)$;

(ii) *for every* $\vartheta > 0$ *the infinite product*

$$\prod_{0 < t < \vartheta} \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\|$$

is convergent.

Proof. (i) We have

$$\int_{v(t-)}^{v(t+)} \mu(s)\, \mathrm{d}s = \int_{v(t-)}^{v(t)} \mu(s)\, \mathrm{d}s + \int_{v(t)}^{v(t+)} \mu(s)\, \mathrm{d}s =$$
$$= [\ln \|\exp [\hat{A}_0(s) - \hat{A}_0(v(t-))]\|]_{v(t-)}^{v(t)} +$$
$$+ [\ln \|\exp [\hat{A}_0(s) - \hat{A}_0(v(t))] \,.\, \exp [\hat{A}_0(v(t)) - \hat{A}_0(v(t-))]\|]_{v(t)}^{v(t+)} =$$
$$= \ln \|\exp [\hat{A}_0(v(t+)) - \hat{A}_0(v(t))] (\exp [\hat{A}_0(v(t-)) - \hat{A}_0(v(t))])^{-1}\| =$$
$$= \ln \|(I + \Delta^+ A(t))(I - \Delta^- A_0(t))^{-1}\| .$$

Since $\hat{A}_0$ is linear on the intervals $[v(t-), v(t)]$ and $[v(t), v(t+)]$, the function μ is evidently measurable over $[0, \infty)$ and integrable over each interval of the form $[v(t-), v(t+)]$.

For every $\sigma > 0$ there is $\vartheta \geqq 0$ such that $\sigma \in [v(\vartheta-), v(\vartheta+)]$. We have

(40) $$\int_0^\sigma \mu(s)\, \mathrm{d}s = \sum_{0 \leqq t < \vartheta} \int_{v(t-)}^{v(t+)} \mu(s)\, \mathrm{d}s + \int_{v(\vartheta-)}^{\sigma} \mu(s)\, \mathrm{d}s =$$
$$= \sum_{0 \leqq t < \vartheta} \ln \|(I + \Delta^+ A_0(t))(I - \Delta^- A_0(t))^{-1}\| + \int_{v(\vartheta-)}^{\sigma} \mu(s)\, \mathrm{d}s .$$

For the integrability of μ over $[0, \sigma]$ it is sufficient to prove that the series above is absolutely convergent.

The set $K = \{t \in (0, \vartheta];\ \|\Delta^- A_0(t)\| > \frac{1}{8}\} \cup \{t \in [0, \vartheta);\ \|\Delta^+ A_0(t)\| > \frac{1}{8}\}$ is finite. From the implication

$$\text{if} \quad C \in M_n(C), \quad \|C\| \leqq \tfrac{1}{2} \quad \text{then} \quad |\ln \|I + C\|\,| \leqq 2 \ln 2 \|C\|$$

we obtain:

$$\text{if} \quad t \in [0, \vartheta) \setminus K \quad \text{then} \quad |\ln \|(I + \Delta^+ A_0(t))(I - \Delta^- A_0(t))^{-1}\|| =$$
$$= |\ln \|I + (A_0(t+) - A_0(t-))(I - \Delta^- A_0(t))^{-1}\|| \leqq$$
$$\leqq 2 . \ln 2 . \|A_0(t+) - A_0(t-)\| . \|\sum_{n=0}^{\infty} (\Delta^- A_0(t))^n\| \leqq$$
$$\leqq 2 . \ln 2 . \|A_0(t+) - A_0(t-)\| . \frac{1}{1 - \|\Delta^- A_0(t)\|} \leqq$$
$$\leqq 4 . \ln 2 . \|A_0(t+) - A_0(t-)\| .$$

Since A_0 has bounded variation over $[0, \vartheta]$, the series in (39) is absolutely convergent.

(ii) We have

$$\sum_{0<t<\vartheta} \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\| =$$
$$= \exp \{\ln \sum_{0<t<\vartheta} \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1})\| =$$
$$= \exp \{ \sum_{\substack{0<t<\vartheta \\ \|\Delta^+ At)\| < 1 \text{ and } \|\Delta^- A(t)\| < 1}} \ln \|(I + \Delta^+ A_0(t))(I - \Delta^- A_0(t))^{-1}\| +$$
$$+ \sum_{\substack{0<t<\vartheta \\ \|\Delta^+ A(t)\| \geqq 1 \text{ or } \|\Delta^- A(t)\| \geqq 1}} \ln \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\| .$$

In the last expression the first series is convergent by virtue of part (i) and the second series consists of a finite number of nonzero values.

3.4. Lemma. *Assume that there is a locally integrable function $\lambda: [0, \infty) \to \mathbb{R}$ such that λ is zero on every interval of the form $[v(t-), v(t))$ or $v(t), v(t+)]$ and*

(41) $\quad$ *if $s = v(t)$ for some $t \in [0, \infty)$ then $(B_0(s), y, y) \leqq \lambda(s) \|y\|^2$ for every $y \in \mathbb{R}^n$*

(the scalar product of vectors $B_0(s) y$ and y). Let us define μ by (38), $\varrho(s) = \lambda(s) + \mu(s)$ for $s \in [0, \infty)$.

If $W(s, s_0)$ is the fundamental matrix of

(42) $$\frac{dy}{ds} = B_0(s) y$$

then

(43) $$\|W(v(t_2-), v(t_1+))\| \leqq \exp \int_{v(t_1+)}^{v(t_2-)} \varrho(s) \, ds \quad \text{for every} \quad 0 \leqq t_1 < t_2 .$$

Proof. Let y be a solution of (42) on $[0, \infty)$, denote $\eta(s) = \|y(s)\|^2$ for $s \in [0, \infty)$.

The relations $y(s) = \exp [\hat{A}_0(s) - \hat{A}_0(v(t-))] y(v(t-))$ for $s \in [v(t-), v(t)]$ and $y(s) = \exp [\hat{A}_0(s) - \hat{A}_0(v(t))] y(v(t))$ for $s \in (v(t), v(t+)]$ yield

(44) $$\eta(v(t+)) = \|\exp [\hat{A}_0(v(t+)) - \hat{A}_0(v(t))] \exp [\hat{A}_0(v(t)) - \hat{A}_0(v(t-))] y(v(t-))\|^2 \leqq \|\exp [\hat{A}_0(v(t+)) - \hat{A}_0(v(t))] .$$

$$. \exp [\hat{A}_0(v(t)) - \hat{A}_0(v(t-))]\|^2 \, \|y(v(t-))\|^2 =$$
$$= \exp [2 \int_{v(t-)}^{v(t+)} \mu(\sigma) \, d\sigma] \, \eta(v(t-)) \quad \text{for} \quad t \in [0, \infty) .$$

Using the notation $\dot{z}(s) = (d/ds) \, z(s)$, we have

$$\dot{\eta}(s) = \frac{d}{ds} (y(s), y(s)) = 2(\dot{y}(s), y(s)) = 2(B_0(s) \, y(s), y(s)) .$$

The assumption (41) implies

$$\text{(45)} \qquad \dot{\eta}(s) \leqq 2 \, \lambda(s) \, \eta(s) \quad \text{if} \quad s = v(t) , \quad t + [0, \infty) .$$

Let us denote

$$\text{(46)} \qquad \xi(s) = \dot{\eta}(s) - 2 \, \varrho(s) \, \eta(s) \quad \text{for} \quad s \in [0, \infty) .$$

(45) implies that if $s = v(t)$ then $\xi(s) = \dot{\eta}(s) - 2 \, \lambda(s) \, \eta(s) \leqq 0$. Let us denote

$$\text{(47)} \qquad M = \{s \in [v(t_1+), v(t_2-)] \, ; \, s = v(t) \text{ for some } t\} .$$

Then

$$\text{(48)} \qquad (v(t_1+), v(t_2-)) \setminus M = \bigcup_{t \in (t_1, t_2)} [v(t-), v(t)) \cup (vt), v(t+)] .$$

From (46) it follows that the function η is a solution of the ordinary differential equation

$$\dot{\eta} = 2 \, \varrho(s) \, \eta + \xi(s)$$

on $[0, \infty)$; the variation-of-constants formula yields

$$\text{(49)} \qquad \eta(v(t_2-)) = \exp [2 \int_{v(t_1+)}^{v(t_2-)} \varrho(\sigma) \, d\sigma] \, \eta(v(t_1+)) +$$
$$+ \int_{v(t_1+)}^{v(t_2-)} \exp [2 \int_s^{v(t_2-)} \varrho(\sigma) \, d\sigma] \, \xi(s) \, ds .$$

By (47), (48) we have

$$\text{(50)} \qquad \int_{v(t_1+)}^{v(t_2-)} \exp [2 \int_s^{v(t_2-)} \varrho(\sigma) \, d\sigma] \, \xi(s) \, ds =$$
$$= \int_M \exp [2 \int_s^{v(t_2-)} \varrho(\sigma) \, d\sigma] \, \xi(s) \, ds +$$
$$+ \sum_{t_1 < t < t_2} \exp [2 \int_{v(t+)}^{v(t_2-)} \varrho(s) \, ds] \int_{v(t-)}^{v(t+)} \exp [2 \int_s^{v(t+)} \varrho(\sigma) \, d\sigma] \, \xi(s) \, ds .$$

The integral over the set M is evidently nonpositive because $\xi(s) \leqq 0$ for $s \in M$.

If we prove that $\int_{v(t-)}^{v(t+)} \exp [2 \int_s^{v(t-)} \varrho(\sigma) \, d\sigma] \, \xi(s) \, ds \leqq 0$ for every $t \in (t_1, t_2)$ then the left-hand side of (50) will be nonpositive and from (49) we will get the inequality

$$\text{(51)} \qquad \eta(v(t_2-)) \leqq \exp [2 \int_{v(t_1+)}^{v(t_2-)} \varrho(\sigma) \, d\sigma] \, \eta(v(t_1+)) .$$

Since $\varrho(s) = \mu(s)$ a.e. on $[v(t-), v(t+)]$, we have

$$\int_{v(t-)}^{v(t+)} \exp [2 \int_s^{v(t+)} \varrho(\sigma) \, d\sigma] \, \xi(s) \, ds =$$
$$= \int_{v(t-)}^{v(t+)} \exp [2 \int_s^{v(t+)} \varrho(\sigma) \, d\sigma] \, (\dot{\eta}(s) - 2 \, \varrho(s) \, \eta(s)) \, ds =$$
$$= \int_{v(t-)}^{v(t+)} \frac{d}{ds} \{\exp [e \int_s^{v(t+)} \varrho(\sigma) \, d\sigma] \, \eta(s)\} \, ds =$$
$$= \eta(v(t+)) - \exp [2 \int_{v(t-)}^{v(t+)} \mu(\sigma) \, d\sigma] \, \eta(v(t-)) .$$

The last expression is nonpositive due to (44). Hence (51) holds.

Since $\|y(s)\|^2 = \eta(s)$, the inequality (51) implies that

$$\|y(v(t_2-))\| \leqq \exp\left[\int_{v(t_1+)}^{v(t_2-)} \varrho(\sigma)\, d\sigma\right] \|y(v(t_1+))\| \tag{52}$$

for every maximal solution y of (42).

By the definition of the norm of a matrix, there is a vector $z \in R^n$ such that $\|z\| \leqq 1$ and $\|W(v(t_2-)), v(t_1+))\, z\| = \|W(v(t_2-), v(t_1+))\|$. The function $y(s) = W(s, v(t_1+))\, z$ is a solution of (42), hence (52) implies that

$$\|W(v(t_2-), v(t_1+))\| = \|W(v(t_2-), v(t_1+))\, z\| =$$
$$= \|y(v(t_2-))\| \leqq \exp\left[\int_{v(t_1+)}^{v(t_2-)} \varrho(\sigma)\, d\sigma\right] \|z\| \leqq \exp\left[\int_{v(t_1+)}^{v(t_2-)} \varrho(\sigma)\, d\sigma\right].$$

3.5. Theorem. *Assume that there is a continuous function $\alpha: [0, \infty) \to R$ with locally bounded variation such that*

$$([A^c(t_2) - A^c(t_1)]\, x, x) \leqq [\alpha(t_2) - \alpha(t_1)]\, \|x\|^2 \quad \textit{for every} \quad x \in R^n, \tag{53}$$
$$0 \leqq t_1 \leqq t_2 < \infty.$$

Let us denote

$$\beta(t_2, t_1) = \|(I - \Delta^- A(t_2))^{-1}\| \,.\, \prod_{t_1 < t < t_2} \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\| \,. \tag{53'}$$
$$.\, \|I + \Delta^+ A(t_1)\|, \quad 0 \leqq t_1 \leqq t_2 < \infty.$$

Then the fundamental matrix $U(t, t_0)$ of (10) *satisfies the estimate*

$$\|U(t, t_0)\| \leqq \exp[\alpha(t) - \alpha(t_0)]\, \beta(t, t_0), \quad 0 \leqq t_0 \leqq t < \infty. \tag{54}$$

Proof. Let us define functions A_1 and B_1 as in the proof of Theorem 1.7. We may assume that the increasing function v is chosen so that

$$|\alpha(t_2) - \alpha(t_1)| \leqq v(t_2) - v(t_1), \quad 0 \leqq t_1 \leqq t_2 < \infty.$$

Let us define $\alpha_1(s) = \alpha(t)$ if $s \in [v(t-), v(t+)]$, $t \in [0, \infty)$. The function α_1 is lipschitzian and has a derivative $\lambda(s) = \alpha_1(s)$ a.e. on $[0, \infty)$. Evidently $\alpha_1(s) = B_1(s) = 0$ if $s \in [v(t-), v(t)) \cup (v(t), v(t+)]$ for some $t \in [0, \infty)$.

From (53) we get the inequality

$$([A_1(s_2) - A_1(s_1)]\, x, x) \leqq$$
$$\leqq [\alpha_1(s_2) - \alpha_1(s_1)]\, \|x\|^2 \quad \text{for} \quad 0 \leqq s_1 < s_2, \quad x \in R^n.$$

Consequently,

$$(B_1(s)\, x, x) \leqq \lambda(s)\, \|x\|^2 \tag{55}$$

a.e. on $[0, \infty)$. If $s \in [0, \infty)$ is such a point that (55) does not hold, let us re-define $\lambda(s) = \|B_1(s)\|$. Then (55) holds for every $s \in [0, \infty)$. Let $\hat{A}_0$ be a logarithmic prolongation of A_0 along v, $B_0(s) = (d/ds)\, \hat{A}_0(s)$.

Let us define μ by (38) and $\varrho(s) = \lambda(s) + \mu(s)$, $s \in [0, \infty)$. Since $B_0(s) = B_1(s)$ provided $s = v(t)$, the assumption (41) of Lemma 3.4 is fulfilled.

Theorem 1.5 implies that $U(t_2-, t_1+) = W(v(t_2-), v(t_1+))$ provided the interval

(t_1, t_2) includes no point t at which $\|\Delta^- A(t)\| \geqq 1$ or $\|\Delta^+ A(t)\| \geqq 1$, because at this interval the equation (10) and

$$dx = d[A_0(t)]\, x$$

have the same solutions. By Lemmas 3.3 and 3.4 we have

$$(56) \qquad \begin{aligned} &\|U(t_2-, t_1+)\| \leqq \exp \textstyle\int_{v(t_1+)}^{v(t_2-)} \varrho(\sigma)\, d\sigma = \\ &= \exp \left[\textstyle\int_{v(t_1+)}^{v(t_2-)} \lambda(\sigma)\, d\sigma + \int_{v(t_1+)}^{v(t_2-)} \mu(\sigma)\, d\sigma\right] = \\ &= \exp \left[\textstyle\int_{v(t_1+)}^{v(t_2-)} \lambda(\sigma)\, d\sigma + \sum\limits_{t_1 < t < t_2} \int_{v(t-)}^{v(t+)} \mu(\sigma)\, d\sigma\right] = \\ &= \exp \{[\alpha_1(v(t_2-)) - \alpha_1(v(t_1+))] + \\ &+ \sum_{t_1 < t < t_2} \ln \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\|\} = \\ &= \exp [\alpha(t_2) - \alpha(t_1)] \prod_{t_1 < t < t_2} \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\| \\ &\text{provided} \quad \|\Delta^- A(t)\| < 1 \quad \text{and} \quad \|\Delta^+ A(t)\| < 1 \\ &\text{for every} \quad t \in (t_1, t_2)\,. \end{aligned}$$

Assume that $0 \leqq t_0 < t < \infty$. There are points $t_0 < t_1 < t_2 < \ldots t_k = t$ such that $\|\Delta^- A(\tau)\| < 1$ and $\|\Delta^+ A(\tau)\| < 1$ for every $\tau \in (t_{i-1}, t_i)$, $i = 1, 2, \ldots, k$. Then

$$\begin{aligned} &\|U(t, t_0)\| = \|U(t_k, t_k-)\, U(t_k-, t_{k-1}+)\, U(t_{k-1}+, t_{k-1})\,. \\ &. \, U(t_{k-1}-, t_{k-2}+) \ldots U(t_1-, t_0+)\, U(t_0+, t_0)\| \leqq \\ &\leqq \|U(t_k, t_k-)\| \,. \prod_{i=1}^{k} \{\exp [\alpha(t_i) - \alpha(t_{i-1})] \\ &. \prod_{t_{i-1} < t < t_i} \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\|\}\,. \\ &. \prod_{i=1}^{k-1} \|U(t_i+, t_i-)\| \,. \|U(t_0+, t_0)\|\,. \end{aligned}$$

By [STV], Th. II.2.10 we have $U(t, t-) = (I - \Delta^- A(t))^{-1}$, $U(t+, t) = I + \Delta^+ A(t)$; hence $U(t+, t-) = U(t+, t)\, U(t, t-) = (I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}$.

We can continue:

$$\begin{aligned} &\|U(t, t_0)\| \leqq \|(I - \Delta^- A(t_k))^{-1}\,. \\ &. \prod_{i=1}^{k} \{\exp [\alpha(t_i) - \alpha(t_{i-1})] \,. \prod_{t_{i-1} < t < t_i} \|(I + \Delta^+ A(t))(I - \Delta^- A(t))^{-1}\|\}\,. \\ &. \prod_{i=1}^{k-1} \|(I + \Delta^+ A(t_i))(I - \Delta^- A(t_i))^{-1}\| \,. \|I + \Delta^+ A(t_0)\| = \\ &= \exp [\alpha(t) - \alpha(t_0)]\, \beta(t, t_0)\,. \end{aligned}$$

3.6. Theorem. *Assume that* (53) *and* (53)′ *hold. If there is* $K > 0$ *such that*

$$\exp\left[\alpha(t) - \alpha(t_0)\right] \beta(t, t_0) \leqq K \quad \textit{for every} \quad 0 \leqq t_0 \leqq t < \infty\,, \tag{57}$$

then the zero solution of (10) *is variationally stable.*

Proof. By Theorem 3.4 we have the estimate $\|U(t, t_0)\| \leqq K$ for $0 \leqq t_0 \leqq t$. In [S2] it is proved that the solution of the equation (10) is variationally stable if and only if the fundamental matrix $U(t, t_0)$ is bounded for $0 \leqq t_0 \leqq t$. The proof is based on the variation-of-constants formula (34).

3.7. Theorem. *Assume that* (53), (53)′ *and* (57) *hold. Assume that* (58) *for every* $\varepsilon > 0$ *there is* $T > 0$ *such that*

$$\exp\left[\alpha(t) - \alpha(t_0)\right] \beta(t, t_0) < \varepsilon \quad \textit{whenever} \quad 0 \leqq t_0 < t\,,\ t - t_0 \geqq T\,.$$

Then the zero solution of (10) *is variationally asymptotically stable.*

Proof. By [S2], Prop. 3 the zero solution of (10) is variationally attracting if and only if there exists $\delta_0 > 0$ and for any $\varepsilon > 0$ there exist $T > 0$ and $\gamma > 0$ such that if $[t_0, t_1] \subset [0, \infty)$, $\operatorname{var}_{t_0}^{t_1} f < \gamma$ and if x is a solution of (1) on $[t_0, t_1]$, $\|x(t_0)\| < \delta_0$, then $\|x(t)\| < \varepsilon$ for all $t \in [t_0, t_1] \cap [t_0 + T, \infty)$.

Let us find $K > 0$ such that (57) holds; then $\|U(t, t_0)\| \leqq K$ for every $0 \leqq t_0 \leqq t$. Let $\delta_0 > 0$ be arbitrary; let $\varepsilon > 0$ be given. By (58) there is $T > 0$ such that

$$\text{if} \quad t - t_0 \geqq T \quad \text{then} \quad \exp\left[\alpha(t) - \alpha(t_0)\right] \beta(t, t_0) < \varepsilon/2\delta_0\,.$$

Denote $\gamma = \varepsilon/2K$. If $\operatorname{var}_{t_0}^{t_1} f < \gamma$ and if x is a solution of (1) on $[t_0, t_1]$, $\|x(t_0)\| < \delta_0$, then by the variation-of-constants formula (34) we have for $t \in [t_0, t_1]$, $t \geqq t_0 + T$:

$$\begin{aligned}
\|x(t)\| &= \Big\|U(t, t_0)\, x(t_0) + \int_{t_0}^{t} U(t, \tau)\, \mathrm{d}f^c(\tau) + \sum_{t_0 < \vartheta \leqq t} U(t, \vartheta-)\, \Delta^- f(\vartheta) + \\
&+ \sum_{t_0 \leqq \vartheta < t} U(t, \vartheta+)\, \Delta^+ f(\vartheta)\Big\| \leqq \|U(t, t_0)\| \,.\, \|x(t_0)\| + \\
&+ \sup_{t_0 \leqq \vartheta \leqq t} \|U(t, \tau)\| \operatorname{var}_{t_0}^{t} f^c + \sup_{t_0 < \vartheta \leqq t} \|U(t, \vartheta-)\| \,.\, \sum_{t_0 < \vartheta \leqq t} \|\Delta^- f(\vartheta)\| + \\
&+ \sup_{t_0 \leqq \vartheta < t} \|U(t, \vartheta+)\| \,.\, \sum_{t_0 \leqq \vartheta < t} \|\Delta^+ f(\vartheta)\| \leqq \\
&\leqq \exp\left[\alpha(t) - \alpha(t_0)\right] \beta(t, t_0)\, \delta_0 + K \operatorname{var}_{t_0}^{t} f < \varepsilon\,.
\end{aligned}$$

We have proved that the zero solution of (10) is variationally attracting. By the previous theorem it is variationally stable; hence the zero solution of (10) is variationally-asymptotically stable.

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Souhrn

SUBSTITUTČNÍ METODA PRO ZOBECNĚNÉ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE

DANA FRAŇKOVÁ

Zobecněnou lineární diferenciální rovnici

$$\mathrm{d}x = \mathrm{d}[A(t)]\, x + \mathrm{d}f\,,$$

kde A, $f \in BV_n^{loc}(J)$ a matice $I - \Delta^- A(t)$, $I + \Delta^+ A(t)$ jsou regulované, lze transformovat na obyčejnou lineární diferenciální rovnici

$$\frac{\mathrm{d}y}{\mathrm{d}s} = B(s)\, y + g(s)$$

pomocí pojmu logaritmického protažení podél rostoucí funkce. Tato metoda umožňuje odvodit různé výsledky o zobecněných LDR ze známých vlastností obyčejných LDR. Jako příklad je vyšetřována variační stabilita zobecněné LDR.

Author's address: Lužická 1054, 250 82 Úvaly u Prahy, Czechoslovakia.